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Algebraic Studies of First-Order Enlargements

Abraham Robinson in memoriam

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This paper may be considered as an axiomatic study of first-order elementary extensions and enlargements of *full* relational systems.

An axiomatization of this sort has been given by Robinson and Zakon [16] for superstructures which form a convenient set-theoretic framework for higher-order logic (cf. also Zakon [20], Keisler [8], Stroyan and Luxemburg [19], and Davis [3]). Most of Robinson's and Zakon's axioms (see (4.1)-(4.4) below) are only first order (Keisler [8]: elementary) in character. The language they use is that of a Boolean homomorphism $\alpha: \mathcal{P}(A) \to \mathcal{P}(B)$; for our purposes we will frequently assume α to preserve just all finite intersections. Based on general lattice-theoretic considerations summarized in Section 0 (and Section 2), we will come up with five equivalent approaches in Sections 1-3. Two of these are the extension and contraction procedures (Section 1) well-known from general ring theory, less known from the theory of algebraic lattices. Another equivalent is the passing from a filter to its monad (Section 2), which constitutes one of the fundamental ideas in the exact foundation of Leibniz's infinitesimals discovered by Robinson [14]. A particularly striking equivalent approach seems that of an arbitrary mapping $\omega: B \to \Phi(A)$ (lattice of filters on A). Booleanity of $\alpha: \mathcal{P}(A) \to \mathcal{P}(B)$ being equivalent (see (2.12)) with $\omega[B] \subset \Omega(A)$ (set of ultrafilters on A), $\Omega(A) \subset \omega[B]$ characterizes (see (2.17) and (2.18)) enlargements in the sense of Robinson [14].

Sections 1-3 deal with the zero-order (Boolean) aspects of elementary extensions; for full first-order logic in Sections 4-6, we have to consider a sequence of at-least-finitely-intersection-preserving mappings $\alpha^n: \mathcal{P}(A^n) \to$

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$\mathcal{P}(B^n)(n \geqq 1)$. Theorem 4.1 translates the first-order axioms given by Robinson and Zakon [16] into category language: The family $\alpha = (\alpha^n)$ is interpreted as a twofold natural transformation between functors connected with transformations of variables (substitutions). One of these naturality conditions can be reinterpreted (Addition 4.2) to the effect that each α^n preserves cylindrifications and the diagonal relations, making α^n a homomorphism between cylindric algebras once it is Boolean. At any rate (Theorem 5.1) each α^n is an embedding (Booleanity is not even needed for that), and the sequence $\alpha = (\alpha^n)$ induces an embedding $h: A \to B$. In the Boolean case (Corollary 5.4) the latter turns out to be an elementary embedding from A endowed with all its finitary relations $F \subset A^n$ into B endowed with the corresponding relations $\alpha^n(F) \subset B^n$. On the other hand, any such elementary embedding h makes the mappings α^n satisfy Robinson's and Zakon's axioms and our equivalents (Corollary 5.8). In Section 6, translations of the naturality conditions into the five other languages (extension, contraction, etc.) are given.

0 Extension and contraction of semilattice ideals We start from the following well-known result:

Proposition 0.1 *Given complete lattices I and J, there is a one-to-one correspondence between the completely join-preserving mappings $\phi: I \to J$ and the completely meet-preserving mappings $\psi: J \to I$, established by the formulas*

(0.1) $\phi(x) = \bigwedge\{y \in J | x \leqq \psi(y)\}$ and $\psi(y) = \bigvee\{x \in I | \phi(x) \leqq y\}$

for each $x \in I$, $y \in J$. The ordered pairs (ϕ, ψ) occurring here are exactly the ordered pairs of mappings $\phi: I \to J$, $\psi: J \to I$ such that

(0.2) $\phi(x) \leqq y$ iff $x \leqq \psi(y)$

for each $x \in I$, $y \in J$.

Such an ordered pair (ϕ, ψ) is called a *Galois connection of mixed type* or an *adjoint situation;* ϕ is its *left adjoint,* ψ its *right adjoint.* Note that ϕ preserves zero (least element, join of the empty set), $\phi(0) = 0$, while ψ preserves the identity (greatest element, meet of the empty set), $\psi(e) = e$. (Note that $\phi(e) = e$ iff $\psi(y) = e$ implies $y = e$ for each $y \in J$; it suffices for each y in a meet-dense subset of J. And dually.) It is well-known that $\psi \circ \phi: I \to I$ is a closure operator in I, with image $\psi[J]$; $\phi \circ \psi: J \to J$ is a kernel operator in J, with image $\phi[I]$. The restrictions $\phi | \psi[J]: \psi[J] \to \phi[I]$, $\psi | \phi[I]: \phi[I] \to \psi[J]$ are lattice isomorphisms which are the inverses of each other. Note that ϕ is one-to-one iff ψ is onto I, i.e., iff $\psi \circ \phi = \mathrm{id}_I$.

Let now I be algebraic, i.e., the join-subsemilattice $C(I)$ of compact elements of I (which contains 0) is join-dense in I. Or to put it this way: I is canonically isomorphic to the ideal lattice of the join-semilattice (with 0) $C = C(I)$, the canonical isomorphism assigning to each $x \in I$ the ideal $C \cap [0, x] = \{c \in C | c \leqq x\}$ of C. We now have

Proposition 0.2 *Given complete lattices I and J, with I algebraic. Then each finitely join-preserving mapping α: $C(I) \to J$ can be uniquely extended to a completely join-preserving mapping $\phi: I \to J$. One so gets one-to-one correspondences between all finitely join-preserving mappings α: $C(I) \to J$, all* *completely join-preserving mappings $\phi: I \to J$, and all completely meet-preserving mappings $\psi: J \to I$, established by the formulas*

$$(0.3) \qquad \begin{cases} \alpha = \phi | C(I) \\ \phi(x) = \bigvee\{\alpha(c) | c \in C, c \leqq x\} \\ \psi(y) = \bigvee\{c \in C | \alpha(c) \leqq y\} \end{cases}$$

(and (0.1) or (0.2)).

Actually, the inclusion mapping from $C(I)$ into I is the universal finitely join-preserving mapping of the join-semilattice $C = C(I)$ into a complete lattice—the category of complete lattices here taken with the completely join-preserving mappings as morphisms.

The following is usually stated for the case that J is algebraic too, which turns out to be superfluous.

Proposition 0.3 *Let (ϕ, ψ) be an adjoint situation between the complete lattices I and J, let I be algebraic. Then ϕ is compact, $\phi[C(I)] \subset C(J)$, i.e., $\alpha = \phi | C(I)$: $C(I) \to C(J)$, iff ψ is continuous. In this case, both im ψ = im $\psi \circ \phi$ and im ϕ = im $\phi \circ \psi$ are algebraic.*

One calls $\psi: J \to I$ *(up-)continuous* iff ψ preserves the joins of (up-)directed sets, i.e., ψ is continuous as a mapping from the T_0-space J into the T_0-space I, both J and I endowed with Dana Scott's topology of d-closed lower ends (lower ends closed under joins of (up-)directed sets).

Proposition 0.4 *Let (ϕ, ψ) be an adjoint situation between the complete lattices I and J, let I be algebraic and ϕ compact. Then ϕ is one-to-one iff so is* $\alpha = \phi | C(I)$.

Assuming α one-to-one, hence an order-embedding, one shows that $\phi(x) \leqq \phi(x')$ implies $x \leqq x'$.

Here is some analogue of Proposition 0.3 that will be used in the sequel. A complete lattice J is isomorphic to the lattice of open sets of some topological space (which may be chosen T_0) iff its *spectrum* $P = P(J)$ is meet-dense in J. The spectrum is the (partially ordered) sets of all *(finitely meet-)prime elements,* shortly *primes,* of J. We will call such a lattice a T_0*-lattice.* It is well-known that any *distributive* algebraic lattice J is T_0.

Proposition 0.5 *Let (ϕ, ψ) be an adjoint situation between the complete lattices I and J, let J be T_0. Then ψ is prime, $\psi[P(J)] \subset P(I)$, iff ϕ is finitely meet-preserving (in particular, preserves e). In this case, both* im ϕ = im $\phi \circ \psi$ *and* im ψ = im $\psi \circ \phi$ *are* T_0.

If ϕ is finitely meet-preserving, then both the closure operator $\psi \circ \phi: I \to I$ and the kernel operator $\phi \circ \psi$: $J \to J$ are finitely meet-preserving too. While finitely meet-preserving kernel operators are of quasi-topological character, finitely meet-preserving operators are comparatively rare (but intensively studied more recently).

Proposition 0.6 *Let (ϕ, ψ) be an adjoint situation between the complete lattices I and J, let both I and J be algebraic. Let $C(I)$ be a sublattice of I and*

$e \in C(I)$ (it suffices to assume that $C(I)$ is a lattice with e). Then ϕ is finitely meet-preserving iff $\alpha = \phi \mid C(I)$ is also.

Note that we did not assume compactness of ϕ. Let α be finitely meet-preserving, i.e., a lattice homomorphism preserving both 0 and e. Consider finitely many elements $x_t \in I (t \in T)$. Trivially, $\phi\left(\bigwedge_t x_t\right) \leqq \bigwedge_t \phi(x_t)$. Let now $d \leqq \bigwedge_t \phi(x_t)$, d compact. So $d \leqq \phi(x_s)$ for each $s \in T$. However, x_s is the join in I of all compact elements $c \leqq x_s$. Hence $\phi(x_s)$ is the join in J of their images $\phi(c) = \alpha(c)$. The latter forming a directed set, $d \leqq \alpha(c_s)$ for some compact element $c_s \leqq x_s$. Hence

$$d \leqq \bigwedge_t \alpha(c_t) = \alpha\left(\bigwedge_t c_t\right) = \phi\left(\bigwedge_t c_t\right) \leqq \phi\left(\bigwedge_t x_t\right).$$

This being true for each compact $d \leqq \bigwedge_t \phi(x_t)$, we get $\bigwedge_t \phi(x_t) \leqq \phi\left(\bigwedge_t x_t\right)$, completing the proof.

In our applications, both I and J will be algebraic. By virtue of Proposition 0.2 (and (0.1), (0.2), and (0.3)) we have one-to-one correspondences between

$$(0.4) \quad \begin{cases} \text{the finitely join-preserving mappings } \alpha: C(I) \to J \\ \text{the completely join-preserving mappings } \phi: I \to J \\ \text{the completely meet-preserving mappings } \psi: J \to I. \end{cases}$$

If $C(I)$ is a lattice with identity and J distributive (hence T_0), then Propositions 0.6 and 0.5 make the following equivalent:

$$(0.5) \quad \begin{cases} \alpha: C(I) \to J \text{ is finitely meet-preserving, i.e., a lattice homomorphism} \\ \qquad\qquad \text{preserving 0 and } e \\ \phi: I \to J \text{ is finitely meet-preserving} \\ \psi: J \to I \text{ is prime.} \end{cases}$$

Proposition 0.3 makes the following equivalent:

$$(0.6) \quad \begin{cases} \alpha: C(I) \to C(J) \\ \phi: I \to J \text{ is compact} \\ \psi: J \to I \text{ is continuous.} \end{cases}$$

We may here, in the case (0.6), interpret I and J as the concrete ideal lattices of $C(I)$ and $C(J)$ respectively. $\phi: I \to J$ then turns out to be the well-known (at least in ring theory) *extension of ideals,* and $\psi: J \to I$ the *contraction of ideals* induced by the semilattice homomorphism $\alpha: C(I) \to C(J)$. As a matter of fact, $\psi(y)$ becomes the preimage, under α, of the ideal y of $C(J)$, while $\phi(x)$ becomes the ideal of $C(J)$ generated by the image, under α, of the ideal x of $C(I)$. Correspondingly, $\psi(\phi(x))$ is the *contracted closure of* x, and $\phi(\psi(y))$ the *extended kernel of* y. Proposition 0.4 tells us—under the assumption (0.6)—the equivalence of the following statements:

$$(0.7) \quad \begin{cases} \alpha: C(I) \to C(J) \text{ is one-to-one} \\ \phi: I \to J \text{ is one-to-one} \\ \psi: J \to I \text{ is onto } I \\ \psi \circ \phi = \mathrm{id}_I. \end{cases}$$

In this case, then, each $x \in I$ is contracted.

1 Extension and contraction of filters over sets We now consider sets A and B and the filter lattices $I = \Phi(A)$ and $J = \Phi(B)$, i.e., the ideal lattices of the power sets $\mathcal{P}(A)$ and $\mathcal{P}(B)$, the latter ordered by the dual of inclusion. We then have (from (0.4) and (0.6)) one-to-one correspondences between

$$(1.1)\quad \begin{cases} \text{the finitely meet-preserving mappings } \alpha\colon \mathcal{P}(A) \to \mathcal{P}(B) \\ \text{the completely join-preserving compact mappings } \phi\colon \Phi(A) \to \Phi(B) \\ \text{the completely meet-preserving continuous mappings} \\ \qquad \psi\colon \Phi(B) \to \Phi(A). \end{cases}$$

Here all meets are intersections. Joins in (1.1) are joins of filters, union of the filters form a directed set. Note that the mappings α, ϕ, ψ above satisfy:

$$(1.2)\qquad \alpha(A) = B,\ \phi(\{A\}) = \{B\},\ \psi(\mathcal{P}(B)) = \mathcal{P}(A).$$

Compactness above means the preservation of principal filters, as explicitly shown in the formula

$$(1.3)\qquad \phi([F, A]) = [\alpha(F), B]$$

for each $F \subset A$. In particular,

$$(1.4)\qquad \phi(\mathcal{P}(A)) = [\alpha(\phi), B].$$

With that, we are in the reformulation of the transition formulas (0.2) and (0.3). They now run

$$(1.5)\qquad \phi(\mathcal{F}) \subset \mathcal{B} \text{ iff } \mathcal{F} \subset \psi(\mathcal{B})$$

for each $\mathcal{F} \in \Phi(A)$, $\mathcal{B} \in \Phi(B)$. Moreover,

$$(1.6)\quad \begin{cases} \alpha(F) = \bigcap \phi([F, A]) \\ \phi(\mathcal{F}) = \{G \subset B \mid \alpha(F) \subset G \text{ for some } F \in \mathcal{F}\} \\ \psi(\mathcal{B}) = \{F \subset A \mid \alpha(F) \in \mathcal{B}\}. \end{cases}$$

As a consequence,

$$(1.7)\qquad \phi(\mathcal{F}) = \bigcup \{\phi([F, A]) \mid F \in \mathcal{F}\},$$

i.e., ϕ preserves the representation of a filter $\mathcal{F}$ as the join of a directed family of principal filters.

In model theory, $\alpha(F)$ is usually denoted by $^*F$, and the sets $^*F$ are known as the *standard* (sub)*sets* of B. By virtue of (1.6), each *extended filter* $\mathcal{B} = \phi(\mathcal{F})$ has a basis of standard sets. Conversely, each filter $\mathcal{B} \in \Phi(B)$ admitting a basis of standard sets is extended, $\mathcal{B} = \phi(\psi(\mathcal{B}))$, since, by (1.3), each standard set generates an extended filter and since, by (1.1), the closure operator $\phi \circ \psi$ is continuous.

As a special case of (0.5), the following are now equivalent:

$$(1.8)\quad \begin{cases} \alpha\colon \mathcal{P}(A) \to \mathcal{P}(B) \text{ is finitely join-preserving, i.e., a Boolean homomorphism} \\ \phi\colon \Phi(A) \to \Phi(B) \text{ is finitely meet-preserving} \\ \psi\colon \Phi(B) \to \Phi(A) \text{ is prime} \end{cases}$$

Joins in the power sets are unions, of course. As a consequence of (1.8), we have

(1.9) $\alpha(\phi) = \phi$ and $\phi(\mathcal{P}(A)) = \mathcal{P}(B)$

(and, equivalently, $\psi(\mathcal{G}) \neq \mathcal{P}(A)$ for each filter $\mathcal{G} \neq \mathcal{P}(B)$). Note that the prime elements of the filter lattice are the prime or ultrafilters. So this Boolean case (1.8) is, indeed, characterized by the condition that ψ maps ultrafilters onto ultrafilters. Note that always

(1.10) $\psi(\mathcal{G}) = \bigcap \{\psi(\mathcal{U}) \mid \mathcal{G} \subset \mathcal{U} \in \Omega(B)\}$,

where $\Omega(B)$ denotes the spectrum of $\Phi(B)$, i.e., the set of ultrafilters over B. Indeed, as the *extension mapping* ϕ: $\Phi(A) \to \Phi(B)$ was completely determined, from (1.7), by its restriction to principal filters, i.e., by α, so is now the *contraction mapping* ψ: $\Phi(B) \to \Phi(A)$ completely determined, from (1.10), by its restriction to $\Omega(B)$. So we might add this restriction as an analogue of α to our list of fundamental notions, α, ϕ, ψ (with more to follow in Section 2). But we are not going to do that. All we are saying here is that the Boolean case is characterized by the fact that ψ (its restriction) maps $\Omega(B)$ into $\Omega(A)$: ψ preserves the representation of a filter $\mathcal{G} \in \Phi(B)$ as the intersection of ultrafilters.

As a special case of (0.7), the following are equivalent:

(1.11) $$\begin{cases} \alpha\colon \mathcal{P}(A) \to \mathcal{P}(B) \text{ is one-to-one} \\ \phi\colon \Phi(A) \to \Phi(B) \text{ is one-to-one} \\ \psi\colon \Phi(B) \to \Phi(A) \text{ is onto } \Phi(A) \\ \psi \circ \phi = \mathrm{id}_{\Phi(A)}. \end{cases}$$

The first two statements can be formally weakened as follows:

(1.12) $$\begin{cases} \alpha(F) \neq \alpha(\phi) \text{ for each } F \neq \phi,\ F \subset A \\ \alpha(\{a\}) \neq \alpha(\phi) \text{ for each } a \in A \\ \phi(\mathcal{F}) \neq \phi(\mathcal{P}(A)) \text{ for each } \mathcal{F} \neq \mathcal{P}(A),\ \mathcal{F} \in \Phi(A) \\ \phi(\mathcal{A}) \neq \phi(\mathcal{P}(A)) \text{ for each } \mathcal{A} \in \Omega(A). \end{cases}$$

Indeed, these four statements are equivalent since α and ϕ are order-preserving and because of (1.3) and (1.6). If α is Boolean, i.e., a ring homomorphism, then (1.12) certainly implies (1.11). The point is that Booleanity is not needed here, not even $\alpha(\phi) = \phi$. For let $\alpha(F_1) \subset \alpha(F_2)$. We get

$$\alpha(F_1 - F_2) = \alpha(F_1) \cap \alpha(A - F_2) \subset \alpha(F_2) \cap \alpha(A - F_2) = \alpha(\phi).$$

Then (1.12) makes $F_1 - F_2 = \phi$, i.e., $F_1 \subset F_2$, thereby proving (1.11).

2 Monadization of filters We arrive at another extension of α: $\mathcal{P}(A) \to \mathcal{P}(B)$ by specializing $I = \Phi(A)$, $J = \mathcal{P}(B)$. We so get one-to-one correspondences between

(2.1) $$\begin{cases} \text{the finitely meet-preserving mappings } \alpha\colon \mathcal{P}(A) \to \mathcal{P}(B) \\ \text{the completely join-reversing mappings } \mu\colon \Phi(A) \to \mathcal{P}(B) \\ \text{the completely join-reversing mappings } \gamma\colon \mathcal{P}(B) \to \Phi(A) \\ \text{the mappings } \omega\colon B \to \Phi(A). \end{cases}$$

A mapping is completely join-reversing if it turns arbitrary joins into meets. Note that the mappings α, μ, and ν above satisfy

(2.2) $\alpha(A) = B,\ \mu(\{A\}) = B,\ \nu(\phi) = \mathcal{P}(A)$.

As far as these mappings are concerned, here are the transition formulas obtained from (0.2) and (0.3) (and Section 1):

(2.3) $G \subset \mu(\mathcal{F})$ iff $\mathcal{F} \subset \nu(G)$,

for each $\mathcal{F} \in \Phi(A)$, $G \subset B$. Moreover,

(2.4) $$\begin{cases} \alpha(F) = \mu([F, A]) \\ \mu(\mathcal{F}) = \bigcap \{\alpha(F) | F \in \mathcal{F}\} = \bigcap \phi(\mathcal{F}) \\ \nu(G) = \{F \subset A | G \subset \alpha(F)\} = \psi([G, B]), \end{cases}$$

for each $F \subset A$, $G \subset B$, $\mathcal{F} \in \Phi(A)$. In particular,

(2.5) $\alpha(\phi) = \mu(\mathcal{P}(A))$.

As another consequence of (2.4),

(2.6) $\mu(\mathcal{F}) = \bigcap \{\mu([F, A]) | F \in \mathcal{F}\}$;

i.e., μ reverses the representation of a filter as the join of a directed family of principal filters.

In his foundation of nonstandard analysis, Robinson [14] gave the first satisfactory justification of infinitesimals since Leibniz. It is in this context that he introduced $\mu(\mathcal{F})$ as the *monad of the filter* $\mathcal{F}$, first for a neighborhood filter of a point on the real line, in the complex plane, and in an arbitrary topological space A. Note that the filter monads are exactly the meets (intersections) of standard sets; in fact, the intersections of extended filters. $\nu(G)$ may be called the *comonad of the set* G (it occurs, without a name, in Luxemburg [10], Theorem 2.5.1, as $\mathcal{D}_G$). The comonads are exactly the contractions of principal filters. (2.3) makes (μ, ν) a *Galois connection of Ore type* or a *contravariant adjoint situation between* $\Phi(A)$ *and* $\mathcal{P}(B)$. Accordingly, $\nu \circ \mu$: $\Phi(A) \to \Phi(A)$ is a closure operator in $\Phi(A)$. Since each filter is contained in its *principal closure*, we have in particular $\phi(\mathcal{F}) \subset \left[\bigcap \phi(\mathcal{F}), B\right] = [\mu(\mathcal{F}), B]$, whence

(2.7) $\mathcal{F} \subset \psi(\phi(\mathcal{F})) \subset \nu(\mu(\mathcal{F}))$.

Hence if $\mathcal{F}$ is closed under $\nu \circ \mu$ (a comonad), then $\mathcal{F}$ is closed under $\psi \circ \phi$ (contracted). $\mu \circ \nu$: $\mathcal{P}(B) \to \mathcal{P}(B)$ is a closure operator in $\mathcal{P}(B)$, assigning to each set $G \subset B$ *the monad generated by* G, known as the *discrete monad of* G (see [10], p. 46).

Concerning the mappings ω: $B \to \Phi(A)$, let us observe the following. An element y of a complete lattice J may be called *supercompact* or *completely join-prime* once it satisfies the strongest possible covering condition: For every subset $S \subset J$ (not necessarily directed or finite), $y \leqq \bigvee S$ implies $y \leqq s$ for some $s \in S$. Clearly y is supercompact iff y is compact (S directed) and finitely join-prime (S finite). An analogue of Proposition 0.2 states the following: Suppose the set of supercompact elements of J is join-dense in J. Then each order-preserving mapping from the (partially ordered) set of supercompact elements into a complete lattice I can be uniquely extended to a completely join-preserving mapping from J into I. Observing that the atoms (singletons) of $\mathcal{P}(B)$ (ordered by inclusion) are exactly the supercompact elements of $\mathcal{P}(B)$, one gets the above one-to-one correspondence between arbitrary mappings $\omega: B \to \Phi(A)$ and completely join-reversing mappings $\nu: \mathcal{P}(B) \to \Phi(A)$. One gets formulas analogous to (0.3), (1.4), and (2.4):

$$(2.8) \qquad \begin{cases} \omega(b) = \nu(\{b\}) \\ \nu(G) = \bigcap \{\omega(b) \mid b \in G\} = \bigcap \omega[G] \\ \mu(\mathcal{F}) = \{b \in B \mid \mathcal{F} \subset \omega(b)\} \end{cases}$$

for each $b \in B$, $G \subset B$, $\mathcal{F} \in \Phi(A)$.

As a consequence of (2.3), (2.4), and (2.8), we get now a neat direct relationship between $\alpha: \mathcal{P}(A) \to \mathcal{P}(B)$ and $\omega: B \to \Phi(A)$, namely

$$(2.9) \qquad b \in \alpha(F) \text{ iff } F \in \omega(b)$$

for each $F \subset A$, $b \in B$. Equivalently,

$$(2.10) \qquad \begin{cases} \alpha(F) = \{b \in B \mid F \in \omega(b)\} \\ \omega(b) = \{F \subset A \mid b \in \alpha(F)\}. \end{cases}$$

In particular,

$$(2.11) \qquad \alpha(\phi) = \{b \in B \mid \omega(b) = \mathcal{P}(A)\}.$$

(Of the $6 \times 5 = 30$ transition formulas between α, ϕ, ψ, μ, ν, ω, Formulas (1.4), (1.5), (2.3), (2.4), (2.8), and (2.9) represent 17. The missing 13 formulas would not be very attractive.)

By (0.5), the following are now equivalent:

$$(2.12) \qquad \begin{cases} \alpha: \mathcal{P}(A) \to \mathcal{P}(B) & \text{is finitely join-preserving, i.e., a Boolean homomorphism} \\ \mu: \Phi(A) \to \mathcal{P}(B) & \text{is finitely meet-reversing} \\ \nu: \mathcal{P}(B) \to \Phi(A) & \text{maps atoms onto ultrafilters} \\ \omega: B \to \Omega(A). \end{cases}$$

In particular, one then has

$$(2.13) \qquad \alpha(\phi) = \mu(\mathcal{P}(A)) = \phi$$

(equivalently, $\nu(G) \neq \mathcal{P}(A)$ for each $G \neq \phi$, $\omega(b) \neq \mathcal{P}(A)$ for each $b \in B$). Note that the atoms of the power set $\mathcal{P}(B)$, i.e., its completely join-prime elements, are also its finitely join-prime elements, hence the finitely meet-prime elements of the dual. So ν is indeed prime in the sense of (0.5) iff $\omega[B] \subset \Omega(A)$. Note that always

$$(2.14) \qquad \nu(G) = \bigcap \{\nu(\{b\}) \mid b \in G\}$$

(see (2.8)). In the Boolean case (2.12), ν turns the representation of G as the join of atoms into a representation of $\nu(G)$ as the meet of ultrafilters. As another consequence of Booleanity, we have (cf. Luxemburg [8], Thm. 2.4.3):

$$(2.15) \qquad \mu(\mathcal{F}) = \bigcup \{\mu(\mathcal{U}) \mid \mathcal{F} \subset \mathcal{U} \in \Omega(A)\}.$$

For let $b \in \mu(\mathcal{F})$, then $\mathcal{F} \subset \nu(\{b\}) \in \Omega(A)$, whence

$$b \in \mu(\nu(\{b\})) \subset \bigcup \{\mu(\mathcal{U}) | \mathcal{F} \subset \mathcal{U} \in \Omega(A)\},$$

establishing the nontrivial part of (2.15). As a special case of (2.15), (2.2) yields

(2.16) $\quad B = \bigcup \{\mu(\mathcal{U}) | \mathcal{U} \in \Omega(A)\}$

By (2.13), the monads of different ultrafilters $\mathcal{U}_1$, $\mathcal{U}_2$ are disjoint. In the Boolean case, (2.16) makes B the union of pairwise disjoint sets. Some of the latter may still be empty, however; cf. below.

In the Boolean case (2.12), $\nu \circ \mu$: $\Phi(A) \to \Phi(A)$ is now a finitely meet-preserving closure operator, like $\psi \circ \phi$: $\Phi(A) \to \Phi(A)$. While $\phi \circ \psi$: $\Phi(B) \to \Phi(B)$ was a finitely meet-preserving, i.e., quasi-topological, kernel operator, $\mu \circ \nu$: $\mathcal{P}(B) \to \mathcal{P}(B)$ is now a finitely join-preserving closure operator, i.e., establishes a genuine topology in B, to be looked at more closely in Section 3.

The following are now equivalent:

(2.17) $\begin{cases} \mu\colon \Phi(A) \to \mathcal{P}(B) \text{ is one-to-one} \\ \nu\colon \mathcal{P}(B) \to \Phi(A) \text{ is onto } \Phi(A) \\ \nu \circ \mu = \mathrm{id}_{\Phi(A)} \\ \text{each ultrafilter } \mathcal{U} \in \Omega(A) \text{ is in } \omega[B]. \end{cases}$

The equivalence of the first three statements is trivial. If $\Omega(A) \subset \omega[B] \subset \nu[\mathcal{P}(B)]$, then $\nu[\mathcal{P}(B)]$, being completely meet-closed in $\Phi(A)$, must be all of $\Phi(A)$. Conversely, let $\nu \circ \mu = \mathrm{id}_{\Phi(A)}$ and $\mathcal{U} \in \Omega(A)$. So

$$\mathcal{U} = \nu(\mu(\mathcal{U})) = \bigcap \{\omega(B) | b \in \mu(\mathcal{U})\}$$

$\mathcal{U}$ being a dual atom of $\Phi(A)$ (hence completely meet-irreducible), $\mathcal{U} = \omega(b)$ for some $b \in \mu(\mathcal{U})$, establishing $\Omega(A) \subset \omega[B]$. So this inclusion makes every filter $\mathcal{F} \in \Phi(A)$ closed under $\nu \circ \mu$, i.e., a comonad (and with that contracted, see (2.7)), while $\Omega(A) \supset \omega[B]$ characterized the Boolean case.

Again, the first two statements of (2.17) may be formally weakened:

(2.18) $\begin{cases} \mu(\mathcal{F}) \neq \mu(\mathcal{P}(A)) \text{ for each proper filter } \mathcal{F} \in \Phi(A) \\ \mu(\mathcal{U}) \neq \mu(\mathcal{P}(A)) \text{ for each ultrafilter } \mathcal{U} \in \Omega(A). \end{cases}$

Indeed, since μ is order-reversing, these two statements are equivalent anyway. However, the second statement of (2.18) implies (2.17). Actually,

(2.19) $\quad \mathcal{U} = \omega(b) \text{ iff } b \in \mu(\mathcal{U}) - \mu(\mathcal{P}(A))$

for each ultrafilter $\mathcal{U} \in \Omega(A)$, each element $b \in B$. For suppose $b \in \mu(\mathcal{U}) - \mu(\mathcal{P}(A))$. So $\mathcal{U} \subset \nu(\{b\}) = \omega(b)$ and $\omega(b) = \mathcal{U}$ or $\omega(b) = \mathcal{P}(A)$. Since $b \notin \mu(\mathcal{P}(A)) = \alpha(\phi)$, $\omega(b) \neq \mathcal{P}(A)$ by (2.11). With that, (2.18) implies (2.17) indeed. The other direction of (2.19) is easily obtained from (2.11).

If μ is one-to-one, then so are α and ϕ. In this case, we have a pairwise one-to-one triple alliance between

(2.20) $\begin{cases} \text{the filters } \mathcal{F} \in \Phi(A) \\ \text{the extended filters } \mathcal{B} \in \Phi(B) \\ \text{the monads } M \subset B \end{cases}$

described by

$$(2.21)\quad \begin{cases} \mathscr{B} = \phi(\mathscr{F}) \text{ and } \mathscr{F} = \psi(\mathscr{B}) \\ M = \mu(\mathscr{F}) \text{ and } \mathscr{F} = \nu(M) \\ M = \bigcap \mathscr{B} \text{ and } \mathscr{B} = \phi(\psi([M, B])). \end{cases}$$

In particular, $[M, B]$ is the principal closure of $\mathscr{B}$, $\mathscr{B}$ the extended kernel of $[M, B]$. One so arrives at the pairwise disjoint intervals $[\phi(\mathscr{F}), [\mu(\mathscr{F}), B]] \subset \Phi(B)$ corresponding one-to-one to the filters $\mathscr{F} \in \Phi(A)$, whose union is the set of all filters $\mathscr{B} \in \Phi(B)$ so that $\bigcap \mathscr{B} = \bigcap \phi(\psi(\mathscr{B}))$ (which means $\bigcap \mathscr{B} = \operatorname{adh} \mathscr{B}$ in the Boolean case, cf. (3.9)). Indeed, for filters $\mathscr{F} \in \Phi(A)$, $\mathscr{B} \in \Phi(B)$, $\phi(\mathscr{F}) \subset \mathscr{B} \subset [\mu(\mathscr{F}), B]$ is equivalent to $\mathscr{F} = \psi(\mathscr{B})$ and $\bigcap \mathscr{B} = \bigcap \phi(\mathscr{F})(= \mu(\mathscr{F}))$.

The following remarks make it quite clear that α and ϕ may be one-to-one while μ is not. Indeed, (1.12) can be restated as follows:

$$(2.22)\quad \begin{cases} \mu(\mathscr{F}) \neq \mu(\mathscr{P}(A)) \text{ for each proper } \textit{principal} \text{ filter } \mathscr{F} \in \Phi(A) \\ \mu(\mathscr{U}) \neq \mu(\mathscr{P}(A)) \text{ for each } \textit{principal} \text{ ultrafilter } \mathscr{U} \in \Omega(A) \\ \text{each } \textit{principal} \text{ ultrafilter } \mathscr{U} \in \Omega(A) \text{ is in } \omega[B]. \end{cases}$$

The first two statements weaken (2.18), while the third weakens the last statement of (2.17). The equivalence of the last two statements is a special case of the previous argument based on (2.19).

If α (or equivalently ϕ) is one-to-one, each filter $\mathscr{F} \in \Phi(A)$, principal or not, can be recovered from $\phi(\mathscr{F})$ just by virtue of $\psi(\phi(\mathscr{F})) = \mathscr{F}$. In our applications to model theory, however, we will almost exclusively deal with the special situation $A \subset B$ (or at least A embedded in B). In this case, $\omega(a)$ becomes meaningful for each $a \in A$. Note that, as a consequence of (2.11), the following are equivalent:

$$(2.23)\quad \begin{cases} A \cap \alpha(\phi) = \phi; \\ \omega(a) \neq \mathscr{P}(A) \text{ for each } a \in A \end{cases}$$

(whereas $\alpha(\phi) = \phi$ iff $\omega(b) \neq \mathscr{P}(A)$ for each $b \in B$). We are now going to show the equivalence of the following statements:

$$(2.24)\quad \begin{cases} A \cap \alpha(F) = F \text{ for each finite set } F \in A \\ A \cap \alpha(F) = F \text{ for each set } F \in A \\ \{A \cap G \mid G \in \phi(\mathscr{F})\} = \mathscr{F} \text{ for each filter } \mathscr{F} \in \Phi(A) \\ [\{a\}, A] = \omega(a) \text{ for each } a \in A. \end{cases}$$

The last statement refines the last statement of (2.22) and makes α and ϕ one-to-one (Section 1), as the second and third statements make evident enough. With the first statement, we have $A \cap \alpha(\phi) = \phi$ and $A \cap \alpha(\{a\}) = \{a\}$ for each $a \in A$. So $a \in \alpha(\{a\}) - \alpha(\phi)$, i.e., $a \in \mu([\{a\}, A]) - \mu(\mathscr{P}(A))$, and this is in fact all we need (cf. (2.19)) for $[\{a\}, A] = \omega(a)$. Assuming the latter, let now $F \subset A$ be arbitrary. Then for $a \in A$, $a \in \alpha(F)$ iff $F \in \omega(a) = [\{a\}, A]$, i.e., $a \in F$, showing that $A \cap \alpha(F) = F$. By virtue of (1.3), this can be reinterpreted to the effect that each *principal* filter $\mathscr{F} \in \Phi(A)$ is the "trace" in A of the extended filter $\phi(\mathscr{F}) \in \Phi(B)$. The extension to arbitrary filters $\mathscr{F} \in \Phi(A)$ is based on (1.6), making all statements of (2.24) equivalent.

3 Examples Summarizing, we got one-to-one correspondences between

(3.1)
- the Boolean mappings $\alpha: \mathcal{P}(A) \to \mathcal{P}(B)$
- the completely join-preserving, finitely meet-preserving mappings $\phi: \Phi(A) \to \Phi(B)$ mapping principal filters onto principal filters
- the completely meet-preserving, continuous mappings $\psi: \Phi(B) \to \Phi(A)$ mapping ultrafilters onto ultrafilters
- the completely join-reversing, finitely meet-reversing mappings $\mu: \Phi(A) \to \mathcal{P}(B)$
- the completely join-reversing mappings $\nu: \mathcal{P}(B) \to \Phi(A)$ mapping atoms onto ultrafilters
- the mappings $\omega: B \to \Omega(A)$.

One has, in particular,

$$(3.2) \quad \begin{cases} \alpha(A) = B \text{ and } \alpha(\phi) = \phi \\ \phi(\{A\}) = B \text{ and } \phi(\mathcal{P}(A)) = \mathcal{P}(B) \\ \psi(\mathcal{P}(B)) = \mathcal{P}(A) \text{ and } \psi(\mathcal{B}) = \mathcal{P}(A) \Rightarrow \mathcal{B} = \mathcal{P}(B) \\ \mu(\{A\}) = B \text{ and } \mu(\mathcal{P}(A)) = \phi \\ \nu(\phi) = \mathcal{P}(A) \text{ and } \nu(G) = \mathcal{P}(A) \Rightarrow G = \phi. \end{cases}$$

Also, α is one-to-one iff ϕ is one-to-one iff ψ is onto $\Phi(A)$; μ is one-to-one iff ν is onto $\Phi(A)$ iff ω is onto $\Omega(A)$. In the latter case, $\psi \circ \phi = \nu \circ \mu = \mathrm{id}_{\Phi(A)}$. In general, $\phi \circ \psi: \Phi(B) \to \Phi(B)$ is a finitely meet-preserving kernel operator, $\mu \circ \nu: \mathcal{P}(B) \to \mathcal{P}(B)$ is a topological closure operator:

$$(3.3) \qquad \overline{G} = \mu(\nu(G)) = \bigcap \{\alpha(F) | G \subset \alpha(F)\}$$

is the topological closure of $G \subset B$.

As the simplest case, consider $B = A$ and $\alpha = \mathrm{id}_{\mathcal{P}(A)}$. Here $\phi = \psi = \psi \circ \phi = \phi \circ \psi = \mathrm{id}_{\Phi(A)}$, and

$$(3.4) \qquad \mu(\mathcal{F}) = \bigcap \mathcal{F}, \; \nu(G) = [G, A]$$

for each filter $\mathcal{F} \in \Phi(A)$, each set $G \subset A$. So $\nu: \mathcal{P}(A) \to \Phi(A)$ is the natural embedding. In particular,

$$(3.5) \qquad \omega(b) = [\{b\}, A]$$

for each $b \in A$. All subsets of A being standard, they are all closed. Indeed, $\mu \circ \nu = \mathrm{id}_{\mathcal{P}(A)}$ is the discrete topology, while $\nu \circ \mu$ assigns to each filter $\mathcal{F} \in \Phi(A)$ its principal closure $\left[\bigcap \mathcal{F}, A\right]$. Here α is one-to-one, while μ is not.

More generally, consider any mapping $\sigma: B \to A$ and define $\alpha(F) = \sigma^{-1}[F]$ for each $F \subset A$. Then $\alpha: \mathcal{P}(A) \to \mathcal{P}(B)$ is not just a Boolean homomorphism, it even preserves arbitrary meets and joins. For $\mathcal{F} \in \Phi(A)$, $\phi(\mathcal{F})$ is the filter generated by the preimages $\sigma^{-1}[F]$, $F \in \mathcal{F}$, while for $\mathcal{B} \in \Phi(B)$, $\psi(\mathcal{B})$ now turns out to be the filter generated by the images $\sigma[G]$, $G \in \mathcal{B}$. (It is well-known that the image, $\psi(\mathcal{O})$, of an ultrafilter $\mathcal{O} \in \Omega(B)$ is an ultrafilter, as stated in (3.1).) For $\mathcal{F} \in \Phi(A)$, one has $\mu(\mathcal{F}) = \sigma^{-1}\left[\bigcap \mathcal{F}\right]$, while for $G \subset B$, $\nu(G) = [\sigma[G], A]$;

in particular, $\omega(b) = [\{\sigma(b)\},A]$ for each $b \in B$. Note that α is one-to-one iff $\sigma: B \to A$ is onto A. This still does not make μ one-to-one.

Next, let $B = \Omega(A)$ and $\omega = \mathrm{id}_{\Omega(A)}$. So

(3.6) $\alpha(F) = \{b \in \Omega(A) | F \in b\}$

$(F \subset A)$ is the Stone representation of the Boolean lattice $\mathcal{P}(A)$. We have

(3.7) $\mu(\mathcal{F}) = \{b \in \Omega(A) | \mathcal{F} \subset b\}$, $\nu(\Gamma) = \bigcap \Gamma$

for each filter $\mathcal{F} \in \Phi(A)$, each set $\Gamma \subset \Omega(A)$. So μ is the anti-isomorphism of the filter lattice $\Phi(A)$ onto the lattice of closed sets of the Stone space $\Omega(A)$ (the Stone-Čech compaction of the discrete space A). $\nu \circ \mu = \mathrm{id}_{\Phi(A)}$ restates the ultrafilter theorem (which, of course, has been used here several times), while $\mu \circ \nu: \Omega(A) \to \Omega(A)$ assigns to each $\Gamma \subset \Omega(A)$ its topological closure

(3.8) $\overline{\Gamma} = \mu(\nu(\Gamma)) = \{\mathcal{a} \in \Omega(a) | \bigcap \Gamma \subset \mathcal{a}\}$.

This brings us to the consideration of the topology (3.3) in the general case of an arbitrary mapping $\omega: B \to \Omega(A)$. One can extend (3.3) to arbitrary filters $\mathcal{b} \in \Phi(B)$:

(3.9) $\mathrm{adh}\, \mathcal{b} = \mu(\psi(\mathcal{b})) = \bigcap \phi(\psi(\mathcal{b})) = \bigcap \{\alpha(F) | \alpha(F) \in \mathcal{b}\}$.

Indeed, (3.9) holds for principal filters $\mathcal{b} = [G, B]$ where $\mathrm{adh}\, \mathcal{b} = \overline{G}$. Moreover, $\mu \circ \psi: \Phi(B) \to \mathcal{P}(B)$ turns joins (unions) of directed families into intersections, proving (3.9).

The topology of B is now the initial topology, under ω, of the Stone topology of $\Omega(A)$. Indeed, for $b \in B$ and $G \subset B$, $b \in \overline{G} = \mu(\nu(G))$ is equivalent to $\nu(G) \subset \nu(\{b\})$, i.e., (cf. (2.8)) $\bigcap \omega[G] \subset \omega(b)$, and this (cf. (3.8)) is nothing but $\omega(b) \in \overline{\omega[G]}$. With that, the closed (open) sets $G \subset B$ are precisely the preimages $\omega^{-1}[\Gamma]$ of the closed (open) sets $\Gamma \subset \Omega(A)$. Since $\Omega(A)$ is well-known to be zero-dimensional, so now is B. This can be easily seen. Note that each standard set $\alpha(F)$ is clopen since its complement $B - \alpha(F) = \alpha(A - F)$ is closed. Any closed set (monad) being the intersection of clopen (even standard) sets, every open set is the union of clopen (even standard) sets, making B zero-dimensional. (In particular, $\Omega(A)$ (would we not know it) must be zero-dimensional.)

Each zero-dimensional space is uniformizable, in fact it admits a basis of uniformities which are equivalence relations. The quasi-ordering induced by the topology of any uniform space is always symmetric, in fact, the equivalence relation which is the intersection of the filter of uniformities. Again, the symmetry of our quasi-ordering can be easily seen: for points $b, c \in B$, $b \prec c$, which means $b \in \overline{\{c\}}$ or $\overline{\{b\}} \subset \overline{\{c\}}$, is here equivalent to $\omega(c) \subset \omega(b)$, i.e., to $\omega(b) = \omega(c)$. With that, B is T_0 ($\prec$ antisymmetric) iff B is T_1 ($\prec$ the identity), i.e., iff ω is one-to-one; and in that case, B is totally disconnected (its points being separated by clopen sets), hence T_2. (In particular, $\Omega(A)$ is totally disconnected.)

Since the topology of B is the initial topology, under ω, of the T_0-topology of $\Omega(A)$, the relative space $\omega[B] \subset \Omega(A)$ becomes the T_0-contraction of B, i.e., carries the final (identification, quotient) topology under the contraction mapping ω ($\omega: B \to \omega[B]$ is "strongly continuous").

In case $\omega[B] = \Omega(A)$, μ becomes an anti-isomorphism of the filter lattice $\Phi(A)$ onto the lattice of closed sets of B. Putting complementation on top of that, we get an isomorphism of $\Phi(A)$ onto the lattice of open sets of B. Under this isomorphism, complemented elements (principal filters) correspond to complemented elements (clopen sets), so that standard sets (monads of principal filters) and clopen subsets of B coincide. Also, extended filters and clopen filters (filters admitting a basis of clopen sets) $\mathcal{B} \in \Phi(B)$ coincide. Also, under our isomorphism, compact elements (principal filters) correspond to compact elements (compact open subsets of B), so that standard sets and compact open subsets of B coincide. In particular, B itself is compact, and B is a *Stone space* in the wide sense that the compact open sets form a basis. (All this applies, of course, to the special case $B = \Omega(A)$.)

Suppose, finally, that we are given an *index-domain* T and over it a proper filter $\mathcal{F}$, the *index-filter.* Given any statement $S(t)$ involving the indices $t \in T$, we will say that *$S(t)$ holds for almost all $t \in T$* once $\{t \in T \mid S(t)\} \in \mathcal{F}$. Suppose now we are given mappings $p_t: B \to A$, to be referred to as the *projections.* Not assuming anything further yet, we define α: $\mathcal{P}(A) \to \mathcal{P}(B)$ by

(3.10) $\quad \alpha(F) = \{b \in B \mid p_t(b) \in F \text{ for almost all } t \in T\}$

for any subset $F \subset A$. Then *α is finitely meet-preserving and $\alpha(\phi) = \mu(\mathcal{P}(A)) = \phi$ (the latter being even equivalent with $\mathcal{F} \neq \mathcal{P}(T)$). If $\mathcal{F}$ is an ultrafilter, α is even Boolean* (which will not be so important in the subsequent observations).

Recall that α and ϕ are one-to-one iff $\alpha(\{a\}) \neq \alpha(\phi)$ for each $a \in A$. Here, then, α is one-to-one iff $\alpha(\{a\}) \neq \phi$:

FP0 *For each $a \in A$, there is a $b \in B$ so that $p_t(b) = a$ for almost all $t \in T$.*

For many purposes, we will need more, namely:

FP1 *For each mapping a: $T \to A$, there is a $b \in B$ so that $p_t(b) = a(t)$ for almost all $t \in T$.*

This means the following: Let e_t: $A^T \to A$ $(t \in T)$ be the evaluation mappings, p: $B \to A^T$ be such that $e_t \circ p = p_t$ for each $t \in T$. Let k: $A^T \to A^T/\mathcal{F}$ be the natural projection onto the partition of A^T modulo $\mathcal{F}$. Then FP1 states that $k \circ p$: $B \to A^T/\mathcal{F}$ is onto $A^T/\mathcal{F}$. $k \circ p$ is one-to-one iff the following holds:

FP2 *For each $b, b' \in B$, if $p_t(b) = p_t(b')$ for almost all $t \in T$, then $b = b'$.*

Together with FP0, FP2 makes every set $\alpha(\{a\})(a \in A)$ an atom of $\quad(B)$, $\alpha(\{a\}) = \{h(a)\}$ (usually written $*\{a\} = \{*a\}$). In fact, the mapping h: $A \to B$ so obtained must be one-to-one since α is. We may then readjust the whole setting to the effect that $A \subset B$ and h is the inclusion mapping, as we will assume henceforth. FP0 ($\alpha(\{a\}) \neq \phi$) is so strengthened to $a \in \alpha(\{a\})$:

FP3 *For each $a \in A$, $p_t(a) = a$ for almost all $t \in T$.*

FP3 is actually equivalent with $F \subset \alpha(F)$ for each subset $F \subset A$, even (since $\mathcal{F}$ is a proper filter) with each of the equations (2.24). If α is Boolean, FP2 and FP3 permit us to strengthen the first equation of (2.24) to

(3.11) $\quad \alpha(F) = F$ for each finite set $F \subset A$.

(FP2 and FP3 guarantee already $\alpha(\{a\}) = \{a\}$.)

Note that FP1, FP2, and FP3 characterize the Filter Power extension $A^T/\mathscr{J}$ of A up to (unique) isomorphism. It is well-known that for many purposes one needs a particularly neat index-filter. From our viewpoint, the most obvious question is: when is μ one-to-one? This happens, remember, iff $\mu(\mathscr{F}) \neq \mu(\mathscr{P}(A))$ for each filter $\mathscr{F} \neq \mathscr{P}(A)$. Here, then, μ is one-to-one iff $\mu(\mathscr{F}) \neq \phi$ for each proper filter $\mathscr{F}$. In the presence of FP1, one can now show that $\mu(\mathscr{F}) \neq \phi$ iff the index-filter $\mathscr{J}$ is *adequate for* $\mathscr{F}$: There is a mapping $a: T \to A$ so that

$$\mathscr{F} \subset \{G \subset A \mid a[J] \subset G \text{ for some } J \in \mathscr{J}\}$$

or equivalently,

$$\{a^{-1}[F] \mid F \in \mathscr{F}\} \subset \mathscr{J}.$$

Or to put it this way: each $F \in \mathscr{F}$ contains almost all elements $a(t)$. This is certainly so in case $b \in \mu(\mathscr{F})$: Putting $a(t) = p_t(b)$ for each $t \in T$, we indeed have such a mapping $a: T \to A$. Conversely, let a be such a mapping for $\mathscr{F}$. By virtue of FP1, there is a $b \in B$ so that $p_t(b) = a(t)$ for almost all $t \in T$. For any $F \in \mathscr{F}$, we get $b \in \alpha(F)$, whence $b \in \mu(\mathscr{F})$ and $\mu(\mathscr{F}) \neq \phi$. So in the presence of FP1, μ is one-to-one iff

FP4 $\quad$ *$\mathscr{J}$ is adequate for A,*

i.e., $\mathscr{J}$ is adequate for every proper filter $\mathscr{F} \in \Phi(A)$. (For the notion of adequacy, cf. Bruns and Schmidt [1], A. L. Stone [18], and Luxemburg [10].)

The standard (or shall we say nonstandard?) application of the final remarks is to *concurrent* (upwards directed) *relations* $R \subset A \times A$, for which, by definition,

$$A/R = \{R[\{a\}] \mid a \in \operatorname{dom} R\}$$

has the finite intersection property, i.e., generates a proper filter over A (cf. Robinson [14], [15] and Luxemburg [10]).

4 First-order properties: The two transformation functors So far, we have been dealing with the Boolean operations, i.e., the zero-order (propositional) part of logic. For the needs of first-order logic, we will consider all Cartesian powers A^n, B^n ($n \geqq 1$) of our carrier sets A, B. (Alternatively, one might throw all these finite powers into some infinite power, say A^ω, B^ω, as usual in many algebraic treatments of first-order logic.) To be more precise, we will assume that for each $n \geqq 1$, a finitely meet-preserving (if not Boolean) mapping

$$\alpha^n: \mathscr{P}(A^n) \to \mathscr{P}(B^n)$$

is given, or, equivalently, any other of the corresponding mappings ϕ^n, ψ^n, μ^n, ν^n, ω^n studied in full detail in Sections 1-3. For our purpose, these homomorphisms α^n should be interrelated by first-order properties. It would be nice, of course, to define α^n recursively in terms of the preceding mappings α^m ($m < n$). This not being feasible, the interrelations may be put down in axiomatic form.

Such an axiomatization has been given by Robinson and Zakon [16]. Here are their postulates: For every m, $k \geqq 1$, $E \subset A^m$, $E' \subset A^k$, (4.1) $\alpha^{m+k}(E \times E') = \alpha^m(E) \times \alpha^k(E')$.

Moreover, for every $m, k \geqq 1$ and $F \subseteq A^{m+k}$,

(4.2) $\alpha^m(\mathrm{dom}_m F) = \mathrm{dom}_m \alpha^{m+k}(F)$, $\alpha^k(\mathrm{im}_k F) = \mathrm{im}_k \alpha^{m+k}(F)$;

for the definition of dom_m and im_k, cf. below. Furthermore,

(4.3) $\alpha^n(\sigma^{-1}[F]) = \sigma^{-1}[\alpha^n(F)]$

for every $n \geqq 1$, every permutation σ of $\{1, \ldots, n\}$, every $F \subset A^n$. Finally, Robinson and Zakon call $\alpha = (\alpha^n)$ *normal* provided that

(4.4) $\alpha^2(\mathrm{id}_A) = \mathrm{id}_B$.

One may accept these conditions (as, for instance, their analogues in the theory of projective sets, cf. below) without formal challenge. Robinson and Zakon, however, added some dogmatic remarks which are certainly not ill-advised. Defining the ordered pair (a_1, a_2) à la Kuratowski and the ordered n-tuple $(a_1, \ldots, a_n)$ recursively, one has to face nuisances like the fact that $E \times E'$ in (4.1) is not really a subset of A^{m+k}. This is remedied by a natural equivalence between $A^m \times A^k$ and A^{m+k}. Robinson and Zakon introduce now an additional postulate concerning so-called *groupings*, i.e., natural equivalences of that type. However, we can avoid these inessential technicalities still playing it safe. To that end, we will consider the free semigroup over A, FS(A), likewise the free semigroup FS(B). We may think of the elements of FS(A) as *abstract products*

$$a = \prod_{k=1}^{n} a_k$$

with the *length* $n \geqq 1$ and the *factors* $a_1, \ldots, a_n \in A$ uniquely determined. Alternatively, we may regard the elements of FS(A) as *concrete words*

$$a = \langle a_1, \ldots, a_n \rangle,$$

i.e., *functions* a: $\{1, \ldots, n\} \to A$, the function value $a(k) = a_k \in A$ being the *k'th letter*. A^n will then be the set of words of length n, with $A^1 = A$. The associativity of FS(A) makes all those grouping considerations superfluous. For example, $E \times E'$ in (4.1) is now the ("complex") product $E \cdot E'$ in that semigroup, $A^m \times A^k = A^m \cdot A^k = A^{m+k}$, etc. Any $F \subset A^{m+k}$ can now be considered, as in (4.2), as a *binary* relation $F \subset A^m \times A^k$. Its domain will then be the set of m-tuples $\langle a_1, \ldots, a_m \rangle$ such that $\langle a_1, \ldots, a_m, a_{m+1}, \ldots, a_{m+k} \rangle \in F$ for some k-tuple $\langle a_{m+1}, \ldots, a_{m+k} \rangle$, and this is what we mean by $\mathrm{dom}_m F$ (Robinson and Zakon [1] write simply $D(F)$; Kuratowski and Mostowski ([9], Chapter V, Section 5) write $P_m^{m+k}(F)$).

The meaning of the permutation axiom (4.3) is, of course, more or less clear (except that one might mix up σ and σ^{-1}). However, we are going to extend (4.3) to arbitrary mappings (*transformations*)

$$\sigma: \{1, \ldots, m\} \to \{1, \ldots, n\}$$

($m, n \geqq 1$) anyway. (In the sequel, we will simply write σ: $m \to n$.) So we had better say somewhat more here. The set $N^* = \{1, 2, \ldots\}$ together with these transformations forms a category. Given σ above, we get an induced mapping

$$\sigma_A : A^n \to A^m$$

by the definition

(4.5) $\sigma_A(\langle a_1, \ldots, a_n \rangle) = \langle a_{\sigma(1)}, \ldots, a_{\sigma(m)} \rangle,$

for each $a_1, \ldots, a_n \in A$ (i.e., for each functions a: $\{1, \ldots, n\} \to A$, we have $\sigma_A(a) = a \circ \sigma$). With that, we established a contravariant functor. We may put on top of that the contravariant power set functor by passing from σ_A to the Boolean homomorphisms

$$\alpha_\sigma : \mathcal{P}(A^m) \to \mathcal{P}(A^n)$$

described in Section 3; i.e., for $E \subset A^m$,

(4.6) $\alpha_\sigma(E) = \sigma_A^{-1}[E] = \{\langle a_1, \ldots, a_n \rangle \in A^n | \langle a_{\sigma(1)}, \ldots, a_{\sigma(m)} \rangle \in E\}.$

With that, we established *the covariant transformation functor* (a left action) $\sigma \mapsto \alpha_\sigma$. There is reason to use also the covariant power set functor. That is, we consider, for each $F \subset A^n$,

(4.7) $\beta_\sigma(F) = \sigma_A[F] = \{\langle a_{\sigma(1)}, \ldots, a_{\sigma(m)} \rangle | \langle a_1, \ldots, a_n \rangle \in F\}.$

We get a completely join-preserving mapping

$$\beta_\sigma : \mathcal{P}(A^n) \to \mathcal{P}(A^m)$$

and *the contravariant transformation functor* (a right action) $\sigma \mapsto \beta_\sigma$.

We are now going to show

Theorem 4.1 *Let A and B be sets; for each $n \geqq 1$, let $\alpha^n : \mathcal{P}(A^n) \to \mathcal{P}(B^n)$ be a finitely meet-preserving mapping:*

(4.8) $\alpha^n(F_1 \cap F_2) = \alpha^n(F_1) \cap \alpha^n(F_2)$

for each $F_1, F_2 \subset A^n$, and

(4.9) $\alpha^n(A^n) = B^n.$

Then (4.1)-(4.4) *are equivalent to*

(4.10) $\alpha^n(\sigma_A^{-1}[E]) = \sigma_B^{-1}[\alpha^m(E)]$

and

(4.11) $\alpha^m(\sigma_A[F]) = \sigma_B[\alpha^n(F)]$

for each transformation $\sigma: m \to n$ ($m, n \geqq 1$), each $E \subset A^m$, $F \subset A^n$.

According to (4.10) and (4.11), the sequence $\alpha = (\alpha^n)$ can be interpreted as a twofold natural transformation, between the covariant and between the contravariant transformation functors (associated with the carriers A and B) respectively.

Proof of Theorem 4.1: Note that $E \times E' = (E \times A^k) \cap (A^m \times E')$ for each $m, k \geqq 1, E \subset A^m, E' \subset A^k$. With the permutation axiom (4.3), it thus suffices to postulate (4.1) only for the special case $E' = A^k$. However, $\alpha^{m+k}(E \times A^k) = \alpha^m(E) \times B^k$ is simply (4.10) for the special case of the inclusion $\sigma: m \to n = m + k$ (it suffices to postulate this only for $k = 1$). Consequently, (4.1) *and* (4.3) *are equivalent to* (4.10) *for one-to-one transformations* σ. Again, the permutation axiom (4.3) makes the second equation of (4.2) superfluous, and the first equation is simply (4.11) for the special case of the inclusion $\sigma: m \to n = m + k$ (it suffices again to postulate this only for $k = 1$). Consequently, (4.2) *and* (4.3) *are equivalent to* (4.11) *for one-to-one transformations* σ.

(4.4) *is now an immediate consequence of* (4.11) *for onto-transformations* σ. Indeed, with the *total identification* $\Delta^n: n \to 1$.

(4.12) $\Delta^n[A] = \{\langle a, \ldots, a\rangle \in A^n | a \in A\}$

is the *total diagonal of* A^n, which will be mapped by α^n onto the total diagonal of B^n. In particular, $\alpha^1(A) = B$ and $\alpha^2(\mathrm{id}_A) = \mathrm{id}_B$.

Let us now look once more at the *standard inclusion* $\iota_n: n-1 \to n$ $(n \geqq 2)$. It has a left inverse $\mathscr{K}_n: n \to n-1$, the *standard identification* that identifies n and $n-1$. Let $\delta_n: 2 \to n$ $(n \geqq 2)$ be the transformation that throws 1 onto $n-1$ and 2 onto n. We get the *standard partial diagonal*

(4.13) $\delta_n^{-1}[id_A] = \{\langle a_1, \ldots, a_n\rangle \in A^n | a_{n-1} = a_n\}$.

For $E \subset A^{n-1}$, we have

(4.14) $\mathscr{K}_n[E] = \iota_n^{-1}[E] \cap \delta_n^{-1}[\mathrm{id}_A] = (E \times A) \cap \delta_n^{-1}[\mathrm{id}_A]$,

while for $F \subset A^n$,

(4.15) $\mathscr{K}_n^{-1}[F] = \iota_n[F \cap \delta_n^{-1}[\mathrm{id}_A]] = \mathrm{dom}_{n-1}(F \cap \delta_n^{-1}[\mathrm{id}_A])$.

(4.14) and (4.15) express the (right and left) action of the simplest nontrivial onto-transformation, $\mathscr{K}_n$, in terms of the simplest nontrivial one-to-one transformation, ι_n (and the partial diagonals). Since every onto-transformation is a product of standard identifications and permutations, (4.1)-(4.4) *imply* (4.10) *and* (4.11) *also for onto-transformations* σ, *hence for arbitrary transformations* σ. ((4.14) shows that for the proof of (4.11) we can get away without (4.2).) This ends the proof of Theorem 4.1.

Robinson and Zakon's axioms (4.1)-(4.4) are, of course, closely related to well-known set-theoretic or algebraic approaches to first-order calculus. There is a very close relationship, indeed, to the theory of projective sets (for references, cf. Kuratowski and Mostowski [9], Chapter X, Section 5). There one considers a family

$$\mathscr{B} \subset \bigcup_{n=1}^{\infty} \mathscr{P}(A^n)$$

called a *base,* so that each $\mathscr{B}_n = \mathscr{B} \cap \mathscr{P}(A^n)$ $(n \geqq 1)$ is a Boolean subalgebra of $\mathscr{P}(A^n)$. Moreover, $\mathscr{B}$ is assumed closed under Cartesian products, permutations, and covariant (left) actions of identifications, i.e., onto-transformations $\sigma: n \to n-1$ (of which our standard identification $\mathscr{K}_n: n \to n-1$ was the prototype). In other words, a *base* $\mathscr{B}$ *is closed under the covariant transformation* *functor.* Given such a base $\mathcal{B}$ (e.g., the family of Borel sets of some topological space A), one then passes to the least base, $\mathcal{C}$, closed also under taking domains (cf. (4.2)). $\mathcal{C}$ is then called the family of *projective sets over the base* $\mathcal{B}$. So $\mathcal{C}$ is *a projective family* (over some base $\mathcal{B}$) iff $\mathcal{C}$ is *a base closed under the contravariant transformation functor at least for one-to-one transformations* σ. Such a *projective family is unrestrictedly closed under the contravariant transformation functor iff* $\mathrm{id}_A \in \mathcal{C}$, i.e., iff all total diagonals $\Delta^n[A]$ belong to $\mathcal{C}$, equivalently, iff all standard partial diagonals $\delta_n^{-1}[\mathrm{id}_A]$, hence all partial diagonals (cf. below), belong to $\mathcal{C}$. The least projective family $\mathcal{C}$, consisting of all powers A^n ($n \geqq 1$) and the empty set, shows that $\mathrm{id}_A \notin \mathcal{C}$ would be possible (but is it desirable?).

So closure under the contravariant transformation functor can be partly replaced by the more or less strongly formulated *diagonal condition* above. Since the contravariant functor has quite undesirable features, we would like to get rid of it completely. This can be achieved with the unary *cylindric operations* (*cylindrifications*) or *quantifiers*

$$\exists_k^n\colon \mathcal{P}(A^n) \to \mathcal{P}(A^n)$$

($1 \leqq k \leqq n$) which are characteristic for both the *cylindric algebras* (cf. Henkin and Tarski [7]; Henkin, Monk, and Tarski [6]; Henkin and Monk [5]) and the *polyadic algebras* (cf. Halmos [4]). For $n \geqq 2$, we may introduce the *n'th standard quantifier* $\exists_n^n$ as follows: For $F \subset A^n$,

$$(4.16) \qquad \begin{aligned} \exists_n^n(F) &= \iota_n^{-1}[\iota_n[F]] = (\mathrm{dom}_{n-1}F) \times A \\ &= \{\langle a_1, \ldots, a_n\rangle \in A^n \mid \langle a_1, \ldots, a_{n-1}, x\rangle \in F \text{ for some } x \in A\}. \end{aligned}$$

Here $\iota_n\colon n-1 \to n$ ($n \geqq 2$) is the *standard inclusion* used before. Clearly, $\exists_n^n$ is a completely join-preserving closure operator in $\mathcal{P}(A^n)$, the closed subsets of A^n being all sets saturated with respect to the equivalence relation induced by the (contravariant) action of ι_n on A^n. With the *standard identification* $\mathcal{K}_n\colon n \to n-1$ which satisfies $\mathcal{K}_n \circ \iota_n = \mathrm{id}_{n-1}$ ($n \geqq 2$), contravariance yields a converse of (4.16):

$$(4.17) \qquad \iota_n[F] = \mathcal{K}_n^{-1}[\exists_n^n(F)]$$

(cf. also (4.15)). As a consequence of (4.16) and (4.15), we also get

$$(4.18) \qquad (\iota_n \circ \mathcal{K}_n)^{-1}[F] = \exists_n^n(F \cap \delta_n^{-1}[\mathrm{id}_A]),$$

a formula used in the theory of cylindric algebras (cf. [6], Sections 1.5 and 1.11) to express at least some substitutions, i.e., actions of the covariant transformation functor, in terms of the cylindric operations. (The reader may determine the meaning of $(\iota_n \circ \mathcal{K}_n)[F]$.)

As a result of (4.16) and (4.17), a *projective family* is now the same as *a base* $\mathcal{C}$ *closed under all quantifiers.* (Note that polyadic algebras are abstract Boolean algebras with abstract quantifiers and a covariant action of transformations.) And the base $\mathcal{C}$ is completely closed under the contravariant transformation functor iff $\mathcal{C}$ is closed under all quantifiers and contains all partial diagonals: each $\mathcal{C}_n$ ($n \geqq 2$) is a *subalgebra of the cylindric algebra* $\mathcal{P}(A^n)$, i.e., of the Boolean algebra enriched by the quantifiers

(4.19) $\exists_k^n(F) = \{\langle a_1, \ldots, a_n\rangle \in A^n \mid \langle a_1, \ldots, a_{k-1}, x, a_{k+1}, \ldots, a_n\rangle \in F$ for some $x \in A\}$

($1 \leqq k \leqq n$) and the $(n/2)$ *partial diagonals*

(4.20) $D_{ik}^n(A) = \{\langle a_1, \ldots, a_n\rangle \in A^n \mid a_i = a_k\}$

($1 \leqq i < k \leqq n$). Correspondingly, we get

Addition 4.2 *In Theorem 4.1, condition (4.11) may be equivalently replaced by*

(4.21) $\alpha^n(\exists_k^n(F)) = \exists_k^n(\alpha^n(F))$

or shorter $\alpha^n \circ \exists_k^n = \exists_k^n \circ \alpha^n$ ($1 \leqq k \leqq n; n \geqq 2$), *and*

(4.22) $\alpha^n(D_{ik}^n(A)) = D_{ik}^n(B)$

($1 \leqq i < k \leqq n; n \geqq 2$).

In other words, α^n: $\mathcal{P}(A^n) \to \mathcal{P}(B^n)$ ($n \geqq 2$) is not just a Boolean homomorphism, but a *homomorphism between the cylindric algebras.* Note that in (4.21) and (4.22) the type n is kept fixed, in striking contrast to (4.2) and (4.11). It suffices, of course, to postulate (4.21) for the special case $k = n$. (4.22) may be specialized, without loss of generality, to (4.4) ($i = 1, k = n = 2$), and then be generalized to all total diagonals:

(4.23) $\alpha^n(\Delta^n[A]) = \Delta^n[B]$.

*5 **Elementary extensions, enlargements of full relational systems*** The last observations about the partial diagonals $D_{ik}^n(A)$ and the quantifiers $\exists_k^n$ were all made with the assumption $n \geqq 2$ in mind. Of course, we may take (4.20) as the definition of $D_{ik}^n(A)$ without restriction whatsoever, so that conveniently

(5.1) $D_{ik}^n(A) = \sigma^{-1}[\mathrm{id}_A]$,

where σ: $2 \to n$ ($n \geqq 1$) is quite arbitrary, and $\sigma(1) = i$, $\sigma(2) = k$. Hence $D_{ik}^n(A) = D_{ki}^n(A)$ and $D_{kk}^n(A) = A^n$. The latter is still true in case $k = n = 1$, and (4.22) will still hold in this more general case.

It takes somewhat more to show that (4.21) is still true for $n = 1$, where $\exists^1 = \exists_1^1$: $\mathcal{P}(A) \to \mathcal{P}(A)$ is defined by (4.19), but right now without an analogue of (4.16). More generally, we may introduce the *n'th total quantifier*

(5.2) $\exists^n = \exists_1^n \circ \exists_2^n \circ \ldots \circ \exists_n^n$,

which is still a topological closure operator on the set A^n, with ϕ and A^n as the only closed subsets of A^n. So $\exists^n$ can be used as a discriminator to decide whether a subset $F \subset A^n$ is empty or not. We are going to show much more than (4.21) for $n = 1$.

Theorem 5.1 *Suppose the mappings* α^n: $\mathcal{P}(A^n) \to \mathcal{P}(B^n)$ ($n \geqq 1$) *satisfy all conditions of Section 4: (4.1)-(4.4), (4.8)-(4.9), (4.10)-(4.11), (4.21), and (4.22) for* $n \geqq 2$. *Then each* α^n ($n \geqq 1$) *is one-to-one, with*

(5.3) $\alpha^n(\phi) = \phi$.

Moreover, there is a (unique) mapping $h: A \to b$, also one-to-one, such that

(5.4) $$\alpha^n(\{\langle a_1, \ldots, a_n\rangle\}) = \{\langle h(a_1), \ldots, h(a_n)\rangle\}$$

for each $n \geqq 1, a_1, \ldots, a_n \in A$. Finally, (4.21) holds also for $n \geqq 1$, so that

(5.5) $$\alpha^n(\exists^n(F)) = \exists^n(\alpha^n(F))$$

for each $n \geqq 1, F \subset A^n$.

Proof: As a consequence of (4.1) and (4.4), $\alpha^1(\{a\})$ $(a \in A)$ has at most one element, and so has, by another application of (4.1), $\alpha^n(\{\langle a_1, \ldots, a_n\rangle\})$ $(n \geqq 1, a_1, \ldots, a_n \in A)$. Hence $\alpha^n(\phi)$, being a subset of $\alpha^n(\{\langle a_1, \ldots, a_n\rangle\})$, has at most one element too. However, for $n \geqq 2$, (5.5) holds, as a consequence of (4.21). We get $\exists^n(\alpha^n(\phi)) = \alpha^n(\exists^n(\phi)) = \alpha^n(\phi)$, whence $\alpha^n(\phi) = \phi$ or $\alpha^n(\phi) = B^n$. *Assuming that B has at least two elements,* we definitely get $\alpha^n(\phi) = \phi$ $(n \geqq 2)$. Using (4.1) again, we also have $\alpha^1(\phi) = \phi$, proving (5.3) without restriction. Again, for $n \geqq 2$, $F \subset A^n$ and $F \neq \phi$, we get $\exists^n(\alpha^n(F)) = \alpha^n(\exists^n(F)) = \alpha^n(A^n) = B^n$ and $\alpha^n(F) \neq \phi = \alpha^n(\phi)$. As shown in Section 1, $\alpha^n (n \geqq 2)$ is one-to-one. Using (4.1) once more, we get $\alpha^1(F) \neq \phi = \alpha^1(\phi)$ for $F \subset A$, $F \neq \phi$, so that α^1 is one-to-one too. In particular, there is a one-to-one mapping $h: A \to B$ so that (5.4) holds for $n = 1$, hence, by virtue of (4.1), for every $n \geqq 1$. (5.5), i.e., (4.21), holds now for $n = 1$ too, be F empty or not.

Robinson and Zakon [16] assumed α^n one-to-one right from the start and stated (5.4) as an axiom. That this is superfluous here is essentially due to (4.4), which they did not throughout assume in their presentation.

In order to round off our findings, we might extend the considerations of Section 4 to the case $n = 0$. Note that $A^0 = B^0$ is the singleton whose only element is the empty sequence or 0-tuple, i.e., the identity of the free semigroup with identity over A or B respectively. So $\mathcal{P}(A^0) = \mathcal{P}(B^0)$ is the Boolean algebra of two elements, to be identified with the truth-values "true" and "false", if one so wishes. One will then introduce $\alpha^0: \mathcal{P}(A^0) \to \mathcal{P}(B^0)$ as the only automorphism of $\mathcal{P}(A^0) = \mathcal{P}(B^0)$, so that (4.8), (4.9), (5.3), and (5.4) will still be true. Extending (5.2), one may also introduce $\exists^0: \mathcal{P}(A^0) \to \mathcal{P}(A^0)$ as the identity of $\mathcal{P}(A^0)$, so that (5.5) will still hold. One would also admit the empty transformation $\sigma: 0 \to n$, making (4.10) and (4.11) still valid. With $\Delta^0 = \iota_1: 0 \to 1$, we will get $\Delta^0[A] = A^0$ (extending (4.12)), so that (4.23) will still be true. Also, (4.16) will still be valid for $n = 1$ (making the exceptional role of $n = 1$ vanish):

(5.6) $$\exists^1(F) = \exists^1_1(F) = \iota_1^{-1}[\iota_1[F]] = \{a \in A \mid F \neq \phi\}.$$

However, there is no transformation $\mathscr{C}_1: 1 \to 0$, no transformation $\sigma: 2 \to 0$. Correspondingly, none of the formulas involving the partial diagonals can be extended.

We may again think of $h: A \to B$ as genuine inclusion, with the corresponding omission of h in (5.4). With that, we get

Corollary 5.2 *Suppose everything is as in Theorem 5.1. Suppose the mapping $h: A \to B$ is, in fact, inclusion. Then for every $n \geqq 1$:*

(5.7) $\begin{cases} A^n \cap \alpha^n(F) = F \text{ for every subset } F \subset A^n \\ \{A^n \cap G \mid G \in \phi^n(\mathcal{F})\} = \mathcal{F} \text{ for every filter } \mathcal{F} \in \Phi(A^n) \\ [\{a\}, A^n] = \omega^n(a) \text{ for every point } a \in A^n. \end{cases}$

If α^n is Boolean, then

(5.8) $\alpha^n(F) = F$ *for every finite subset* $F \subset A^n$.

Proof: By (5.3) and (5.4), we have $\alpha^n(F) = F$ for each subset $F \subset A^n$ of at most one element. If α^n is Boolean, (5.8) follows. At any rate, we have $A^n \cap \alpha^n(F) = F$ for each subset $F \subset A^n$ of at most one element, and this was all needed to prove the statements (2.24), i.e., (5.7).

Suppose now that we are, in fact, considering a relational system $\mathfrak{A} = (A, (F_i)_{i \in I})$ of type $(m_i)_{i \in I}$. In other words, for each index $i \in I$, a *fundamental relation* $F_i \subset A^{m_i}$ ($m_i \geqq 1$) is given. For each $n \geqq 1$, we introduce the *n-ary elementary relations* as the relations $\sigma^{-1}[F_i]$, where $i \in I$ and $\sigma\colon m_i \to n$ is an arbitrary transformation. The *n-ary derived relations* form, by definition, the subalgebra, $\mathcal{O}_n(\mathfrak{A})$, of the cylindric algebra $\mathcal{P}(A^n)$ generated by the n-ary elementary relations $\sigma^{-1}[F_i]$; i.e., $\mathcal{O}_n(\mathfrak{A})$ is the least Boolean subalgebra of $\mathcal{P}(A^n)$ containing the n-ary elementary relations $\sigma^{-1}[F_i]$, the n-ary partial diagonals $\sigma^{-1}[\mathrm{id}_A]$ ($\sigma\colon 2 \to n$), and closed under the quantifiers $\exists_k^n$ ($1 \leqq k \leqq n$). This corresponds precisely to the definition of the *n-ary algebraic* or *polynomial operations* in an algebraic system (e.g., see Henkin, Monk, and Tarski [6]). As the n-ary polynomial operations may be described (or indicated) by means of the *n-ary polynomials* (frequently called "polynomial symbols", "words", "terms", etc.), so we may use (abuse?) here the first-order language of type $(m_i)_{i \in I}$ in n distinct variables $x_1, \ldots, x_n$ (ordered by their indices) for the sole purpose of describing (indicating) the n-ary derived relations above. This language forms an algebraic system, denoted by $\mathcal{L}_n$, with a binary operation $\wedge$ (*conjunction*), a unary operation $\neg$ (*negation*), n unary operations $\bigvee_k^n$ ($1 \leqq k \leqq n$) (the *quantifiers*), and n^2 nullary operations d_{ik}^n ($1 \leqq i, k \leqq n$) (the *diagonal constants*); i.e., $\mathcal{L}_n$ is, like $\mathcal{P}(A^n)$, an algebra of the first-order logic type

$$(2, 1, \underbrace{1, \ldots, 1}_{n \text{ times}}, \underbrace{0, \ldots, 0}_{n^2 \text{ times}}).$$

The elements of that algebra are usually called *formulas*. As an algebra, $\mathcal{L}_n$ is generated by the *elementary formulas*

$$\langle P_i, x_{\sigma(1)}, \ldots, x_{\sigma(m_i)} \rangle$$

where the *predicates* P_i ($i \in I$) are distinct elements different from the variables $x_1, \ldots, x_n$ and $\sigma\colon m_i \to n$ any transformation. (This way of writing the elementary formulas is only a concession to common usage. It would suffice to define the elementary formulas as ordered pairs $\langle P_i, \sigma \rangle$, or even $\langle i, \sigma \rangle$.) The crucial property of the algebra $\mathcal{L}_n$ is its *absolute freeness* (expressible in terms of *generalized Peano axioms*). As a typical consequence of that, there is exactly one homomorphism

$$\mathrm{nat}_{\mathfrak{A}}^n : \mathcal{L}_n \to \mathcal{P}(A^n)$$

actually onto $\mathcal{O}_n(\mathfrak{A})$, such that, for each $i \in I$, $\sigma: m_i \to n$:

$$\text{(5.9)} \qquad \mathrm{nat}^n_{\mathfrak{A}}(\langle P_i, x_{\sigma(1)}, \ldots, x_{\sigma(m_i)}\rangle) = \sigma^{-1}[F_i].$$

We may call $\mathrm{nat}^n_{\mathfrak{A}}$ *the natural indication of the n-ary derived relations of the relational system* $\mathfrak{A} = (A, (F_i)_{i\in I})$. Rewriting

$$\text{(5.10)} \qquad \mathrm{nat}^n_{\mathfrak{A}}(\Phi) = \Phi_{\mathfrak{A}}$$

for every formula $\Phi \in \mathcal{L}_n$, we can reformulate the definition of $\mathrm{nat}^n_{\mathfrak{A}}$ as follows:

$$\text{(5.11)} \qquad \begin{cases} \langle P_i, x_{\sigma(1)}, \ldots, x_{\sigma(m_i)}\rangle_{\mathfrak{A}} = \sigma^{-1}[F_i] \\ (\Phi \wedge \psi)_{\mathfrak{A}} = \Phi_{\mathfrak{A}} \cap \psi_{\mathfrak{A}} \\ (\neg\Phi)_{\mathfrak{A}} = A^n - \Phi_{\mathfrak{A}} \\ (\vee^n_k \Phi)_{\mathfrak{A}} = \exists^n_k(\Phi_{\mathfrak{A}}) \\ (d^n_{ik})_{\mathfrak{A}} = D^n_{ik}(A) = \sigma^{-1}[\mathrm{id}_A] \end{cases}$$

where σ: $2 \to n$ and $\sigma(1) = i$, $\sigma(2) = k$ (so that the partial diagonals can be reinterpreted as the n-ary elementary relations of the binary relational system (A, id_A)). In a less algebraicized presentation, (5.11) is often referred to as the recursive definition of $\Phi_{\mathfrak{A}}$. We may, indeed, think of any n-tuple $\langle a_1, \ldots, a_n\rangle \in A^n$ as a *valuation* (*interpretation*) v: $\{x_1, \ldots, x_n\} \to A$, defined by $v(x_k) = a_k$ $(1 \leqq k \leqq n)$. In case $\langle a_1, \ldots, a_n\rangle \in \Phi_{\mathfrak{A}}$, one usually says that *the formula* Φ *holds in the relational system* $\mathfrak{A}$ *under the valuation* v and writes $\mathfrak{A} \models_{v} \Phi$. (5.11) is nothing but the well-known recursive definition of validity going back to Tarski. In case $\Phi_{\mathfrak{A}} = A^n$, one says that *the formula* Φ *holds, is valid, in* $\mathfrak{A}$, or *a theorem of* $\mathfrak{A}$.

Like polynomials in universal algebra, the formulas can now be used globally in order to compare, by means of the respective natural indications, the n-ary derived relations of any two relational systems $\mathfrak{A} = (A, (F_i)_{i\in I})$ and $\mathfrak{B} = (B, (G_i)_{i\in I})$, of the same type $(m_i)_{i\in I}$, of course. Here is our main application:

Theorem 5.3 *Suppose we are given two relational systems* $\mathfrak{A}$ *and* $\mathfrak{B}$ *of type* $(m_i)_{i\in I}$ *and a sequence of cylindric homomorphisms (in fact embeddings)* α^n: $\mathcal{P}(A^n) \to \mathcal{P}(B^n)$ $(n \geqq 1)$ *compatible with the covariant transformation functor (4.10). Suppose*

$$\text{(5.12)} \qquad \alpha^{m_i}(F_i) = G_i$$

for every $i \in I$. *Then*

$$\text{(5.13)} \qquad \alpha^n(\Phi_{\mathfrak{A}}) = \Phi_{\mathfrak{B}}$$

for every $n \geqq 1$, *every formula* $\Phi \in \mathcal{L}_n$. *In short,*

$$\text{(5.14)} \qquad \alpha^n \circ \mathrm{nat}^n_{\mathfrak{A}} = \mathrm{nat}^n_{\mathfrak{B}}.$$

Proof: By assumption, using (5.12) and (4.10), (5.13) holds for every elementary formula $\Phi = \langle P_i, x_{\sigma(1)}, \ldots, x_{\sigma(m_i)}\rangle$. The two homomorphisms $\alpha^n \circ \mathrm{nat}^n_{\mathfrak{A}}$ and $\mathrm{nat}^n_{\mathfrak{B}}$: $\mathcal{L}_n \to \mathcal{P}(B^n)$, thus coinciding on the *Peano basis* (the set of n-ary elementary formulas) *of* $\mathcal{L}_n$, are in fact equal.

Together with Corollary 5.2, Theorem 5.3 yields:

Corollary 5.4 *Suppose everything is as in Theorem 5.3. Suppose the mapping $h: A \to B$ of Theorem 5.1 is, in fact, inclusion. Then $\mathfrak{B}$ is an elementary extension of $\mathfrak{A}$, i.e.,*

(5.15) $A^n \cap \Phi_{\mathfrak{B}} = \Phi_{\mathfrak{A}}$

for every $n \geqq 1$, every formula $\Phi \in \mathcal{L}_n$. Hence $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent,

(5.16) $\Phi_{\mathfrak{A}} = A^n$ *iff* $\Phi_{\mathfrak{B}} = B^n$,

for every $n \geqq 1$, every formula $\Phi \in \mathcal{L}_n$.

(5.15) immediately results from (5.13) and (5.7). As a consequence of (5.15), $\Phi_{\mathfrak{B}} = B^n$ implies $\Phi_{\mathfrak{A}} = A^n$. For the converse implication, one may use (5.13). Alternatively, as is well-known, this follows directly from (5.15); one uses the *total quantifier* $\bigvee^n = \bigvee_1^n \circ \ldots \circ \bigvee_n^n$ and its dual $\bigwedge^n = \neg \circ \bigvee^n \circ \neg$.

Corollary 5.4, (5.16) in particular, is a first-order analogue of the Metatheorem 3.2 of Robinson and Zakon [16] (whose proof, without our algebraization in Sections 4 and 5, is done by induction in about two pages). In the complete algebraization achieved here, one can hardly call (5.13), (5.15), or (5.16) a metatheorem any longer (unless one considers polynomial rings as part of the metatheory of rings).

All this applies, of course, if $\mathfrak{A} = (A, (F_i)_{i \in I})$ is a *full relational system,* i.e., if

(5.17) $$\bigcup_{m=1}^{\infty} \mathcal{P}(A^m) = \{F_i | i \in I\}.$$

In this case, we have a converse of Theorem 5.3 (and with that a striking reinterpretation of Robinson and Zakon's axioms (4.1)-(4.4)):

Theorem 5.5 *Suppose $\mathfrak{A} = (A, (F_i)_{i \in I})$ is a full relational system, $\mathfrak{B} = (B, (G_i)_{i \in I})$ a relational system of the same type $(m_i)_{i \in I}$. Let $\alpha^n: \mathcal{P}(A^n) \to \mathcal{P}(B^n)$ $(n \geqq 1)$ be any mappings. Then the latter are cylindric homomorphisms (in fact, embeddings) compatible with the covariant transformation functor (4.10), iff (5.14) holds for each $n \geqq 1$.*

Proof: Since $\mathfrak{A}$ is a full relational system, it is *relationally complete*, i.e., $\mathcal{C}_n(\mathfrak{A}) = \mathcal{P}(A^n)$ for each $n \geqq 1$. (5.14) thus makes each $\alpha^n: \mathcal{P}(A^n) \to \mathcal{P}(B^n)$ a cylindric homomorphism. It remains to show (4.10). We first show (5.12). Let $\Phi = \langle P_i, x_1, \ldots, x_{m_i} \rangle \in \mathcal{L}_{m_i}$, then $\Phi_{\mathfrak{A}} = F_i$ and $\Phi_{\mathfrak{B}} = G_i$, whence $\alpha^{m_i}(F_i) = G_i$ by (5.14). Let now $E \subset A^m$ and $\sigma: m \to n$. If $E = \phi$, $\alpha^m(E) = \phi$ since α^m is Boolean; in this case, (4.10) is trivial. Suppose hence that $E \neq \phi$. Since $\mathfrak{A}$ is full, $E = F_i$ for some $i \in I$. Since $E \neq \phi$, $m = m_i$. Let now $\psi = \langle P_i, x_{\sigma(1)}, \ldots, x_{\sigma(m_i)} \rangle \in \mathcal{L}_n$. We get

$$\begin{aligned}\alpha^n(\sigma^{-1}[E]) = \alpha^n(\sigma^{-1}[F_i]) = \alpha^n(\psi_{\mathfrak{A}}) = \psi_{\mathfrak{B}} = \sigma^{-1}[G_i] &= \sigma^{-1}[\alpha^{m_i}(F_i)] \\ &= \sigma^{-1}[\alpha^m(E)],\end{aligned}$$

completing the proof of Theorem 5.5.

In this context, we observe

Proposition 5.6 *Let* $\mathfrak{A} = (A, (F_i)_{i\in I})$ *and* $\mathfrak{B} = (B, (G_i)_{i\in I})$ *be relational systems of the same type. Then the following are equivalent, for each* $n \geqq 1$:

(i) *for all formulas* $\Phi \in \mathcal{L}_n$, $\Phi_{\mathfrak{A}} = A^n$ *implies* $\Phi_{\mathfrak{B}} = B^n$
(ii) *for all formulas* Φ, $\psi \in \mathcal{L}_n$, $\Phi_{\mathfrak{A}} = \psi_{\mathfrak{A}}$ *implies* $\Phi_{\mathfrak{B}} = \psi_{\mathfrak{B}}$
(iii) *there is a unique cylindric homomorphism* α^n: $\mathcal{C}_n(\mathfrak{A}) \to \mathcal{C}_n(\mathfrak{B})$ *(of course onto* $\mathcal{C}_n(\mathfrak{B})$*) such that (5.14) holds.*

Proof: One has $\Phi_{\mathfrak{A}} \subset \psi_{\mathfrak{A}}$ iff $(\neg(\Phi \wedge \neg\psi))_{\mathfrak{A}} = A^n$, and the same holds in $\mathfrak{B}$, of course. Thus (i) implies (ii). Trivially, (ii) implies (i): Take any tautology ψ, so that $\psi_{\mathfrak{A}} = A^n$ and $\psi_{\mathfrak{B}} = B^n$. (ii) states that the congruence of $\mathcal{L}_n$ induced by the natural homomorphism $\text{nat}^n_{\mathfrak{A}}$: $\mathcal{L}_n \to \mathcal{C}_n(\mathfrak{A})$ is contained in the congruence induced by $\text{nat}^n_{\mathfrak{B}}$: $\mathcal{L}_n \to \mathcal{C}_n(\mathfrak{B})$. The (general) homomorphism theorem thus makes (ii) and (iii) equivalent.

Corollary 5.7 *The relational systems* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are elementarily equivalent iff there are unique cylindric isomorphisms* α^n: $\mathcal{C}_n(\mathfrak{A}) \to \mathcal{C}_n(\mathfrak{B})$ $(n \geqq 1)$ *such that (5.14) holds.*

This applies, of course, to the case that A is a full relational system. We get

Corollary 5.8 *Let* $\mathfrak{A}$ *be a full relational system,* $\mathfrak{B}$ *a relational system of the same type. Then there are unique mappings* α^n: $\mathcal{P}(A^n) \to \mathcal{P}(B^n)$ *satisfying Robinson and Zakon's axioms (4.1)-(4.4) iff* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are elementarily equivalent.*

For the application Robinson and Zakon [16] had in mind, let us again assume, as in Section 3, that we are given projections p_t: $B \to A$ $(t \in T)$ and a proper index-filter $\mathcal{F}$ over the index-domain T. With that, we are also given the n'th powers p_t^n: $B^n \to A^n$ defined by

(5.18) $$p_t^n(\langle b_1, \ldots, b_n\rangle) = \langle p_t(b_1), \ldots, p_t(b_n)\rangle.$$

Applying (3.10), we get mappings α^n: $\mathcal{P}(A^n) \to \mathcal{P}(B^n)$ defined by

(5.19) $$\alpha^n(F) = \{b \in B^n \mid p_t^n(b) \in F \text{ for almost all } t \in T\},$$

for each $n \geqq 1$, $F \subset A^n$. Again, α^n is finitely meet-preserving, also $\alpha^n(\phi) = \phi^n(\mathcal{P}(A^n)) = \phi$; if $\mathcal{F}$ is an ultrafilter, α^n is even Boolean. As an immediate consequence of the definition, the sequence $\alpha = (\alpha^n)$ respects the covariant transformation functor: (4.10) holds. Indeed, for any mapping p: $B \to A$, any transformation σ: $m \to n$ $(m, n \geqq 1)$, the formula

(5.20) $$p^m \circ \sigma_B = \sigma_A \circ p^n$$

states one of the simple facts of life. Applying this to the mappings $p = p_t$, we get (4.10).

In the same way, one gets one half of (4.11), namely

$$\sigma_B[\alpha^n(F)] \subset \alpha^m(\sigma_A[F])$$

for each $F \subset A^n$. So far, none of the properties FP0-FP4 of Section 3 has been used. As a consequence of (4.10), (4.1) and (4.3) hold. (4.4) (Robinson and

Zakon's normality) is nothing but FP2. It suffices to prove the other half of (4.11) for the special case of the inclusion $\sigma = \iota_n$: $m = n-1 \to n$. Let $\langle b_1, \ldots, b_{n-1}\rangle \in \alpha^{n-1}(\iota_n[F])$. Hence $\langle p_t(b_1), \ldots, p_t(b_{n-1})\rangle \in \iota_n[F]$ for almost all $t \in T$. Hence for almost all $t \in T$ there is an element $a_t \in A$ so that $\langle p_t(b_1), \ldots, p_t(b_{n-1}), a_t\rangle \in F$ (we did use the full power of the axiom of choice here). Defining a mapping a: $T \to A$ on the remaining indices quite arbitrarily, we are guaranteed by FP1 the existence of an element $b_n \in B$ such that $p_t(b_n) = a_t$ for almost all $t \in T$. Hence $\langle p_t(b_1), \ldots, p_t(b_{n-1}), p_t(b_n)\rangle \in F$ for almost all $t \in T$; i.e., $\langle b_1, \ldots, b_{n-1}, b_n\rangle \in \alpha^n(F)$, so that $\langle b_1, \ldots, b_{n-1}\rangle \in \iota_n[\alpha^n(F)]$, establishing (4.2).

FP3, not used in this argument, made the mapping h: $A \to B$ of Theorem 5.1 the inclusion mapping and so guarantees (5.7) and (5.8) (see Section 3). The application of Corollary 5.4 to the present case is, of course, Łoś's Ultrapower Theorem. FP4, together with FP1, will be used as in Section 3 to make each monadization mapping μ^n: $\Phi(A^n) \to \mathcal{P}(B^n)$ $(n \geqq 1)$ one-to-one; note that $\mathcal{V}$ is adequate for each power A^n once it is adequate for A. In that case, then, $\mu^n(\mathcal{F}) = \phi$ for each proper filter $\mathcal{F}$ over A^n. This again applies to *concurrent binary relations* $R \subset A^m \times A^n = A^{m+n}$ whose images $R[\{\langle a_1, \ldots, a_m\rangle\}] \subset A^n$ $(\langle a_1, \ldots, a_m\rangle \in \text{dom}_m R)$ have the finite intersection property, i.e., generate a proper filter $\mathcal{F}$ over A^n. Again, with $\langle b_1, \ldots, b_n\rangle \in \mu^n(\mathcal{F})$, we found a point of B^n such that $\langle a_1, \ldots, a_m, b_1, \ldots, b_n\rangle \in \alpha^{m+n}(R)$ for every point $\langle a_1, \ldots, a_m\rangle \in \text{dom}_m R$. In other words, we got (the first-order analogue of) an *enlargement* in the sense of Robinson [14], [15].

6 First-order properties, continued It is left to reexpress the properties of the Boolean mappings α^n that characterized an elementary extension of a full relational structure in terms of the corresponding mappings $\phi^n, \psi^n, \mu^n, \nu^n, \omega^n$. We know from (3.1) how Booleanity translates. What about (4.10) and (4.11)?

Let us look again at any transformation σ: $m \to n$, its induced mapping σ_A: $A^n \to A^m$ defined in (4.5) and the Boolean mapping α_σ: $\mathcal{P}(A^m) \to \mathcal{P}(A^n)$ defined in (4.6). The latter (3.1) induces a completely join-preserving, finitely meet-preserving mapping

$$\phi_\sigma: \Phi(A^m) \to \Phi(A^n)$$

preserving principal filters and a completely meet-preserving, continuous mapping

$$\psi_\sigma: \Phi(A^n) \to \Phi(A^m)$$

preserving ultrafilters. Here are their explicit definitions (cf. (1.6) and the second example in Section 3):

$$(6.1) \qquad \phi_\sigma(\mathcal{F}) = \{G \subset A^n | \sigma_A^{-1}[F] \subset G \text{ for some } F \in \mathcal{F}\}$$

for each filter $\mathcal{F} \in \Phi(A^m)$, and

$$(6.2) \qquad \psi_\sigma(\mathcal{G}) = \{F \subset A^m | \sigma_A^{-1}[F] \in \mathcal{F}\}$$

for each filter $\mathcal{G} \in \Phi(A^n)$. We will also need the completely join-preserving mapping β_σ: $\mathcal{P}(A^n) \to \mathcal{P}(A^m)$ defined in (4.7).

We are going to look at the combined Boolean mappings

$$\alpha^n \circ \alpha_\sigma \text{ and } \alpha_\sigma \circ \alpha^m: \mathcal{P}(A^m) \to \mathcal{P}(B^n).$$

The corresponding mappings (3.1) are:

$$\begin{array}{llll} \phi^n \circ \phi_\sigma & \text{and} & \phi_\sigma \circ \phi^m\colon \Phi(A^m) \to \Phi(B^n), & \text{respectively} \\ \psi_\sigma \circ \psi^n & \text{and} & \psi^m \circ \psi_\sigma\colon \Phi(B^n) \to \Phi(A^m), & \text{respectively} \\ \mu^n \circ \phi_\sigma & \text{and} & \alpha_\sigma \circ \mu^m\colon \Phi(A^m) \to \mathcal{P}(B^n), & \text{respectively} \\ \psi_\sigma \circ \nu^n & \text{and} & \nu^m \circ \beta_\sigma\colon \mathcal{P}(B^n) \to \Phi(A^m), & \text{respectively} \\ \psi_\sigma \circ \omega^n & \text{and} & \omega^m \circ \sigma_B\colon B^n \to \Omega(A^m), & \text{respectively.} \end{array}$$

Indeed, applying (1.3) to both ϕ^n and ϕ_σ, we see that the completely join-preserving mappings $\phi^n \circ \phi_\sigma$ and $\phi_\sigma \circ \phi^m$ "extend" $\alpha^n \circ \alpha_\sigma$ and $\alpha_\sigma \circ \alpha^m$ respectively. (1.5) shows that $\psi_\sigma \circ \psi^n$ and $\psi^m \circ \psi_\sigma$ are the right adjoints of $\phi^n \circ \phi_\sigma$ and $\phi_\sigma \circ \phi^m$ respectively. Let now $F \subset A^m$ and $G \subset B^n$. Then $F \in \psi_\sigma(\nu^n(G))$ is equivalent to $\alpha_\sigma(F) = \sigma_A^{-1}[F] \in \nu^n(G)$ by (6.2), which in turn is equivalent to $G \subset \alpha^n(\alpha_\sigma(F))$ by (2.4). Hence by (2.4), $\psi_\sigma \circ \psi^n$ is the comonadization mapping ν belonging to $\alpha^n \circ \alpha_\sigma$. On the other hand, $F \in \nu^m(\beta_\sigma(G))$ is equivalent to $\sigma_B[G] = \beta_\sigma(G) \subset \alpha^m(F)$ by (2.4), which in turn is equivalent with $G \subset \sigma_B^{-1}[\alpha^m(F)] = \alpha_\sigma(\alpha^m(F))$. Again by (2.4), $\nu^m \circ \beta_\sigma$ is the comonadization mapping ν belonging to $\alpha_\sigma \circ \alpha^m$. (2.3), and the fact that $(\alpha_\sigma, \beta_\sigma)$ is an adjoint situation, shows that the corresponding monadization mappings μ are $\mu^n \circ \phi_\sigma$ and $\alpha_\sigma \circ \mu^m$, respectively. Moreover, for $b \in B^n$, we get $\psi_\sigma(\nu^n(\{b\}) = \psi_\sigma(\omega^n(b))$ and $\nu^m(\beta_\sigma(\{b\})) = \nu^m(\sigma_B[\{b\}]) = \nu^m(\{\sigma_B(b)\}) = \omega^m(\sigma_B(b))$ by (2.8), showing that $\psi_\sigma \circ \omega^n$ and $\omega^m \circ \sigma_B$ are the "restrictions" (2.8) of $\psi_\sigma \circ \nu^n$ and $\nu^m \circ \beta_\sigma$, respectively.

As a consequence of all that, we get

Theorem 6.1 *Let $\alpha^n\colon \mathcal{P}(A^n) \to \mathcal{P}(B^n)$ be Boolean, for each $n \geqq 1$. Then for each transformation $\sigma\colon m \to n$, the following are equivalent:*

(6.3) $\alpha^n \circ \alpha_\sigma = \alpha_\sigma \circ \alpha^m$

(6.4) $\phi^n \circ \phi_\sigma = \phi_\sigma \circ \phi^m$

(6.5) $\psi_\sigma \circ \psi^n = \psi^m \circ \psi_\sigma$

(6.6) $\mu^n \circ \phi_\sigma = \alpha_\sigma \circ \mu^m$

(6.7) $\psi_\sigma \circ \nu^n = \nu^m \circ \beta_\sigma$

(6.8) $\psi_\sigma \circ \omega^n = \omega^m \circ \sigma_B$.

This takes care of the compatibility of the sequence $\alpha = (\alpha^n)$ with the covariant transformation functor (4.10). Among the equivalent conditions above, the last two have a particular neat concrete meaning for, as a consequence of (6.2), (6.7) states that:

(6.9) $F \in \nu^m(\sigma_B[G])$ iff $\sigma_A^{-1}[F] \in \nu^n(G)$

for each $F \subset A^m$, $G \subset B^n$, while (6.8) states that

(6.10) $F \in \omega^m(\sigma_B(b))$ iff $\sigma_A^{-1}[F] \in \omega^n(b)$

for each $F \subset A^m$, $b \in B^n$.

As it comes to reformulate (4.11) in the language of the "derived" operators ϕ^n, etc., we are in a somewhat less fortunate position since β_σ: $\mathcal{P}(A^n) \to \mathcal{P}(A^m)$ is not finitely meet-preserving in general. We hence pass to $\overline{\alpha_\sigma}\colon \mathcal{P}(A^n) \to \mathcal{P}(A^m)$, defined by

(6.11) $\overline{\alpha_\sigma}(F) = A^m - \sigma_A[A^n - F]$

for each $F \subset A^n$. Indeed, this mapping being completely meet-preserving, our theory (Sections 1 and 2) applies. In particular, we have the "extended" completely join-preserving filter mapping $\overline{\phi_\sigma}$: $\Phi(A^n) \to \Phi(A^m)$, defined by

(6.12) $\overline{\phi_\sigma}(\mathcal{F}) = \{G \subset A^m | \overline{\alpha_\sigma}(F) \subset G \text{ for some } F \in \mathcal{F}\}$

for each filter $\mathcal{F} \in \Phi(A^n)$, and its right adjoint, the completely meet-preserving mapping $\overline{\psi_\sigma}$: $\Phi(A^m) \to \Phi(A^n)$, defined by

(6.13) $\overline{\psi_\sigma}(\mathcal{G}) = \{F \subset A^n | \overline{\alpha_\sigma}(F) \in \mathcal{G}\}$

for each filter $\mathcal{G} \in \Phi(A^m)$. We also need the monadization mapping $\overline{\mu_\sigma}$: $\Phi(A^n) \to \mathcal{P}(A^m)$, defined by

(6.14) $\overline{\mu_\sigma}(\mathcal{F}) = \bigcap \{\overline{\alpha_\sigma}(F) | F \in \mathcal{F}\} = \bigcap \overline{\phi_\sigma}(\mathcal{F})$

for each filter $\mathcal{F} \in \Phi(A^n)$, the corresponding comonadization mapping $\overline{\nu_\sigma}$: $\mathcal{P}(A^m) \to \Phi(A^n)$, defined by

(6.15) $\overline{\nu_\sigma}(G) = \{F \subset A^n | G \subset \overline{\alpha_\sigma}(F)\} = \overline{\psi_\sigma}([G, A^m])$

for each subset $G \subset A^m$, finally its restriction to points, $\overline{\omega_\sigma}$: $A^m \to \Phi(A^n)$, defined by

(6.16) $\overline{\omega_\sigma}(a) = \{F \subset A^n | a \in \overline{\alpha_\sigma}(F)\} = \overline{\nu_\sigma}(\{a\})$

for each point $a \in A^m$.

Again, we are going to look at the finitely meet-preserving mappings

$$\alpha^m \circ \overline{\alpha_\sigma} \text{ and } \overline{\alpha_\sigma} \circ \alpha^n: \mathcal{P}(A^n) \to \mathcal{P}(B^m).$$

The corresponding mappings (1.1) and (2.1) turn out to be

$$\begin{array}{lll}
\phi^m \circ \overline{\phi_\sigma} & \text{and} \quad \overline{\phi_\sigma} \circ \phi^n\colon \Phi(A^n) & \to \Phi(B^m), \text{ respectively} \\
\overline{\psi_\sigma} \circ \psi^m & \text{and} \quad \psi^n \circ \overline{\psi_\sigma}\colon \Phi(B^m) & \to \Phi(A^n), \text{ respectively} \\
\mu^m \circ \overline{\phi_\sigma} & \text{and} \quad \overline{\mu_\sigma} \circ \phi^n\colon \Phi(A^n) & \to \mathcal{P}(B^m), \text{ respectively} \\
\overline{\psi_\sigma} \circ \nu^m & \text{and} \quad \psi^n \circ \overline{\nu_\sigma}\colon \mathcal{P}(B^m) & \to \Phi(A^n), \text{ respectively} \\
\overline{\psi_\sigma} \circ \omega^m & \text{and} \quad \psi^n \circ \overline{\omega_\sigma}\colon B^m & \to \Phi(A^n), \text{ respectively.}
\end{array}$$

We leave the details to the reader, except for the comparatively exotic cases. Let again $F \subset A^n$ and $G \subset B^m$. Then $F \in \psi^n(\overline{\nu_\sigma}(G))$ is equivalent to $\alpha^n(F) \in \overline{\nu_\sigma}(G)$ by (1.6), which in turn is equivalent to $G \subset \overline{\alpha_\sigma}(\alpha^n(F))$ by (2.4), making $\psi^n \circ \overline{\nu_\sigma}$ the comonadization mapping ν belonging to $\overline{\alpha_\sigma} \circ \alpha^n$, and $\overline{\mu_\sigma} \circ \phi^n$ its monadization mapping by virtue of (1.5) and (2.3), and moreover, $\psi^n \circ \overline{\omega_\sigma}$ its restriction to points $b \in B^n$. We get

Theorem 6.2 *Let α^n: $\mathcal{P}(A^n) \to \mathcal{P}(B^n)$ be finitely meet-preserving mappings, for each $n \geqq 1$. Then for each transformation σ: $m \to n$, the following are equivalent:*

(6.17) $\alpha^m \circ \overline{\alpha_\sigma} = \overline{\alpha_\sigma} \circ \alpha^n$
(6.18) $\phi^m \circ \overline{\phi_\sigma} = \overline{\phi_\sigma} \circ \phi^n$
(6.19) $\overline{\psi_\sigma} \circ \psi^m = \psi^n \circ \overline{\psi_\sigma}$
(6.20) $\mu^m \circ \overline{\phi_\sigma} = \overline{\mu_\sigma} \circ \phi^n$
(6.21) $\overline{\psi_\sigma} \circ \nu^m = \psi^n \circ \overline{\nu_\sigma}$
(6.22) $\overline{\psi_\sigma} \circ \omega^m = \psi^n \circ \overline{\omega_\sigma}$.

If each α^n is Boolean, then (6.17) is equivalent to $\alpha^m \circ \beta_\sigma = \beta_\sigma \circ \alpha^n$ which is (4.11). Unfortunately, (6.20)-(6.22) are no longer conditions just on the sequences $\mu = (\mu^n)$, $\nu = (\nu^n)$, or $\omega = (\omega^n)$, hence not really as nice translations as were (6.6)-(6.8).

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