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On Uncountable Boolean Algebras With No Uncountable Pairwise Comparable or Incomparable Sets of Elements

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Elements $a, b,$ of a Boolean algebra are said to be *comparable* iff either $a \leqslant b$ or $b \leqslant a$, otherwise *incomparable*. A *chain* in a Boolean algebra is a set of pairwise comparable elements, while a *pie* is a set of pairwise incomparable elements.

In [2] Baumgartner and Komjath proved, using $\diamondsuit_{\aleph_1}$:

Theorem 1 (Baumgartner-Komjath) *Assume $\diamondsuit_{\aleph_1}$. There is an uncountable Boolean algebra with no uncountable chain or pie.*

In [6] Rubin, also using $\diamondsuit_{\aleph_1}$, proved:

Theorem 2 (Rubin) *Assume $\diamondsuit_{\aleph_1}$. There is a Boolean algebra B, with $\overline{\overline{B}} = \aleph_1$, in which every ideal is $\aleph_0$-generated and every subalgebra is generated by an ideal and $\aleph_0$ elements. Thus, B has only $\aleph_1$ ideals and subalgebras.*

Using only *CH*, Berney and Nyckos [3] and Bonnet [4] proved:

Theorem 3 *Assume CH. There is an uncountable Boolean algebra with no uncountable pie.*

They chose a set A of reals of cardinality $\aleph_1$, and the Boolean algebra is the Boolean algebra of subsets of the reals generated by (r, s), $r, s \in A$.

In the opposite direction, Baumgartner [1] showed:

Theorem 4 *It is consistent with ZFC that $2^{\aleph_0} = \aleph_2$, Martin's axiom holds, and every Boolean algebra of cardinality $\aleph_1$ contains an uncountable pie.*

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In fact, the above follows from Martin's axiom + any $\aleph_1$-dense sets of reals are isomorphic.

The main result of this paper is (for generalizing to higher powers see the end)

Theorem 5 *Assume CH. There is a Boolean algebra B, $\overline{\overline{B}} = \aleph_1$ such that*

(i) *B has no uncountable pies.*
(ii) *B has no uncountable chains.*
(iii) *Every ideal of B is generated by $\aleph_0$ elements.*
(iv) *Among any $\aleph_1$-elements of B there are four elements x_0, x_1, x_2, x_3 such that $x_0 \wedge x_1 = x_2 \wedge x_3$.*

In [6] Rubin used $\diamondsuit_{\aleph_1}$ to show that there is a Boolean algebra B, $\overline{\overline{B}} = \aleph_1$, such that for every $I \subseteq B$, $\overline{\overline{I}} = \aleph_1$, there is a partition of $1, b_0, \ldots, b_n \in B$, $n > 1$, such that for every $0 < b_l' < b_l'' \leqslant b_l$, $l = 2, \ldots, n$, there is some $x \in I$ such that $x \wedge b_0 = b_0$, $x \wedge b_1 = 0$, and $b_l' < x \wedge b_l < b_l''$ for $l = 2, \ldots, n$. We will obtain a similar result as Lemma 7 below which will lead directly to Theorem 5.

1 We first introduce a Boolean algebra B, and then in a series of lemmas show that B satisfies the conditions of Theorem 5. Though our original treatment was somewhat different, here, at the suggestion of the referee, we use forcing to construct B.

We begin with a countable atomless Boolean algebra B_0. We think of B_0 as being embedded in its completion, and form B by adding some elements of the completion, and taking the closure.

As our set of forcing conditions we take

$$P = \{(a, b) \colon a, b \in B_0 \text{ and } a < b\}.$$

A condition (a_1, b_1) extends a condition (a_2, b_2), written $(a_1, b_1) \leqslant (a_2, b_2)$ iff $a_2 \leqslant a_1 < b_1 \leqslant b_2$. We think of a condition (a, b) as giving information about an element x of the completion, with (a, b) specifying that $a < x < b$. Thus, as conditions are extended, the value of x is squeezed from below and above.

Let HC denote the set of hereditarily countable sets. We define a sequence $\langle N_\alpha \colon \alpha < \omega_1 \rangle$ satisfying

a. $B_0 \in N_0$
b. For $\alpha < \beta < \omega_1$, $\langle N_\alpha, \in \rangle \prec \langle N_\beta, \in \rangle \prec \langle HC, \in \rangle$.
c. For δ a limit ordinal, $N_\delta = \bigcup_{\alpha < \delta} N_\alpha$.
d. $\bigcup_{\alpha < \omega_1} N_\alpha = HC$.
e. Each N_α, $\alpha < \omega_1$, is countable.
f. For each $\alpha < \omega_1$, there is $G_\alpha \in N_{\alpha+1}$, P-generic over N_α.

It is very easy to construct such a sequence, but only, of course, if CH holds.

Now, for each G_α, we let

$$x_\alpha = \bigvee \{a \colon \exists b [(a, b) \in G_\alpha]\}.$$

The supremum in the above is taken in the completion of B_0. We may now define B as the subalgebra of the completion generated by $B_0 \cup \{x_\alpha \colon \alpha < \omega_1\}$.

2 We now begin the process of showing that B, as just defined, satisfies the conditions of Theorem 5. First, for each $\alpha < \omega_1$ we define

$$I_\alpha = \{b \in B_0 : x_\alpha \wedge b \in B_0\}.$$

Lemma 1 *I_α is a proper ideal.*

Proof: The proof is easy. First suppose $b \in I_\alpha$ and $c < b$, $c \in B_0$. Then $x_\alpha \wedge c = (x_\alpha \wedge b \wedge c) = (x_\alpha \wedge b) \wedge c \in B_0$, since $x_\alpha \wedge b \in B_0$. Now if $b_1, b_2 \in I_\alpha$, then $x_\alpha \wedge (b_1 \vee b_2) = (x_\alpha \wedge b_1) \vee (x_\alpha \wedge b_2) \in B_0$, since both $x_\alpha \wedge b_1, x_\alpha \wedge b_2 \in B_0$. This shows I_α is an ideal. To see that it is proper, simply note that since $1 \wedge x_\alpha = x_\alpha$, $1 \in I_\alpha$ iff $x_\alpha \in B_0$. It is easy to see by genericity that $x_\alpha \notin B_0$.

Lemma 2 *I_α is maximal.*

Proof: First we must give an alternate description of I_α. We claim that

$$I_\alpha = \{a \vee \bar{b} : (a, b) \in G_\alpha\},$$

where $\bar{b}$ denotes the complement of b. First, if $(a, b) \in G_\alpha$, then $x_\alpha \wedge (a \vee \bar{b}) = a \in B_0$, so $a \vee \bar{b} \in I_\alpha$. To obtain the reverse inclusion, suppose $b \in I_\alpha$, i.e., $x_\alpha \wedge b \in B_0$. Denote $x_\alpha \wedge b$ by c. Now, for some $(d, e) \in G_\alpha$, $(d, e) \Vdash x_\alpha \wedge \check{b} = \check{c}$. Then we must have $b \wedge e \leqslant d$, or $x_\alpha \wedge \check{b}$ could be "made" smaller by a stronger condition. Then trivially, $(b \wedge e) \vee \bar{e} \leqslant d \vee \bar{e}$, whence $b \leqslant d \vee \bar{e}$. Now, it is also trivial to verify that both d and $\bar{e} \in I_\alpha$, viz., $d \wedge x_\alpha = d$, $\bar{e} \wedge x_\alpha = 0$. Now, since I_α is an ideal $d \vee \bar{e} \in I_\alpha$, and since $b \leqslant d \vee \bar{e}$, $b \in I_\alpha$. This finishes the proof of the claim.

Next, fix $c \in B_0$ and consider the set

$$D = \{(a, b) : (a, b) \leqslant (c, 1) \text{ or } (a, b) \text{ and } (c, 1) \text{ are incompatible}\}.$$

D is, as usual, dense in P, and obviously an element of N_α. Thus there is some $(a, b) \in G_\alpha \cap D$. Now, if (a, b) is incompatible with $(c, 1)$, this must mean $a \vee c \geqslant b$. Then $\bar{a} \wedge (a \vee c) \geqslant \bar{a} \wedge b$, again leading to $\bar{c} \leqslant a \vee \bar{b}$, which puts $\bar{c}$ in the ideal I_α. If, on the other hand, $(a, b) \leqslant (c, 1)$, then $c \leqslant a \leqslant x_\alpha$. Then, $c \wedge x_\alpha = c \in B_0$, so $c \in I_\alpha$. This shows that I_α is maximal.

Lemma 3 (i) *For $\alpha < \beta < \omega_1$, $I_\alpha \neq I_\beta$.* (ii) *For $\alpha < \omega_1$, $G = \{(a, b) : a \leqslant x_\alpha \leqslant b, a, b \in B_0\}$ is P-generic over N_α (i.e., $\bar{x}_\alpha$ is also "generic".)*

Proof: (i) Suppose $I_\alpha = I_\beta$, $\alpha < \beta$. Then for some condition (a, b), $(a, b) \Vdash I_\beta = \check{I}_\alpha$, in the forcing for constructing x_β. Since B_0 is atomless, we can choose $c \in B_0$ such that $a < c < b$. Suppose $c \in I_\alpha$, the opposite case being similar. Then $(a, b) \Vdash$ "$\check{c} \in I_\beta$". However, by choosing $d \in B_0$ such that $a < d < c$, we have $(a, d) \Vdash$ "$\check{c} \notin I_\beta$". This contradicts the fact that $(a, d) < (a, b)$ and so $(a, d) \Vdash$ "$\check{c} \in I_\beta$".

(ii) Suppose x_α is generated by the generic subset G_α of P. Let $\bar{G}_\alpha = \{(\bar{b}, \bar{a}) : (a, b) \in G\}$. Then it is easy to check that $\bar{G}_\alpha$ is generic (e.g., if D is dense so is $\bar{D}$, etc.) and that $\bar{G}_\alpha$ generates $\bar{x}_\alpha$.

The next lemma, the "Product Theorem", is well-known to those familiar with forcing.

Lemma 4 *Suppose $\alpha_1 < \alpha_2 < \ldots < \alpha_n < \omega_1$. Then $G_{\alpha_1} \times \ldots \times G_{\alpha_n}$ is P^n-generic over N_{α_1}.*

Proof: By induction on n, with the case of $n = 1$ trivial. Let $n \geqslant 1$ and assume the lemma holds for n. The only nonimmediate clause to verify concerns intersections with dense sets.

Let $D \in N_{\alpha_1}$ be dense in P^{n+1}. We must show $G_1 \times \ldots \times G_n \times G_{n+1} \cap D \neq 0$. This amounts to showing that $E \cap G_{\alpha_{n+1}} \neq 0$, where

$$E = \{p\colon \exists (p_1, \ldots, p_n) \in G_{\alpha_1} \times \ldots \times G_{\alpha_n} [(p_1, \ldots, p_n, p) \in D]\}.$$

Since $E \in N_{\alpha_{n+1}}$ it suffices to show that E is dense in P. Thus, given $q \in P$ we must find $p \leqslant q$ and $(p_1, \ldots, p_n) \in G_{\alpha_1} \times \ldots \times G_{\alpha_n}$ such that $(p_1, \ldots, p_n, p) \in D$. To see this it suffices to notice that

$$F = \{(p_1, \ldots, p_n)\colon \exists p \leqslant q [(p_1, \ldots, p_n, p) \in D]\}$$

is dense in P^n, since it is also in N_{α_1}.

For the purposes of the next lemma we define for $A = \{\alpha_1, \ldots, \alpha_n\}$, $\alpha_1 < \ldots < \alpha_n < \omega_1$, $G(A) = G_{\alpha_1} \times \ldots \times G_{\alpha_n}$. It is here that we make use of the choice of the N_α.

Lemma 5 *Let F be an uncountable collection of pairwise disjoint n-element subsets of ω_1. Let $E = \{p \in P^n\colon \{A \in F\colon p \in G(A)\}$ is countable$\}$. Then E is not dense in P^n.*

Proof: Suppose E were dense in P^n. Since $E \in HC$, there is some $\beta < \omega_1$ such that $E \in N_\beta$. However, if $A \in F$ and each element of A is greater than β, then $G(A) \cap E \neq 0$ by the obvious generalization of Lemma 4. Since there are uncountably many such $A \in F$, and E is countable, there must be some $x \in E$ such that $x \in G(A)$ for uncountably many $A \in F$, a contradiction.

The next lemma shows that elements of B can be represented in a special way.

Lemma 6 *If $x \in B$ then there are $\alpha_1 < \ldots < \alpha_n$ and disjoint $b_0, b_1, \ldots, b_n \in B_0$, such that for $i \geqslant 1$, $b_i \notin I_{\alpha_i}$, and $x = b_0 \vee \bigvee_{i=1}^{n} y_i$ where either $y_i = x_{\alpha_i} \wedge b_i$ or $y_i = \bar{x}_{\alpha_i} \wedge b_i$.*

Proof: First, choose a minimal $n \in \omega$ such that there are some $\beta_1, \ldots, \beta_n$ such that x is a Boolean combination of $x_{\beta 1}, \ldots, x_{\beta n}$, with elements of B_0. Next, choose a partition $d_1, \ldots, d_n$ of B_0 such that $d_l \in I_{\beta m}$ iff $l = m$. To see how this may be done first choose, for each $i \leqslant n$, $d_i \in \bigcap_{j \neq i} I_{\beta_j} \backslash I_{\beta_i}$. This can be done since the I_β's are distinct maximal ideals by Lemmas 1-3. Just choose for $j \neq i$, $d_{ij} \in I_{\beta_j} \backslash I_{\beta_i}$ and take $d_i = \bigwedge_{j \neq i} d_{ij}$. Now increase one of the d_i's, if necessary, to get a partition. This is no problem since $\overline{\bigvee_{i<n} d_i} \in \bigcap_{i \leqslant n} I_i$.

Suppose $x = \tau(\ldots x_{\beta_l} \ldots, \ldots c_j \ldots)$, for some Boolean term τ, with the c_j's in B_0. Then $x \wedge d_i = \tau(\ldots x_{\beta_l} \wedge d_i \ldots, \ldots c_j \wedge d_i \ldots) \wedge b_i = \tau'(x_{\beta_i} \wedge d_i, \ldots$ elements of $B_0 \ldots) \wedge d_i$, for some other term τ'. It is easy to find disjoint $d^0, d_i^1, d_i^2 \leqslant d_i$ such that $x \wedge d_i = d_i^0 \vee (x_{\beta_i} \wedge d_i^1) \vee (\bar{x}_{\beta_i} \wedge d_i^2)$. We can also arrange this so that if $x_{\beta_i} \wedge d_i^1 \in B_0$, then $d_i^1 = 0$, and similarly if $\bar{x}_{\beta_i} \wedge d_i^2 \in B_0$. Since $d_i^1 \in I_{\beta_i}$ or $d_i^2 \in I_{\beta_i}$, $x_{\beta_i} \wedge d_i^1 \in B_0$ or $\bar{x}_{\beta_i} \wedge d_i^2 \in B_0$. Thus, $d_i^1 = 0$, or $d_i^2 = 0$ (or both). Now, let $b_0 = \bigvee_{i=1}^{n} d_i^0$, $b_i = d_i^1 \vee d_i^2$, and $y_i = (x_{\beta_i} \wedge d_i^1) \vee (\bar{x}_{\beta_i} \wedge d_i^2)$, for $i = 1, \ldots, m$. Note that $b_0, b_1, \ldots, b_n$ are disjoint (since $d_1, \ldots d_n$ were disjoint and $d_i^0, d_i^1, d_i^2 \leqslant d_i$ were disjoint). Now $x = \bigvee_{1=i}^{n} (x \wedge d_i) = \bigvee_{1=i}^{n} (d_i^0 \vee (x_{\beta_i} \wedge d_i^1) \vee (\bar{x}_{\beta_i} \wedge d_i^2)) = \bigvee_{1=i}^{n} (d_i^0 \vee y_i) = \bigvee_{1=i}^{n} d_i^0 \vee \bigvee_{1=i}^{n} y_i = b_0 \vee y_1 \vee, \ldots \vee y_n$. For each i, $y_i \neq 0$ since otherwise n would not be minimal. Since either $d_i^1 = 0$ or d_i^2, but $y_i \neq 0$, y_i is either $x_{\beta_i} \wedge d_i^1$ or $\bar{x}_{\beta_i} \wedge d_i^2$. Therefore, the y_i's are as required.

We now come to the key lemma.

Lemma 7 *For every uncountable $I \subseteq B$, there is a partition of* 1, $c_0, \ldots, c_n \in B_0$, *and* $c \in B$, *with* $c \leqslant c_0$, *such that for every* $b_l' < b_l'' \leqslant c_l$ *in* B_0, $l = 1, \ldots, n$, *there is some* $x \in I$ *such that* $x \wedge c_0 = c$ *and* $b_l' < x \cap c_l < b_l''$, $l = 1, \ldots, n$. *In fact, there are* $\aleph_1$ *such* x.

Proof: Let $I \subseteq B$ be uncountable. We apply Lemma 6 to each $x \in I$. Since B_0 is countable we may thin I down to some uncountable J, such that each $x \in J$ determines the same sequence $b_0, \ldots, b_n$, and so that the sets $A_x = \{\alpha_1, \ldots, \alpha_n\}$ form a Δ-system (cf. [5]). By appealing to Lemma 3(ii) and a further thinning of J to some uncountable K, we may assume that for each $x \in K$, only y_i of the form $x_{\alpha_i} \wedge b_i$ occur.

Let F be the kernel of the Δ-system. For simplicity, let us first assume that $F = 0$. We apply Lemma 5 to K to see that $E = \{p \in P^n\colon \{x \in K\colon p \in G(A_x)\}$ is countable$\}$ is not dense in P^n. Fix some $p \in P^n$ such that if $q \leqslant p$, then $q \notin E$. Suppose $p = (p_1, \ldots, p_n)$, where $p_i = (a_i^1, a_i^2)$, $i = 1, \ldots, n$. Now define $c_i = b_i \wedge a_i^2 \wedge \bar{a}_i^1$ for $i = 1, \ldots, n$ and let $c_0 = \overline{c_1 \vee \ldots \vee c_n}$. Finally, let $c = b_0 \vee \left(\bigvee_{1=i}^{n} a_i^1\right)$.

We show that if $p \in G(A_x)$ then $x \wedge c_0 = c$. Computing, we get $x \wedge c_0 = \left(b_0 \vee \bigvee_{1=i}^{n} (b_i \wedge x_{\alpha_i})\right) \wedge c_0$. Now, taking the meet of c_0 with each member of the join separately, and recalling that $a_i^1 \leqslant x_{\alpha_i} \leqslant a_i^2$, we get, tracing back the definition c_0, the result $b_0 \vee \bigvee_{i=1}^{n} a_i^1 = c$, each term in this join coming from the corresponding term in the original join. (To see this it is easiest just to draw a Venn diagram.)

Now suppose $b_i' < b_i'' < c_i$, $i = 1, \ldots, n$, b_i', $b_2'' \in B_0$. Let $q_i = (a_i^1 \vee b_i', a_i^1 \vee b_i'')$ for $i = 1, \ldots, n$. Then $q_i \leqslant p$ so there are uncountably many $x \in K$ with

$(q_1, \ldots, q_n) \in G(A_x)$. Let $A_x = \{\alpha_1, \ldots, \alpha_n\}$ be such a set and so $x = b_0 \vee \bigvee_{i=1}^{n} x_{\alpha_i} \wedge b_i$. Computing again, we get by disjointness $x \wedge c_i = b_0 \vee \left(\bigvee_{j=1}^{n} b_j \wedge x_{\alpha_j}\right) \wedge c_i = c_i \wedge x_i$. Expanding this last term we have $b_i \wedge a_i^2 \wedge \bar{a}_i^1 \wedge x_{\alpha_i}$. Now, since $a_i^1 \vee b_i' < x_{\alpha_i} < a_i^1 \vee b_i''$ and since $b_i' < b_i'' \leqslant b_i \wedge a_i^2 \wedge \bar{a}_i^1$, we get $b_i < b_i \wedge a_i^2 \wedge \bar{a}_i^1 \wedge x_{\alpha_i} < b_i''$, and this concludes the calculation.

Finally, if the kernel $D \neq 0$, then we define instead $c = b_0 \vee \bigvee_{\alpha_i \in D} (x_{\alpha_i} \wedge b_i)$, and apply Lemma 3(ii) to $\{A \backslash D: A \in K\}$.

This concludes the sequence of lemmas.

Proof of Theorem 5: (i) Let $I \subseteq B$ be uncountable. Let $c_0, \ldots, c_n, c$, be as in Lemma 7. Since B_0 is atomless we may choose $0 < b_l' < b_l'' < b_l''' < c_l$, $l = 1, \ldots, n$ all in B_0. Now, applying Lemma 7 for b_l', b_l'', there is some $x_1 \in I$ with $x_1 \wedge c_0 = c$ and $b_l' < x_1 \wedge c_l < b_l''$, $l = 1, \ldots, n$. Similarly, applying Lemma 7 to b_l'', b_l''', we obtain $x_2 \in I$ with $x_2 \wedge c_0 = c$ and $b_l'' < x_2 \wedge c_l < b_l'''$. Trivially $x_1 \neq x_2$.

Now, since $\bigvee_{l=0}^{n} c_l = 1$, $x_1 = x_1 \wedge \left(\bigvee_{l=0}^{n} c_l\right) = (x_1 \wedge c_0) \vee \bigvee_{l=1}^{n} (x_1 \wedge c_l) \leqslant c \vee \bigvee_{l=1}^{n} b_l'' \leqslant (x_2 \wedge c_0) \vee \bigvee_{l=1}^{n} (x_2 \wedge c_l) = x_2 \wedge \left(\bigvee_{l=1}^{n} c_l\right) = x_2$. Thus, $x_1 < x_2$, and so I is not a pie.

(ii) Next choose $b_l', b_l'' \in B_0$ such that $b_l' \wedge b_l'' = 0$ and $0 < b_l' < c_l$, $0 < b_l'' < c_l$. Now find $x_1 \in I$ such that $x_1 \wedge c_0 = c$ and $b_l' < x_1 \wedge c_l < b_l' \vee b_l''$, $l = 1, \ldots, n$. Similarly find $x_2 \in I$ such that $x_2 \wedge c_0 = c$ and $b_l'' < x_2 \wedge c_l \leqslant b_l' \vee b_l''$, $l = 1, \ldots, n$. Now, $b_1' \leqslant x_1$, but $b_1'' \nleqslant x_1$ or else $b_1' \vee b_1'' \leqslant x_1$, whence $b_1' \vee b_1'' \leqslant x \wedge c_1$. Similarly $b_1'' \leqslant x_2$, but $b_1' \nleqslant x_2$. Thus neither $x_1 \leqslant x_2$ nor $x_2 \leqslant x_1$, and so I is not a chain.

(iii) Suppose I is an ideal of B not generated by $\aleph_0$ elements. Choose inductively a set $J = \{a_\alpha: \alpha < \omega_1\}$ such that $a_\alpha \in I$ and a_α is not in the ideal generated by $\{a_\beta: \beta < \alpha\}$. We will apply Lemma 7 to J choosing $c_0, \ldots, c_n, c$ as described. Next, choose $0 < b_l' < b_l'' < b_l''' < c_l$, $l = 1, \ldots, n$. By Lemma 7, for some $\alpha < \omega_1$, $a_\alpha \wedge c_0 = c$ and $b_l'' < a_\alpha \wedge c_l < b_l'''$, $l = 1, \ldots, n$. Now, applying the last sentence of Lemma 7, we know that for $\aleph_1$ different $\beta < \omega_1$, $a_\beta \wedge c_0 = c$, and $b_l' < a_\beta \wedge c_l < b_l''$. In particular this is true for some $\beta > \alpha$. But, now, arguing as in the proof of part (i) we get $a_\beta \leqslant a_\alpha$, so that a_β is in the ideal generated by $\{a_\alpha: \alpha < \beta\}$, a contradiction.

(iv) Begin by choosing $I, c_0, \ldots, c_n, c, b_l', b_l''$, $l = 1, \ldots, n$ as in (ii) above. There are, by Lemma 7, $\aleph_1$ elements $x_i \in I$ such that $x_i \wedge c_0 = c$ and $x_i \wedge c_l < b_l'$, $l = 1, \ldots, n$. Similarly, there are $\aleph_1$, $y_i \in I$ such that $y_i \wedge c_0 = c$ and $y_i \wedge c_l < b_l''$, $l = 1, \ldots, n$. Then, for each i, $x_i \wedge y_i \wedge c_0 = (x_i \wedge c_0) \wedge (y_2 \wedge c_0) = c$, and for any j, $x_i \wedge y_j \wedge c_l = (x_i \wedge c_l) \wedge (y_i \wedge c_l) \leqslant b_l' \wedge b_l'' = 0$. Now, since $\bigvee_{l=0}^{n} c_l = 1$, we must have $x_i \wedge y_j = c$.

Concluding Remarks

(1) Suppose $\lambda = \lambda^{<\lambda}$ and $2^\lambda = \lambda^+$ (so λ is regular.) We can find a saturated atomless Boolean algebra B_0 of power λ. Letting $H(\lambda^+)$ be the family of sets of

hereditary power $\leqslant \lambda$, we can find $N_\alpha(\alpha < \lambda)$, increasing continuous, $\|N_\alpha\| \leqslant \lambda$, $H(\lambda^+) = \bigcup_{\alpha<\lambda^+} N_\alpha$, $N_{\alpha+1}$ is λ-closed (i.e., $a \subseteq N_{\alpha+1}$, $|a| < \lambda$ implies $a \in N_{\alpha+1}$). We now can define P, P^n, and even P^α as in Lemma 4. Moreover, we can define inductively x_i, $G_i(i < \lambda^+)$ such that for every $\alpha_0 < \ldots < \alpha_{n-1} < \lambda^+$, $(n < \omega)$, $G_{\alpha_0} \times \ldots \times G_{\alpha_{n-1}}$ is P^n-generic over N_{α_0}, and all the lemmas still hold as well the consequences (replacing $\aleph_0$, $\aleph_1$ by λ, λ^+).

(2) However the construction in (1) has a defect: we would like to demand that if $\xi < \lambda$, $\alpha_0 < \ldots < \alpha_i < \ldots (i < \xi)$, then $\prod_{i<\xi} G_{\alpha_i}$ is P^{ξ_0}-generic over N_{α_0}. This is possible (by [7]) if we assume $\diamondsuit_\lambda$, or even $(Dl)_\lambda$, which follows from "λ strongly inaccessible" hence holds for any $\lambda \neq \aleph_1$ when GCH holds (recall we are assuming $\lambda = \lambda^{<\lambda}$ and $2^\lambda = \lambda^+$), (see [7] and [8]).

(3) The need for the strengthening mentioned in (2) arises when we asked our Boolean algebra to be, e.g., σ-closed. For this it is natural to let B be the Boolean algebra from(2), B^c its completion, and B^* the closure of B in B^c under countable meets and complementation.

If $a \in B^*$, clearly there is a Boolean term τ (countable) such that $a = \tau(\bar{b}, x_{\alpha_0}, \ldots, x_{\alpha_i}, \ldots)_{i<\xi}$ $\xi < \aleph_1$, $\bar{b}$ a countable sequence of elements of B_0. Because $\prod_{i<\xi} G_{\alpha_i}$ is N_{α_i}-generic, without loss of generality $b_i(i < \xi)$ are pairwise disjoint, and $a = \tau(\bar{b}, \ldots, b_i \cap x_{\alpha_i}, \ldots)_{i<\xi}$. Now, as in Lemma 7, we can prove:

$(*)_1$ For every $I \subseteq B^*$ of power λ^+, there are $\xi < \omega_1$, pairwise disjoint $b_i \in B_0\,(i < \xi)$, and $J \subseteq I$, $|J| = \lambda^+$ and for every $a \in J$, $i < \xi$, $J(a, i) < \lambda^+$ pairwise distinct, such that:

(i) for some b', for every $a \in J$, $a - \bigvee_{i<\xi} b_i = b'$

(ii) for each β either $(\forall a \in J)\ a \wedge b_\beta = x_{J(a,i)} \wedge b_\beta$
or $(\forall a \in J)\ a \wedge b_\beta = b_\beta - x_{J(a,i)}$

(iii) for every $b_i' < b_i'' \leqslant b_i (i < \xi)$ there is $a \in J$ such that
$b_i' < a \cap b_i < b_i''$.

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