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Completeness Theorems for Two Propositional Logics in Which Identity Diverges from Mutual Entailment

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1 Introduction In [1] Anderson and Belnap devise a model theory for entailment, on which propositional coentailment equals propositional identity. This feature can be reasonably questioned. Here we devise two extensions of Anderson and Belnap's model theory. Both systems, S and T, preserve Anderson and Belnap's results for entailment, but distinguish coentailment from identity.

The system S is the strongest, satisfying the following plausible principle for propositional substitution: If compound sentences express the same proposition and differ only with respect to the interchange of component sentences A and B, then A and B also express the same proposition. We present a model theory for S and prove soundness and completeness. T results from S by adding just an associativity axiom. The principle for propositional substitution cited above does not hold in T. The model theory for S is extended to T, with soundness and completeness again established.

2 Preliminaries We consider the formal language E_{fde} formulated in [1], Chapter III. The set of purely truth functional formulas (ptfs) consists of propositional variables plus $\sim A$, $(A \;\&\; B)$, $(A \vee B)$ where A and B are ptfs. The set of formulas consists of ptfs plus $(A \rightarrow B)$ where A and B are ptfs.

In the following, L ranges over intensional lattices ([1], p. 193); $\wedge$, $\vee$, and $-$ denote respectively the meet, join, and complementation operations on L; $\leqslant$ denotes the partial ordering relation on L; $A, B, C \ldots$ range over ptfs.

Q is a model for E_{fde} iff $Q = \langle L, s\rangle$, where s assigns elements of L to each of the variables. If Q is a model for E_{fde} then Val_Q assigns elements of L to ptfs as follows:

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if A is a variable, $Val_Q(A) = s(A)$
if A is $\sim B$, $Val_Q(A) = \overline{Val_Q(B)}$
if A is $(B \,\&\, C)$, $Val_Q(A) = Val_Q(B) \wedge Val_Q(C)$
if A is $(B \vee C)$, $Val_Q(A) = Val_Q(B) \vee Val_Q(C)$.

$(A \rightarrow B)$ is true in Q iff $Val_Q(A) \leqslant Val_Q(B)$. Finally, $(A \rightarrow B)$ is valid in E_{fde} iff $(A \rightarrow B)$ is true in all models Q.

This list of postulates is given ([1], p. 158):

Entailment:
Rule from $A \rightarrow B$ and $B \rightarrow C$ to infer $A \rightarrow C$

Conjunction:
Axiom $(A \,\&\, B) \rightarrow A$
Axiom $(A \,\&\, B) \rightarrow B$
Rule from $A \rightarrow B$ and $A \rightarrow C$ to infer $A \rightarrow (B \,\&\, C)$

Disjunction:
Axiom $A \rightarrow (A \vee B)$
Axiom $B \rightarrow (A \vee B)$
Rule from $A \rightarrow C$ and $B \rightarrow C$ to infer $(A \vee B) \rightarrow C$

Distribution:
Axiom $A \,\&\, (B \vee C) \rightarrow (A \,\&\, B) \vee C$

Negation:
Axiom $A \rightarrow \sim\sim A$
Axiom $\sim\sim A \rightarrow A$
Rule from $A \rightarrow B$ to infer $\sim B \rightarrow \sim A$.

The theorems generated by this list are shown to be the same as the valid entailments.

The elements of L are thought of as propositions; $\wedge$, $\vee$, $-$ are thought of as propositional conjunction, disjunction, and negation, respectively; and $\leqslant$ is thought of as entailment between propositions. Note that when $(A \rightarrow B)$ and $(B \rightarrow A)$ are both true in Q, $Val_Q(A) = Val_Q(B)$. So, on this model theory, coentailment and propositional identity are the same.

The symbol for disjunction in E_{fde} is dispensible, since DeMorgan's laws hold for intensional lattices. We will find it convenient to dispense with it. Let E'_{fde} be the same as E_{fde} except that all occurrences of $(A \vee B)$ are replaced by $\sim(\sim A \,\&\, \sim B)$. The symbols of S and T are the symbols of E'_{fde} plus '='. The formulas of S and T are as follows: The ptfs of both systems consist of propositional variables plus $\sim A$ and $(A \,\&\, B)$ where A and B are ptfs. The formulas are ptfs plus $(A \rightarrow B)$ and $(A = B)$ where A and B are ptfs.

3 The postulates and semantics of system S The postulates of S are determined by these nine items:

Axioms for *S*
I. Every axiom of E'_{fde} is an axiom of S
II. $A = A$
III. $A = \sim\sim A$

IV. $A = (A \& A)$
V. $(A \& B) = (B \& A)$

Rules of inference for S

VI. Every rule of inference of E'_{fde} is a rule of inference of S
VII. from $A = B$ to infer $B = A$
VIII. from $A = B$, $B = C$ to infer $A = C$
IX. from $A = B$ to infer $C = C^A_B$ where C^A_B differs from C only in that one occurrence of A in C has been replaced by B.

A formula $(A = B)$ or $(A \rightarrow B)$ is a theorem of S iff it is an axiom of S or follows from theorems of S by a rule of inference of S.

A model for S is a quadruple $\langle Q, D, p, f\rangle$ satisfying the conditions (Q) (D) (p) (f) below:

(Q) $Q = \langle L, s\rangle$ is a model of E'_{fde}.

(D) $D = U\{D_0, \ldots, D_i, \ldots\}$, where these conditions are satisfied:

(1) $D_0 = L$.
(2) $x \in D_n$ iff $x \in D_{n-1}$; or
$x = \{y, -\}$ where $y \in D_{n-1}$ and $-$ is the intensional complement operation defined on L; or
$x = \{y, z, \wedge\}$ where y and z are distinct elements of D_{n-1} and $\wedge$ is the meet operation defined on L.

(p) p is a function defined on ptfs as follows:

(1) $p(v) = s(v)$ where v is a propositional variable.

(2) $p(\sim A) = \begin{cases} x & \text{if } p(A) = \{x, -\} \text{ where } x \in D, \\ \{p(A), -\} & \text{otherwise.} \end{cases}$

(3) $p(A \& B) = \begin{cases} p(A) & \text{if } p(A) = p(B), \\ \{p(A), p(B), \wedge\} & \text{otherwise.} \end{cases}$

(f) f is a function defined on D as follows:

$$f(x) = \begin{cases} \overline{x} & \text{if } x \in L, \\ \overline{f(y)} & \text{if } x = \{y, -\} \text{ and } f(y) \in L, \\ f(y) \wedge f(z) & \text{if } x = \{y, z, \wedge\} \text{ and } f(y), f(z) \in L, \\ x & \text{otherwise.} \end{cases}$$

Intuitively, D is to be thought of as a set of propositions. Construe $\{x, -\}$ as a negative proposition, and $\{x, y, \wedge\}$ as a conjunctive proposition. The function p assigns propositions to the ptfs. The peculiarities of its construction are motivated by the double negation axiom $A = \sim\sim A$, the reduction axiom $A = (A \& A)$ and the commutation axiom $(A \& B) = (B \& A)$. The function f maps propositions into lattice elements. Its construction is motivated by our desire that valid entailments in E'_{fde} remain valid.

We take the semantics of E'_{fde} as developed within a framework whose set theory satisfies the axiom of regularity; in particular, every intensional lattice satisfies that axiom. We further note that every intensional lattice is non-degenerate (i.e., has at least two numbers). This being so we have the following theorem:

T1 *For any model $\langle Q, D, p, f\rangle$ of S: $-, \wedge \notin D$.*

Proof: We let x range over elements of D. *Case 1.* $x \in L$. Note that

(a) $- = \{\langle a, b\rangle: a, b \in L \text{ and } b = \overline{a}\}$;

and that

(b) $\wedge = \{\langle a, b, c\rangle: a, b, c \in L \text{ and } c = a \wedge b\}$.

It follows from (a) and (b) plus the axiom of regularity that $x \neq -$ and $x \neq \wedge$.

Case 2. $x = \{y, -\}$. Since L has at least two members, $\wedge$ has at least four members. Hence $\wedge \neq \{y, -\}$. And by the axiom of regularity $\{y, -\} \neq -$.
Case 3. $x = \{y, z, \wedge\}$ where $y \neq z$. By the axiom of regularity $\{y, z, \wedge\} \neq \wedge$. Now if $\{y, z, \wedge\} = -$ then $\wedge \in -$. But, since L has at least two members, $\wedge$ is more than two-membered. But every element in $-$ is two-membered. So $\wedge \notin -$.

Before proceeding to the definitions of truth and validity, we establish that $p(A)$ is always in D and note that $f(x)$, for $x \in D$, is always in L.

T2 *For any model of S $\langle Q, D, p, f\rangle$ and $x \in D$: $f(x) \in L$.*

Proof: By an obvious induction on the construction of D.

T3 *For any model $\langle Q, D, p, f\rangle$ of S, $p(C) \in D$.*

Proof: As hypothesis of the induction we have, for any C, if $l(C) < n$ then $p(C) \in D$ where $l(C)$ = the number occurrences of $\sim$ and & in C. *Case 1.* C is a variable v. Then $p(v) \in L$. *Case 2.* C is $\sim A$. *Subcase 1.* $p(A) = \{x, -\}$ for some $x \in D$. But then $p(\sim A) = x$. Hence $p(\sim A) \in D$. *Subcase 2.* $p(\sim A) = \{p(A), -\}$. By hypothesis of the induction $p(A) \in D$. Hence, for some positive integer i, $p(A) \in D_{i-1}$. Hence $\{p(A), -\} \in D_i$. Hence $\{p(A), -\} \in D$. *Case 3.* C is $(A \,\&\, B)$, is dealt with similarly.

T3 establishes a fundamental condition of the adequacy of the definition of a model. Its use in subsequent developments is pervasive, although sometimes implicit.

We now define truth in a model of S and validity in S. $(A \rightarrow B)$ is true in a model $\langle Q, D, p, f\rangle$ of S iff $f(p(A)) \leqslant f(p(B))$. $(A = B)$ is true in a model $\langle Q, D, p, f\rangle$ of S iff $p(A) = p(B)$. $(A \rightarrow B)$ is valid in S iff $(A \rightarrow B)$ is true in all models of S. $(A = B)$ is valid in S iff $(A = B)$ is true in all models of S.

4 The soundness of S We establish: (i) $(A \rightarrow B)$ is a theorem of S only if it is valid in S, and (ii) $(A = B)$ is a theorem of S only if it is valid in S.

Item (i). Fundamental to establishing item (i) is this carryover result: $(A \rightarrow B)$ is valid in S iff it is valid in E'_{fde}. To get this result we need:

T4 *For any model $\langle Q, D, f, p\rangle$ of S, $f(p(C)) = Val_Q(C)$.*

Proof: The hypothesis of induction is: if $l(A) < n$ then $f(p(A)) = Val_Q(A)$, for all ptfs A. If C is a variable then T4 is easily shown to hold. Consider the case where C is $\sim A$. *Subcase 1.* $p(A) = \{x, -\}$, for some $x \in D$. Then we have:

(a) $p(\sim A) = x$.
(b) $f(p(A)) = \overline{f(x)}$.

From (b) and the hypothesis of induction we have:

(c) $\overline{f(x)} = Val_Q(A)$.

And from (c) we get $\overline{\overline{f(x)}} = \overline{Val_Q(A)}$, which yields:

(d) $f(x) = \overline{Val_Q(A)} = Val_Q(\sim A)$,

since, by T2, $f(x) \epsilon L$ and since, for all $a \epsilon L$, $\overline{\overline{a}} = a$. But (a) yields:

(e) $f(p(\sim A)) = f(x)$.

And, from (d) and (e), we get $f(p(\sim A)) = Val_Q(\sim A)$. *Subcase 2.* $p(A) \neq \{x, -\}$ for any $x \epsilon D$. Then $p(\sim A) = \{p(A), -\}$. So $f(p(\sim A)) = \overline{f(p(A))} = \overline{Val_Q(A)} = Val_Q(\sim A)$. The same methods show T4 holds where C is $(A \ \&\ B)$.

T5 *$(A \rightarrow B)$ is valid in S iff it is valid in E'_{fde}.*

Proof: Suppose $(A \rightarrow B)$ is valid in E'_{fde}. Now consider any model $M = \langle Q, D, p, f\rangle$ of S. Since $Val_Q(A) \leqslant Val_Q(B)$, it follows by T4 that $(A \rightarrow B)$ is true in M. Since M is an arbitrarily chosen model of S it follows that $(A \rightarrow B)$ is valid in S. Suppose $(A \rightarrow B)$ is valid in S. Now consider any model Q of E'_{fde}. $(A \rightarrow B)$ is true in $\langle Q, D, p, f\rangle$. That is, $f(p(A)) \leqslant f(p(B))$. So it follows by T4 that $(A \rightarrow B)$ is true in Q. Since Q is arbitrarily chosen it follows that $(A \rightarrow B)$ is valid in E'_{fde}.

Two more theorems yield item (i) of soundness.

T6 *$(A \rightarrow B)$ is a theorem of E'_{fde} iff it is a theorem of S.*

Proof: The implication from left to right follows from the fact that all postulates of E'_{fde} are postulates of S. The implication from right to left follows from the fact that none of the postulates of S that contain '=' enter into a derivation of $(A \rightarrow B)$.

T7 *$(A \rightarrow B)$ is a theorem of S iff it is valid in S.*

Proof: $(A \rightarrow B)$ is valid in E_{fde} iff $(A \rightarrow B)$ is a theorem of E_{fde}. It follows from this that, for all ptfs of E'_{fde} A and B, $(A \rightarrow B)$ is valid in E'_{fde} iff $(A \rightarrow B)$ is a theorem of E'_{fde}. T7 follows from this fact plus T6 and T5.

Item (ii). As a first step in establishing the second part of soundness, we note:

T8 *If $\langle Q, D, p, f\rangle$ is a model of S, each of these hold trivially:*

(1) $p(A) = p(A)$
(2) $p(A) = p(A \ \&\ A)$
(3) $p(A \ \&\ B) = p(B \ \&\ A)$
(4) *if* $p(A) = p(B)$, *then* $p(B) = p(A)$
(5) *if* $p(A) = p(B)$ *and* $p(B) = p(C)$, *then* $p(A) = p(C)$.

More difficult to show are:

(6) $p(A) = p(\sim\sim A)$
(7) $p(A) = p(B)$ *only if* $p(C) = p(C^{A}_{B})$, *where* A, B, C *are as stated in Rule IX.*

To these we now turn.

T9 *For any model $\langle Q, D, p, f\rangle$ of S, $p(A) = p(B)$ only if $p(C) = p(C^A_B)$.*

Proof: If C is a variable, T9 holds trivially. Suppose C is a negation $\sim E$. *Subcase 1.* $p(E) = \{x, -\}$. By hypothesis of induction $p(E^A_B) = \{x, -\}$. Hence, $p(\sim E) = x = p(\sim E^A_B)$. *Subcase 2.* $p(E) \neq \{x, -\}$, for all $x \in D$. By hypothesis of induction $p(E^A_B) \neq \{x, -\}$, for all $x \in D$. So we have: $p(\sim E) = \{p(E), -\}$ and $p(\sim E^A_B) = \{p(E^A_B), -\}$. So, by hypothesis of induction, T9 holds in this case. In the case where C is $(E \& F)$, $(E \& F)^A_B$ is $(E^A_B \& F)$, or it is $(E \& F^A_B)$. The proof that T9 holds in each instance is straightforward, using the same methods as in the negation case.

This leaves the double negation axiom. To establish its validity we need two preliminary theorems.

T10 *For any model $\langle Q, D, p, f\rangle$ of S, $\{p(A), p(B), \wedge\} \neq \{x, -\}$ if $p(A) \neq p(B)$.*

Proof: By T1 $\wedge$ is not in D. By T3 $p(A)$ and $p(B)$ are in D. So if $p(A)$ and $p(B)$ are distinct then $\{p(A), p(B), \wedge\}$ is three-membered, while $\{x, -\}$ is at most two-membered.

T11 *For any model $\langle Q, D, p, f\rangle$ of S and $a \in L$, $a \neq \{x, -\}$.*

Proof: T11 follows from the fact that $- = \{\ldots \langle a, \bar{a}\rangle \ldots\}$ plus the axiom of regularity.

T12 *For any model $\langle Q, D, p, f\rangle$ of S, $p(A) = p(\sim\sim A)$.*

Proof: Case 1. A is a variable, v. Then, for some $a \in L$, $p(v) = a$. Now, by T11, $a \neq \{x, -\}$ for all $x \in D$; so $p(\sim v) = \{a, -\}$. Thus $p(\sim\sim v) = a$. *Case 2.* A is $\sim B$. One subcase is $p(B) = \{x, -\}$, for some $x \in D$. Here $p(\sim B) = x$. Now, by the hypothesis of induction, $p(\sim\sim B) = p(B) = \{x, -\}$. But then $p(\sim\sim\sim B) = x = p(\sim B)$. The other subcase is $p(B) \neq \{x, -\}$, for all $x \in D$. Using this plus the hypothesis of induction we get

(a) $p(\sim B) = \{p(B), -\}$
(b) $p(\sim\sim B) \neq \{x, -\}$.

Using (a) and (b) plus the hypothesis of induction again we get

(c) $p(\sim\sim\sim B) = \{p(\sim\sim B), -\} = p(\sim B)$.

The proof that T12 holds in the conjunction case employs the same general idea.

From Theorems T7, T8, T9 and T12 we get soundness:

T13 *A formula $(A \rightarrow B)$ is a theorem of S only if it is valid in S. A formula $(A = B)$ is a theorem of S only if it is valid in S.*

*5 **The completeness of S*** We now want to show $(A = B)$ is valid only if it is a theorem, i.e., $\Vdash (A = B)$ only if $\vdash (A = B)$. We first define a predicate R on ptfs (to be read: 'is reduced'). Then we show: (i) if RA, RB, and $\Vdash (A = B)$ then $\vdash (A = B)$; (ii) if not RA there is a B such that RB and $\vdash A = B$. Completeness follows from (i) and (ii). Accordingly, we divide this section into two parts, corresponding to (i) and (ii), respectively.

5.1 Say that A is part of B iff either A is B or A is a proper part of B. We say A is double negation free (dnf) iff, for all B, $\sim\sim B$ is not part of A. We say A is stammering free (sf) iff, for all B, $(B \;\&\; B)$ is not part of A. We note the following lemma for later reference.

L1 (1) *$\sim A$ is dnf only if A is dnf.*
(2) *$\sim A$ is sf iff A is sf.*
(3) *$(A \;\&\; B)$ is dnf iff A is dnf and B is dnf.*
(4) *$(A \;\&\; B)$ is sf iff A is not B and A is sf and B is sf.*
(5) *$(\sim A \;\&\; \sim B)$ is sf iff $(A \;\&\; B)$ is sf.*
(6) *$\sim A$ is dnf and $\sim B$ is dnf only if $(A \;\&\; B)$ is dnf.*

Definition 1 $|A| = U\{(A)^0, \ldots, (A)^i, \ldots\}$, where:

(1) $(A)^0 = \{A\}$,

(2) $x \in (A)^n$ iff $x = B\dfrac{(P \;\&\; Q)}{(Q \;\&\; P)}$ for some $B \in (A)^{n-1}$.

Definition 2 RA iff (i) A is dnf and (ii) every element in $|A|$ is sf.

To establish the result of this part of the completeness proof we need some additional lemmas. (We omit proofs unless they seem called for.)

L2 *If $A \in |X|$ so is $A\dfrac{(P \;\&\; Q)}{(Q \;\&\; P)}$.*

L3 *$A \in |X|$ iff $\sim A \in |\sim X|$.*

L4 *$(U \;\&\; V) \in |(A \;\&\; B)|$ iff* (i) *$U \in |A|$ and $V \in |B|$ or* (ii) *$U \in |B|$ and $V \in |A|$.*

Proof of L4: The left-right implication is by induction on $|(A \;\&\; B)|$. The right to left implication we show here. We show if $U \in |A|$ and $V \in |B|$ then $(U \;\&\; V) \in |A \;\&\; B|$. (Parallel reasoning will show $U \in |B|$ and $V \in |A|$ implies $(U \;\&\; V) \in |(A \;\&\; B)|$).) This is our hypothesis of induction:

if $i + j < n$ then $U \in (A)^i$ and $V \in (B)^j$ implies $(U \;\&\; V) \in |(A \;\&\; B)|$.

Case 1. $n = 0$. Then $U \in (A)^0$ and $V \in (B)^0$. Then $(U \;\&\; V)$ is $(A \;\&\; B)$. But $(A \;\&\; B) \in |(A \;\&\; B)|$. *Case 2.* $n > 0$. *Subcase 1.* $i > 0$ and $j = 0$. Then for some $Z \in (A)^{i-1}$.

$$U = Z\frac{(P \;\&\; Q)}{(Q \;\&\; P)}$$

holds for the commutation of some h^{th} occurrence of $(P \;\&\; Q)$ in Z. Since $(i - 1) + j < i + j$,

$$(Z \;\&\; V) \in (A \;\&\; B)^k$$

holds for some k. And, thus,

$$\left(Z\frac{(P \;\&\; Q)}{(Q \;\&\; P)} \;\&\; V\right) \in (A \;\&\; B)^{k+1}$$

holds for the h^{th} occurrence of $(P \;\&\; Q)$ in Z. Thus, we have

$$(U \;\&\; V) \in |(A \;\&\; B)|.$$

Analogous reasoning applies in the other two subcases.

Using L1-L4 we get the following additional lemmas:

L5 $(\sim U \,\&\, \sim V) \in |(\sim A \,\&\, \sim B)|$ *iff* $(U \,\&\, V) \in |(A \,\&\, B)|$. (By L3, L4.)
L6 *If* $R \sim A$ *then* RA. (By L1(1), L1(2), L3.)
L7 *If* $R(A \,\&\, B)$ *then* RA *and* RB. (By L1(3), L1(4), L4.)
L8 *If* $R(\sim A \,\&\, \sim B)$ *then* $R(A \,\&\, B)$. (By L1(3), L1(5), L1(6), L5.)
L9 *If* RA *and* RB *and some element in* $|(A \,\&\, B)|$ *is not sf, then for some* X, $(X \,\&\, X) \in |(A \,\&\, B)|$. (By L4.)
L10 *If* $X \in |A| \cap |C|$ *and* $Y \in |B| \cap |E|$, *then* $((X \,\&\, Y) \,\&\, (X \,\&\, Y)) \in |((A \,\&\, B) \,\&\, (C \,\&\, E))|$. (By repeated uses of L4.)
L11 *For any model* $\langle Q, D, p, f\rangle$ *of* S *and* $x \in D$, $p(A) \neq \{\{x, -\}, -\}$.

Proof of L11: The axiom of regularity shows L11 holds if A is a variable. The hypothesis of induction and T12 shows it holds if A is a negation. The same hypothesis plus T10 shows it holds if A is a conjunction.

L12 *For any model* $\langle Q, D, p, f\rangle$ *of* S, $p(\sim A) = p(\sim B)$ *only if* $p(A) = p(B)$.

Proof: L11 shows the following conjunction is impossible, given $p(\sim A) = p(\sim B)$: There is an $x \in D$ such that $p(A) = \{x, -\}$ and there is no $y \in D$ such that $p(B) = \{y, -\}$. The impossibility of this case leaves just two possibilities to consider: (i) There is an $x \in D$ such that $p(A) = \{x, -\}$, and a $y \in D$ such that $p(B) = \{y, -\}$. Then $p(\sim A) = p(\sim B)$ only if $x = y$, in which case $p(A) = p(B)$. (ii) There is no $x \in D$ such that $p(A) = \{x, -\}$, and no $y \in D$ such that $p(B) = \{y, -\}$. Then $p(\sim A) = \{p(A), -\}$ and $p(\sim B) = \{p(B), -\}$. Thus, $p(\sim A) = p(\sim B)$ only if $p(A) = p(B)$.

L1-L12 prove instrumental in establishing the following key theorem.

T14 *Let* $\langle Q, D, p^*, f\rangle$ *be any model of* S *such that* p^* *assigns distinct elements of* D *to distinct variables of* S. *Then if* RA, RB, $R(A \,\&\, B)$ *then* $p^*(A) \neq p^*(B)$.

Proof: Our hypothesis of induction, (H), is: if $l(A) + l(B) < n$ and RA, RB, $R(A \,\&\, B)$ then $p^*(A) \neq p^*(B)$. There are six cases. *Case 1.* A and B are variables. Then $p^*(A) \neq p^*(B)$ by stipulation on p^*. *Case 2.* A is a variable and B is a negation $\sim C$. This divides up into two subcases. If C is a variable, then T11 can be used to show T14 holds. If C is a conjunction, then L6, L7, (H), T10, and T11 can be used to show T14 holds. *Case 3.* A is a variable and B is a conjunction. Then L7, (H), and the axiom of regularity suffice. *Case 4.* A is a negation and B is a negation. L6, L8, (H), and L12 suffice. *Case 5.* A is a negation and B is a conjunction. L6, L7, (H), T11, T10 suffice. *Case 6.* A is a conjunction, $(X \,\&\, Y)$; B is a conjunction, $(Z \,\&\, W)$. We detail the proof in this case:

(1) $R(X \,\&\, Y)$ and $R(Z \,\&\, W)$, given.
(2) RX, RY, RZ, RW, from (1) using L7.
(3) $p^*(X) \neq p^*(Y)$, from (1) and (2) using (H).
(4) $p^*(X \,\&\, Y) = \{p^*(X), p^*(Y), \wedge\}$, from (3).
(5) $p^*(Z \,\&\, W) = \{p^*(Z), p^*(W), \wedge\}$, using analogous reasoning.

(6) Suppose $p^*(X) = p^*(Z)$ and $p^*(Y) = p^*(W)$.
(7) Not $R(X \& Z)$ and not $R(Y \& W)$, from (6) and (1) using (H).
(8) $R((X \& Y) \& (Z \& W))$, given.
(9) Some member of $|(X \& Z)|$ is not sf, from (7), (2) using L1.
(10) For some C, $(C \& C) \in |(X \& Z)|$, from (9) using L9.
(11) $C \in |X|$ and $C \in |Z|$, from (10) using L4.
(12) For some E, $E \in |Y|$ and $E \in |W|$, by reasoning analogous to (7)-(11).
(13) $((C \& E) \& (C \& E)) \in |(X \& Y) \& (Z \& W)|$, from (11) and (12) using L10.
(14) So (6) is false since (13) contradicts (8).
(15) By reasoning analogous to (6)-(14) it follows that the following conjunction also is false: $p^*(X) = p^*(W)$ and $p^*(Y) = p^*(Z)$.
(16) $p^*(X \& Y) \neq p^*(Z \& W)$, from (4), (5), (14), (15).

The main result of this part of the completeness proof is gotten from T14 plus the following lemma, which is obtained from the construction of $|A|$ plus the postulates of S:

L13 *For all $X, Y \in |A|$: $\vdash(X = Y)$.*
T15 *$RA, RB, \Vdash(A = B)$ imply $\vdash(A = B)$.*

Proof of T15: Using T14 not $R(A \& B)$ follows from the hypotheses of T15. It further follows that, for some C, $(C \& C) \in |(A \& B)|$, (L1 and L9); whence it follows that $C \in |A|$ and $C \in |B|$, (L4). Hence, by L13, both of these propositions follow:

$$\vdash(A = C)$$
$$\vdash(C = B).$$

So, by postulate VIII,

$$\vdash(A = B)$$

follows.

5.2 The idea is now to show that for every nonreduced A there is a reduced B such that $\vdash(A = B)$. We begin with some definitions.

Definition 3 $\approx{A}$ is the result of deleting each occurrence of $\sim\sim$ in A.

Definition 4 $A^s = U\{A^s_0, \ldots, A^s_i, \ldots\}$ where

(1) $A^s_0 = \{A\}$

(2) $X \in A^s_n$ iff for some $Y \in A^s_{n-1}$ and C, $X \dfrac{C}{(C \& C)} \in |Y|$.

Definition 5 B maximally simplifies A iff $B \in \overset{\approx}{A}{}^s$ and, for all X if $X \in \overset{\approx}{A}{}^s$ then $l(X) \geqslant l(B)$.

We use the following lemmas to prove the main theorem.

L14 *If $E \in |G|$ and $F \in |G|$ then $l(E) = l(F)$.*
L15 *If $Z \in \overset{\approx}{A}{}^s$ then $\vdash(A = Z)$.*

$\vdash(A = \overset{\approx}{A})$, from the definition of $\overset{\approx}{A}$ and the postulates of S. $\vdash(\overset{\approx}{A} = Z)$ if $Z \in \overset{\approx}{A}{}^s$,

from the construction of $\tilde{\tilde{A}}^s$ *and the postulates of S. Thus also,* $\vdash(A = Z)$, *from Postulate VIII.*

T16 *If not RA there exists a B such that RB and* $\vdash(A = B)$.

Proof: For any A there exists a B such that B maximally simplifies A. By L15 we have $\vdash A = B$. So what has to be shown is that RB; that is: (i) that B is dnf, and (ii) every element in $|B|$ is sf. It is easily seen that every element in $\tilde{\tilde{A}}^s$ is dnf. So item (i) is trivial. We turn to item (ii).

(1) Suppose, for some Z and C, $Z \in |B|$ and $(C \& C)$ is part of Z.

(2) Let the occurrence of $(C \& C)$ in Z and C in $Z\dfrac{C \& C}{C}$ be so chosen that Z is $Z\dfrac{C \& C}{C}\dfrac{C}{C \& C}$.

(3) $B \in A^s_{n-1}$, for some n, since $B \in A^s$.

(4) $Z\dfrac{C \& C}{C}\dfrac{C}{C \& C} \in |B|$ since $Z \in |B|$.

(5) $Z\dfrac{C \& C}{C} \in A^s_n$, from (3) and (4), and so $Z\dfrac{(C \& C)}{C} \in A^s$.

(6) $l\left(Z\dfrac{(C \& C)}{C}\right) \not> l(Z)$.

(7) $l(Z) = l(B)$, by L14 since $Z \in |B|$.

(8) (6) and (7) contradict the assumption that B maximally specifies A. So, on this assumption, (1) is false.

Using T15 and T16 the following theorem is easy to prove:

T17 *If not RA and not RB and* $\Vdash(A = B)$ *then* $\vdash(A = B)$.

Thus also,

T18 *If not RA and RB and* $\Vdash(A = B)$ *then* $\vdash(A = B)$.

(Just note that $\Vdash(A = B)$ implies $\Vdash(A = {\sim\sim}B)$ and that not $R \sim\sim B$.)

From T15, T17, T18, we get:

T19 *If* $\Vdash(A = B)$ *then* $\vdash(A = B)$.

6 The system T The semantics of S validates the following principle:

(1) from $C\frac{A}{B} = C$ to infer $A = B$.

Intuitively, this says that if two sentences express the same proposition and differ only in the interchange of component sentences A and B, then A and B also express the same proposition. E.g., if $({\sim}A = {\sim}B)$ then $(A = B)$; if $(A \& B) = (A \& C)$ then $(B = C)$. This principle of proposition substitution is very plausible.

But this principle is also plausible:

(2) $A \& (B \& C) = (A \& B) \& C$,

and it turns out that, given that a system contains S, the system cannot contain both (1) and (2). For, using (2) plus the postulates of S, one can derive:

(3) $((q \,\&\, r) \,\&\, s) = ((q \,\&\, r) \,\&\, (r \,\&\, s))$.

Using (1),

(4) $s = (r \,\&\, s)$

is derivable.[1] But (4) is not a theorem of S; nor should it be a theorem of any reasonable system of propositional identity. So (1) and (2) are incompatible in this sense: Given that one accepts the postulates of S, one has to choose between (1) and (2).

Some will find (2) to be more plausible on intuitive grounds than (1). Thus it is worthwhile investigating the system which results from adding (2) to S. We call such a system T. Our first goal is to devise an appropriate semantics for T; we then focus on the main theorems needed to carry over results from S so as to get soundness and completeness for T.

A model for T is a quintuple $\langle Q, E, g, q, h\rangle$ satisfying conditions (Q), (E), (g), (q), (h) below:

(Q) $Q = \langle L, s\rangle$ is a model for E'_{fde}.

(E) $E = U\{E_0, \ldots, E_i, \ldots\}$ where

(1) $E_0 = L$

(2) $x \in E_n$ iff $x \in E_{n-1}$; or
$x = \{y, -\}$ where $y \in E_{n-1}$ and $-$ is the intensional complement operation defined on L; or
$x = \{y_1, \ldots, y_k, \wedge\}$ where $y_1, \ldots, y_k \in E_{n-1}$ and $\wedge$ is the meet operation defined on L.

(g) g is a function defined on E as follows:

$$g(x) = \begin{cases} \{x\} & \text{if } x \in L \text{ or } x = \{y, -\}, \\ \{y_1, \ldots, y_k\} & \text{if } x = \{y_1, \ldots, y_k, \wedge\}. \end{cases}$$

(q) q is a function defined on the ptfs as follows:

(1) Where A is a variable or a negation, $q(A)$ is defined in the same way as $p(A)$.

(2) If A is $(B \,\&\, C)$,

$$q(B \,\&\, C) = \begin{cases} q(B) & \text{if } q(B) = q(C), \\ g(q(A))\ U\ g(q(B))\ U\ \{\wedge\} & \text{otherwise.} \end{cases}$$

(h) h is a function defined on E as follows:

$$h(x) = \begin{cases} x & \text{if } x \in L, \\ \overline{h(y)} & \text{if } x = \{y, -\} \text{ and } h(y) \in L, \\ \wedge\{h(y_1), \ldots, h(y_k)\} & \text{if } x = \{y_1, \ldots, y_k, \wedge\} \\ & \text{and } h(y_1), \ldots, h(y_k) \in L, \\ x & \text{otherwise.} \end{cases}$$

The definitions of truth in a model for T and validity in T parallel corresponding definitions for S.

The soundness and completeness of T can be established along the same lines provided for S. The most difficult thing to prove in carrying over results is this:

T20 *If* $\langle Q, E, g, q, h\rangle$ *is a model of* T *and* A *is any ptf of* T, *then* $h(q(A)) = Val_Q(A)$.

In the case where A is a variable or a negation the reasoning given to show that T20 holds of A is the same as was given in the proof of T4; this is also true where A is a conjunction $(B \,\&\, C)$ and $q(B) = q(C)$. So we only consider the case where A is $(B \,\&\, C)$ and $q(B) \neq q(C)$. To show T20 holds in this case we need the following lemma:

L16 $h(q(A)) = \bigwedge\{h(x): x \in g(q(A))\}$.

Proof of L16: Case 1. $q(A) = a$, for some $a \in L$; whence we obtain:

(a) $h(q(A)) = a = \bigwedge\{a\}$,
(b) $g(q(A)) = \{a\}$.

From (b) we get $\{h(x): x \in g(q(A))\} = \{h(a)\}$. It follows from this and (a) that L16 holds in this case. *Case 2.* $q(A) = \{y, -\}$, for some $y \in E$. From this we get:

(a) $h(q(A)) = \overline{h(y)} = \bigwedge\{\overline{h(y)}\}$

using the T analogue of T2. We also get

(b) $g(q(A)) = \{\{y, -\}\}$;

whence we obtain

(c) $\{h(x): x \in g(q(A))\} = \{h(x): x \in \{\{y, -\}\} = \{\overline{h(y)}\}$.

From (a) and (c) it follows that L16 holds in this case. *Case 3.* $q(A) = \{y_1, \ldots, y_k, \wedge\}$, for some $y_1, \ldots, y_k \in E$. Using the T analogue of T2

(a) $h(q(A)) = \bigwedge\{h(y_1), \ldots, h(y_k)\}$

follows; we also obtain

(b) $g(q(A)) = \{y_1, \ldots, y_k\}$.

From (b)

(c) $\{h(x): x \in g(q(A))\} = \{h(x): x \in \{y_1, \ldots, y_k\}\} = \{h(y_1), \ldots, h(y_k)\}$

follows. And from (a) and (c) L16 holds in this case.

Proof of T20: We just consider the case where A is $(B \,\&\, C)$ and where $q(B) \neq q(C)$. From this we get: $q(B \,\&\, C) = g(q(B)) \; U \; g(q(C)) \; U \; \{\wedge\}$. Hence, $g(q(B \,\&\, C)) = g(q(B)) \; U \; g(q(C))$. From L16 we get: $h(q(B \,\&\, C)) = \bigwedge\{\bigwedge\{h(x): x \in g(q(B))\}, \bigwedge\{h(x): x \in g(q(C))\}\}$. Using L16 again we get: $h(q(B \,\&\, C)) = \bigwedge\{h(q(B), q(C)\} = h(q(B)) \wedge h(q(C))$. Using the hypothesis of induction we then get: $h(q(B \,\&\, C)) = Val_Q(B) \wedge Val_Q(C) = Val_Q(B \,\&\, C)$.

We conclude with this theorem which establishes the validity of the association axiom:

T21 *If* $\langle Q, E, G, q, h\rangle$ *is a model of* T *and* A, B, C *are any ptfs of* T *then* $q((A \,\&\, B) \,\&\, C) = q(A \,\&\, (B \,\&\, C))$.

Proof of T21: The trick here is to see what cases to work with. T21 can be readily shown to hold in each of these cases: *Case 1.* $q(A) = q(B) = q(C)$. *Case 2.* $q(A) = q(B)$ and $q(A) \neq q(C)$. *Case 3.* $q(A) = q(C)$ and $q(B) \neq q(C)$. *Case 4.* $q(B) = q(C)$ and $q(B) \neq q(A)$. *Case 5.* $q(A) \neq q(B)$, $q(C) \neq q(A)$ and $q(B) \neq q(C)$.

7 ***Concluding remarks*** For a nonindexical langauge, reasonable criteria for sentences having the same propositional content fall somewhere between the sufficient condition of syntactic identity and the necessary condition of material equivalence. Each such criterion partitions the class of sentences into equivalence classes the members of which coincide in propositional content by the criterion. The equivalence classes thus produced by different criteria can be roughly ordered with respect to the degree of syntactical homogeneity of their members.

On one philosophical view, sentences informationally the same are also the same in propositional content and only sentences logically equivalent convey the same information. The sentences thus classed together are syntactically very dissimilar.

On another philosophical view only logically equivalent sentences in a certain sense "relevant" to one another coincide in propositional content. For E_{fde} this "relevance connection" is coentailment. The sentences thus classed together are syntactically more homogeneous than those classed together by the criterion of logical equivalence.

But further criteria, which tune propositional identity more finely to syntactical structure, can also be philosophically supported. For example, it is plausible to assert that, in many cases, one can grasp the content of A without grasping that of B, but cannot grasp the content of $(A \vee (A \,\&\, B))$ without grasping that of B. Yet, with respect to E_{fde} coentailment holds between A and $(A \vee (A \,\&\, B))$ for any ptfs A and B.

A natural "lower limit" for a system in the notation of E_{fde} would be given by so extending the model theory for E_{fde} as to verify the law that $C^{A}_{B} = C$ if and only if $A = B$. This is our system S, which allows for identity despite syntactic differences by reduction, double negation, and commutation, but no more. $A = B$ holds in S only if A and B are coentailing in E_{fde}, but not conversely. In particular, conjunction is associative for coentailment, but not for propositional identity in S. S yields T by adding an association axiom and adjusting the model theory, but only at the cost of falsifying the law cited above. T is less finely tuned to syntactic structure than is S, but remains more so than is E_{fde}. In particular, distribution of conjunction over disjunction holds for coentailment, but not for propositional identity in T. It is a plausible conjecture that adding an axiom for distribution of conjunction over disjunction to T regains E_{fde} in the sense that in the resulting system (call it W) propositional identity and coentailment again coincide. By a similar conjecture, adding the "identity element" axiom $(\sim(A \,\&\, \sim A) \,\&\, B) = B$ to W yields a system in which propositional identity coincides with logical equivalence.[2]

NOTES

1. We owe examples (3) and (4) to Peter Geach.

2. These two conjectures were suggested to us by an anonymous referee of this journal.

REFERENCE

[1] Anderson, A. R. and N. D. Belnap, Jr., *Entailment,* Vol. 1, Princeton University Press, Princeton, New Jersey, 1975 (pp. 180-206 were contributed by Michael Dunn).

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