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Probabilistic Considerations on Modal Semantics

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Introduction Since the nineteenth century probability and modal logic have made very uneasy bedfellows. This is because the two areas seem to offer competing rather than complementary accounts of the necessary/contingent distinction. De Morgan, for example, offers the following remarks concerning modal assertions:

> "Probability is . . . the unknown god whom the schoolmen ignorantly worshipped when they so dealt with this species of ennunciation, that it was said to be beyond human determination whether they most tortured the modals, or the modals them." ([2], p. 232)

Another equally pointed polemical salvo is fired by Venn:

> "The logicians have failed, after having had a long and fair trial, to make anything satisfactory out of this subject of modals by their methods of inquiry and treatment . . . It ought, therefore, to be banished entirely from that science, and relegated to probability." ([12], p. 296)

Of course, since those days logicians have developed new methods, in particular set-theoretic semantics for the treatment of modals. Many feel that, at last, a satisfactory account can be given. Others continue to think that modal logicians persist with studies which may properly be described as tortuous. It is scarcely surprising that, with the growth of probabilistic semantics, modal logic as an independent subject should once again be hard pressed.

There are two distinct motivations behind recent work in the application of probabilistic semantics to the analysis of modality. The first is a lust for

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unity. Logic now comprises many diverse areas and it would be very satisfying to rewrite these as so many chapters in the general theory of probability. In these terms, a probabilistic analysis of modal logic would be seen as returning one more straying sheep to the fold.

The second motivation involves a desire to provide adequate informal foundations for modal semantics. According to writers of this persuasion a study of the contemporary lore of modal semantics can be a rather caprificative experience. Their complaint is not with the purely *formal* aspects; who would take exception to a completeness or decidability proof? It is rather the metaphysical politicizing which the formalism is so often made to undergo, which gives offense. Those practicing modal semantics in its current form frequently talk as if they were committed to recognizing the existence of entities called possible worlds. Some even talk as if there were informal sense to be made of a relation of "relative possibility" or "accessibility" defined over the set of possible worlds. These locutions stand, as Quine might say, in need of an account, and some practitioners of probabilistic semantics are keen to supply it. Thus, on the second motivation, the aim is not so much to unify as to explain; particularly to explain the semantics of modal logic in a politically satisfactory, i.e., nonmetaphysical, way.

In this essay we shall consider the recent work of Charles Morgan, concentrating on his paper "Probabilistic semantics for propositional modal logics" [6]. This paper serves as a particularly lucid example of a program involving the second motivation. We go on to offer some suggestions of our own concerning the application of probabilistic methods in modal semantics.

Worlds away? There isn't much doubt that the language of possible worlds easily becomes overblown. Several recent authors who invoke the concept profess a style of realism which could only be called extreme. Against these there are others who embrace realism of a more restrained sort (see, e.g., [10]). Without wishing to commit ourselves either to thoroughgoing or to moderate realism we recognize that these philosophers advance arguments which must be met by their opponents. Anyone who wishes to eliminate talk of possible worlds in favor of some more respectable entity must do two things. First, he must show that his alternative can do the same job as the possible worlds framework. Second, it must be shown that the new ideolect *is* more respectable.

Morgan's paper goes a little way toward allaying our fears on the first score. He shows for some standard modal logics that possible worlds semantics is not a prerequisite for definitions of logical truth and entailment in terms of which soundness and completeness proofs can be given. All we really need is the concept of an indexed set of extended classical valuations, that is, an indexed set of functions from wffs of modal propositional logic to the set of truth-values ($\{T,F\}$ say). So much for metaphysical seepage from other possible worlds. We shall return to "accessibility" later. For the moment we must consider whether Morgan's reformulation in the language of valuations really is the mathematical equal of the original account.

Perhaps the greatest single advance in contemporary modal semantics since the pioneering work of Kanger, Smiley, Kripke, and Hintikka has been the shift in emphasis from talk of *models* to talk of *frames*. We read in [4] that the logic $\underline{T}$ is complete with respect to the class of all $\underline{T}$-models (or, more accurately, that a wff is a $\underline{T}$-theorem iff it is $\underline{T}$-valid). This style of exposition continues right through into the bible of contemporary modal logic, viz., [3]. However, by the time of Segerberg's classic work on completeness theory, we read instead that $\underline{T}$ is determined by the class of reflexive frames (see [9]). Now, formally speaking, a frame is simply a pair, $(\underline{U}, R)$ with $\underline{U}$ a nonempty set, and R a binary relation defined on $\underline{U}$. A *model* is just a frame together with a valuation, i.e., a function V: $\underline{At} \to 2^{\underline{U}}$, which assigns to every atomic wff and member of $\underline{U}$ a truth-value. By employing, by now standard, truth-conditions, V may be extended uniquely to a function which evaluates every wff at every point of $\underline{U}$. To say that $\underline{T}$ is determined by the class of reflexive frames is just to say that a wff is a theorem of $\underline{T}$ iff it is true at every point of every model on every frame in which the relation R is reflexive. How important is this change in style of presentation? The answer is: very important. We can now see that the modal axiom $[T]\ \Box p \to p$ *corresponds* to the first-order condition $\forall x{:}xRx$. Since we talk in terms of frames rather than models we have a way of precisely characterizing this correspondence. For a frame is a model of a first-order theory as well as being a semantic structure for a modal logic. Thus we may write

$$(\underline{U}, R) \models \Box p \to p \Leftrightarrow (\underline{U}, R) \models \forall x{:}xRx.$$

Without frames the whole vital area of modal correspondence theory cannot, for lack of the proper terminology, even arise. Few modal logicians would meekly give up correspondence theory which has spawned so much mathematically interesting research. When Morgan urges us to think of $\underline{U}$ as simply an index set for a set of extended valuations and of R as a relation defined on such a set, he is asking us to take a long step backwards so far as the mathematical analysis of modal logic is concerned. On his reconstruction, the concept of a frame has disappeared and the basic semantic object is once again a model. Completeness theory may remain intact, but that is cold comfort to those who have ventured beyond into deeper semantical territory. Some might be tempted to argue here that any theoretical embellishments beyond soundness and completeness serve no genuine semantical needs. Rather than pointing to the obvious provincialism of such an argument we can appeal instead to the desire for unity in logic. It is indisputable that the recent work in such areas as definability theory and incompleteness theory has done a great deal to bring together modal semantics and that branch of mathematical logic which goes under the heading of model theory.[1]

Of course a relatively small change will bring back frames. All we need do is say that R is defined, not on an indexed set of valuations, but on the index set. However, such a restatement does not sit comfortably with the way in which Morgan wishes to interpret the formalism. The main difficulty is that the index set for our set of valuations must now appear independently (it is suppressed in Morgan's presentation). "What is this index set supposed to represent?" cry the realists. The reply must be along the lines of: "The indices simply stand for the valuations, the index i, for example, represents the valuation Vi. There is no need to go beyond talk of functions here at all."

Unfortunately, the realists have a reply to this move:

> We may grant, for the sake of argument, that valuations are in 1:1 correspondence with worlds, but that is a long way from saying that worlds can be eliminated. If we consider just the atomic sentences, the valuation *Vi* assigns each of them T or F but we can always ask why *p* should receive the assignment F rather than T. It is no use saying that *Vi* is just a function which happens to have the value F for the argument *p*. *Functions* don't make sentences true or false, they can only *represent* the truth-making properties of other objects. These objects we call possible worlds and they are represented in your formalism by the index set.

It seems then that we must have recourse to a version of nominalism which is less naive than Morgan's if we really are to avoid worlds. It is all very well to admire desert landscapes, but we must not achieve them by painting the shrubbery brown.

When we move from standard models to probabilistic ones, the realists' arguments continue to bite. Associating a conditional probability function with an index still leaves open the question of what makes the function with that index take on just those values. To say that the function represents the belief structure of some individual (perhaps an "idealized" belief structure) doesn't really provide as much illumination as Morgan suggests. There is a slippery slope from epistemology back into metaphysics. Why, we want to know, does the individual in question have just those beliefs? What is it that accounts for the formation of beliefs if not an encounter with the (a) world?

Unnatural relations One of the most interesting parts of Morgan's paper is his interpretation of that relation sometimes called "accessibility" in modal semantics. Since worlds have become for him belief structures, the "relative possibility" of one world to another has changed to the rational evolution of one belief structure into another.

Although this doxastic interpretation gives more content to accessibility than is usual, there are some drawbacks. The most obvious is that the concepts of "degree of belief" and "rational evolution of belief systems", while initially plausible and attractive, are difficult and imprecise. Consider the former. At the first stage of such a reconstruction of modal semantics, probability explicates necessity and at the second stage, belief systems explicate probability. Clearly, the second stage is vital for informal motivation, for otherwise we merely explain one formal calculus by means of another. But many writers (and not just those of the "frequentist" persuasion) have raised questions about the adequacy of the belief interpretation of probability. A perennial problem for this interpretation is that the empirical evidence does not support a probabilistic account of belief. If we attempt to evade this difficulty by talking about "ideal believers" or some such, we lose much of the intuitive appeal. We all have some immediate understanding of belief but not so many appreciate ideal belief except as something or other which satisfies the axioms of the probability calculus. These criticisms have been continued to the point where some, Kyburg, for example, have called into question whether or not there are such things as degrees of belief (cf. [5]). Here is irony: we must make our choice between commitment to worlds, and commitment to degrees of belief, both of them dicey kinds of entity.

The problems attending the notion of "rational evolution of belief" are no less vexing. Given a certain belief structure (represented, for the sake of argument, by a conditional probability function), the impact of new information should give rise to some change in the structure. It is noncontroversial to say that such change should be rational, but what is the cash-value of such an assertion? This question is notoriously difficult to answer if only because the concept of rationality is vague. In fact, it would seem that the concept of accessibility which most closely matches the "rational evolution" model is not a binary relation. This is because given a belief system x, and new information I, we would say that x should evolve into one of $y_1, y_2, \ldots, y_n$, where y_1 is the most rational on criterion $1, \ldots,$ and y_n is the most rational on criterion n. We consider nonbinary access relations in more detail below, where their connection with probability functions is discussed.

None of this is to say that probability should not be used to explicate the concept of accessibility. Morgan's comment to the effect that most accounts of this relation have unfortunate overtones of astrogation is very well taken. Only temporal logic, in which accessibility can plausibly be interpreted as the before-after relation, can claim to have told a satisfactory story about this part of the formalism. Elsewhere in modal logic, the story-telling rarely rises above the level of pulp fiction.

We shall now take up the matter of the relationship between probability and accessibility in more detail. We shall not, however, subscribe to the belief interpretation of probability, preferring a more neutral stance.

Accessibility from probability In this section we propose to analyze the modal semanticist's accessibility relation in terms of probability. We shall be concerned with a notion of accessibility which is understood in physical rather than metaphysical terms. Thus the possibility and necessity which we try to characterize are physical rather than logical.

The most plausible probabilistic account of accessibility is what we call the *transition* approach. Although there are a number of guises in which such an approach can appear, all involve probability functions, interpreted as transition probabilities, as part of the semantic structures. In the sequel we use such functions in concert with a sort of semantics appropriate to temporal logic.

Rather than worlds, we use a set $\underline{S}$ of states. These are interpreted as (possible) physical states, i.e., complete determinations of the values of all physical quantities of some physical system.[2] We also require a set $\underline{T}$ of times, which are thought of as times at which measurements are carried out on a physical system to determine its state. Under this interpretation there will be a relation '$<$', of temporal priority which is a discrete, total, strict ordering of $\underline{T}$. Relative to some physical system, explicit mention of which we suppress, there will be a function $\underline{A}:\underline{T} \to 2^{\underline{S}}$. This function assigns to every $t \in \underline{T}$ a collection of states, interpreted as the set of states which are possible for the system at t (i.e., by time t the system could have evolved into a state belonging to $\underline{A}(t)$). Finally, we introduce a function

$$\langle\,,\,\rangle_t : \underline{A}(t) \times \underline{A}(t') \to J$$

where t' is the immediate $<$-successor of t and J is the closed unit interval. We stipulate that $\langle\, ,\rangle_t$ be a transition probability function which means that it satisfies:

(1) $0 \leqslant \langle u,v\rangle_t \leqslant 1$

(2) $\sum_{v \in \underline{A}(t')} \langle u,v\rangle_t = 1.$

$\langle u,v\rangle_t$ is interpreted as the probability of the system evolving from the state u at t, to the state v at t'.

Although this semantic machinery may at first seem overelaborate, it does answer to certain philosophical needs which are readily apparent to the discerning eye. As we shall show we may define several different approaches to accessibility. For the present we choose to do this within a framework which is appropriate to the analysis of *dynamic modal logic*. By this we understand a logic in which a mixture of temporal with other (e.g., alethic) modalities occurs. Philosophical, as well as ordinary, discourse abounds with such blends. An example: In 1940 it was possible for Germany to win the second world war, but by 1943 this was no longer possible.

Traditionally, such examples are formalized within some more or less standard temporal logic but this presupposes a definition of "possible" in temporal terms. A typical definition has the form: "α is possible" iff "α is now true or α will (at some point in the future) be true". The difficulty with the definition is that to the uninitiated (or to Diodorus Chronus) such an account makes the example false. There is no point in the future of 1940 in which Germany did win.

In order to respect our intuitions that the example is true we must say that there *is* some time after 1940 in which Germany does win, although such a time is never "actualized". Some take the dangerous course here of introducing talk of "branching time" thus trampling another equally central intuition, viz., that temporal priority is a linear order. Others, talk rather of branching courses of events or branching histories thereby avoiding the appearance of nonsense if not its substance. In any case, it is clear that the example and other sentences of like kind require that some type of branching process take place. The semantics which we propose is in a position to explain such a process.

If we define accessibility (at t) as nonzero transition probability then we may allow the relation to vary over time in any way we choose. More precisely, to every $t \in \underline{T}$ we associate a binary relation R_t, defined on

$$\underline{A}(t) \cup \bigcup_{t'>t} \underline{A}(t').$$

The definition of R_t requires a notion of transition probability which applies across temporal sequences. This may be set up as follows[3]: If $u \in \underline{A}(t)$, $v \in \underline{A}(t_n)$, and $t < t_1 < \ldots < t_n$, then a *u,v-sequence* is an element, $\langle u, x_1, \ldots, x_{n-1}, v\rangle$ in $\underline{A}(t)\ \underline{A}(t_1) \ldots \underline{A}(t_n)$, such that $\langle u,x_1\rangle_t \neq 0$ and $\langle x_1,x_2\rangle_{t_1} \neq 0$ and ... and $\langle x_{n-1},v\rangle_{t_{n-1}} \neq 0$. The collection of all such is denoted by seq(u,v) with $\underline{a}$, $\underline{b}$, etc., denoting members of seq(u,v). By $\pi\underline{a}$ we understand the product of all the transition probabilities $\langle x_i, x_j \rangle_{t_i}$, of adjacent members in the sequence $\underline{a}$. We now define:

$$\langle u, v \rangle = \sum_{\underline{a} \in \text{seq}(u,v)} \pi \underline{a}$$

with the usual convention that if seq(u, v) is empty then $\langle u, v \rangle = 0$. Finally we may define accessibility:

$$uR_t v \text{ iff } u\underline{A}(t) \text{ and } \langle u, v \rangle \neq 0.$$

Thus a state v is *possible relative to* or accessible from a state u iff the physical system in question has a nonzero probability of evolving from u to v.

The properties of R_t fall into two classes: the dynamic properties, as well as those which do not involve changes over time. We see at once that for every $t \in \underline{T}$, R_t is transitive although not necessarily reflexive or symmetric. To obtain the latter and other relational properties we must place certain restrictions on $\underline{A}$ or $\langle \, , \rangle_t$, or both. For example, to get reflexivity we must insist that $\underline{A}$ be nondecreasing with respect to $<$, i.e., that possible states persist over time, and that the "steady state" transition $\langle u, u \rangle_t$ always has nonzero probability. For symmetry we must stipulate that transition probabilities are symmetric, at least in the sense that if $\langle x_i, x_j \rangle_{t_i} \neq 0$ then $\langle x_j, x_i \rangle_{t_j} \neq 0$. Additionally, for this condition to work, it must be the case that later states exist at all earlier times, i.e., that $\underline{A}$ be nonincreasing. The conditions reflexivity and symmetry, imposed simultaneously upon R_t, imply that $\underline{A}$ is a constant function.

In case we wish to avoid automatic commitment to the transitivity of R_t, we must change its definition. This could be effected by choosing some positive r, and requiring:

$$uR_t v \text{ iff } \langle u, v \rangle \geqslant r.$$

The problem with such a redefinition is that it would make universal relational properties (those holding everywhere in $\underline{T}$) very hard to come by. Certainly none of the well-known relational restrictions mentioned above would hold universally.

A brief inspection of relational dynamics shows that we can now provide a plausible account of sentences like our example. The quickest analysis goes like this: let $\underline{r}$ be the state in $\underline{A}(1940)$ which was actualized and let $\underline{g}$ be any state in $\underline{A}(1942)$ such that if the physical system had made the transition from $\underline{r}$ to $\underline{g}$ then Germany would have won the second world war. If $\underline{g} \notin \underline{A}(1943)$ then the possibility in question has evaporated in a very straightforward way. If this analysis were prevented by a restriction making $\underline{A}$ nondecreasing, we would have to complicate things only a little. One would simply say that while $\langle \underline{r}, \underline{g} \rangle \neq 0$ by 1943 we have that $\langle \underline{r}^*, \underline{g} \rangle = 0$, where $\underline{r}^*$ is the actual state in 1943. But, more sensibly, we would reject, for ordinary purposes, any such restriction.

Obviously there are many interesting conditions upon transition probabilities and $\underline{A}$ apart from those we have considered, but the important point is that the accessibility relation has been *defined*. It is not a primitive in the semantics which stands in need of informal interpretation to render it plausible.

In addition to the account sketched, others suggest themselves which are also of interest. One such is motivated by the habit among some philosophers of reading "uRv" as "v is a u-alternative". This suggests that we define R_t on $\underline{A}(t)$ (which gets around difficulties associated with long stretches of time). A definition which is quick to suggest itself is:

$$uR_tv \text{ iff } \forall x \in \underline{A}(t'): \langle x, u\rangle_{t'} \leqslant \langle x, v\rangle_{t'}.$$

On this approach, for v to be a u-alternative (at t), v must have "at least as good a chance" of being the state which is realized (at t) as u.

Understood in this way, the alternativeness relation is both reflexive and transitive, without necessarily being symmetric. Such properties holding, as it were, for free might be thought a drawback; on the other hand our account has certain advantages in spite of its particularity. These advantages relate to the imposition of dynamic properties.

In some cases it is advantageous to restrict the way in which alternativeness may vary over time. We might require, for some project or other, a principle relating temporal and alethic modalities, e.g.:

[$G\Box$] $$G\Box\alpha \rightarrow \Box G\alpha$$

where $G\alpha$ means "it will always be the case that α". Such a principle will only be sound in those structures in which the "amount of alternativeness" is nondecreasing. This means that we must assume that $\underline{A}$ is nondecreasing and that:

$$\forall t,t': t < t' \Rightarrow \forall u: R_t(u) \subseteq R_{t'}(u)$$

where $R_t(u) = \{x \mid uR_tx\}$. We may instead be interested in the condition:

$$\forall t, t': t < t' \Rightarrow \forall u: R_t(u) \supseteq R_{t'}(u)$$

to the effect that alternativeness is nonincreasing (given that $\underline{A}$ is). This condition validates:

[$\Box G$] $$\Box G\alpha \rightarrow G\Box\alpha.$$

It is worth-while remarking here that in the standard temporal reconstruction of $\Box$:

$$\Box\alpha =_{df} \alpha \wedge G\alpha.$$

We have:

$$\begin{aligned} G\Box\beta &\leftrightarrow G(\alpha \wedge G\alpha) \\ &\leftrightarrow G\alpha \wedge GG\alpha \\ \Box(G\alpha) &\leftrightarrow G\alpha \wedge GG\alpha. \end{aligned}$$

So $G\Box\alpha \leftrightarrow \Box G\alpha$ holds automatically.

In order to impose the condition appropriate to [$G\Box$], we begin from the definition:

$$v \in R_t(u) \Leftrightarrow \forall x\underline{A}(t'): \langle x, u\rangle \leqslant \langle x, v\rangle_{t'}$$

so that if $v \in R_t(u)$ but $v \notin R_{t'}(u)$ (still assuming that $\underline{A}$ is nondecreasing), then

$\exists x \in \underline{A}(t')$ and $x \in \underline{A}(t'')$ where t'' is the immediate predecessor of t' such that:

$$\langle x,u\rangle_{t'} \leqslant \langle x,v\rangle_{t'} \text{ but } \langle x,v\rangle_{t''} \not\leqslant \langle x,u\rangle_{t''}.$$

We may prevent this by the probabilistic condition:

$$\forall t,t' : t' < t \Rightarrow \forall x,u,v : \langle x,u\rangle_{t'} \leqslant \langle x,v\rangle_{t'} \Rightarrow \langle x,u\rangle_{t''} \leqslant \langle x,v\rangle_{t''}.$$

This is really a condition to the effect that inequalities among the columns of the transition matrix between two successive times are preserved over time. The corresponding condition which secures the validity of $[\Box G]$ is obvious.

Probability from accessibility Not only may accessibility relations be defined by means of transition probability functions, but the converse is also true.[4] To be sure, in order to accomplish this we must be more sophisticated than usual in our choice of an access relation. In particular, we must drop the restriction to binary relations. This goes beyond the standard approach to the semantics of modal logic and has turned out to have independent interest in other studies. The question as to which modal logics are determined by structures of this type cannot be answered here. This is not to say that no answer has been given. In general terms, these logics resemble the usual ones except that the strong aggregation principle

$$[K] \qquad \Box p \wedge \Box q \rightarrow \Box(p \wedge q)$$

is not automatically valid. Accounts of both the formal aspects and philosophical applications of such logics have been offered elsewhere.[5]

We shall confine ourselves to frames $(\underline{U},R)$ where $\underline{U}$ is a nonempty set and R is an $n+1$-ary relation defined on $\underline{U}$. Such a restriction is dictated by considerations of simplicity of exposition rather than necessity. In general, we may allow each point in $\underline{U}$ to have its own relation and these need not all have the same arity.

By $R(u)$ we understand $\{\langle x_1,\ldots,x_n\rangle | uRx_1\ldots x_n\}$, sequences in $R(u)$ being denoted by $\underline{x},\underline{y}$ etc. We use $R_x(u)$ to denote the subset X of $R(u)$ such that if $\underline{x} \in X$ then $x \in \underline{x}$. Finally, $\%x(u)$ denotes the average number of times that x occurs in $R_x(u)$. For finite cases then, $\%x(u)$ is the total number of occurrences of x divided by the cardinality of $R_x(u)$. In terms of these definitions we define (u,v), the probability of the transition from u to v (relative to our frame), as $(\text{card}(R_v(u))/\text{card}(R(u))\ \%v(u))$.

It is a relatively easy matter to pass from transition probabilities to the probabilities of statements. We may associate with every $u \in \underline{U}$ a probability function PR_u, provided we know of every statement α and point x whether or not α is true at x. This knowledge is summarized by a valuation (which evaluates atomic statements at points) together with truth conditions which extend valuations to the set of all statements. The set of points at which α is true is called the truth set of α indicated by $\|\alpha\|$. Obviously such sets are relative to valuations but we suppress mention of any particular one. Now we may define

$$PR_u(\alpha) = \sum_{v \in \|\alpha\|} (u,v)$$

which, intuitively, represents the probability of the transition from u to some point where α is true. To obtain a conditional probability function of the classical sort we simply use the standard definition

$$PR_u(\alpha/\beta) = PR_u(\alpha \wedge \beta)/PR_u(\beta).$$

This is undefined when $PR_u(\beta) = 0$ and, as several writers have observed, such a concept of conditional probability has many drawbacks. Fortunately, we may also construct a function which, like Popper's, is everywhere defined.

Since any frame/valuation pair will give us a family of probability functions we may associate a Popper function with each. The construction which allows this is due to van Fraassen ([11]) and requires only that $\{PR_u | u \in \underline{U}\}$ be well-ordered, i.e., that $\underline{U}$ be well ordered. This does not introduce into our framework a new element itself requiring intepretation. In the first place, any ordering will do for the purpose. In the second, the well-ordering of $\underline{U}$ serves no semantical purpose at all. As matters stand nothing much hangs on an interpretation of the ordering. All we need to be able to do is to say which PR_u function is the first, which the second and so on, for the purpose of carrying out a purely formal construction. In this respect the ordering of $\underline{U}$ is like the ordering of all statements, required to extend a consistent set of statements to a maximally consistent one.

Conclusions We have argued that probabilistic methods have application in the semantic analysis of modal logic but that their impact in a "reductionist" program has been overestimated by some workers. In particular, the concept of a "possible world" or a "possible case" or one of its cognates cannot be satisfactorily explicated in probabilistic terms.

The same is not true of the concept of accessibility, a relation between worlds essential to the semantic differentiation of the usual modal logics. The relation may be reconstructed from the primitive idea of a transition probability in different ways depending upon one's philosophical goals. Our major example concerned a semantics appropriate to a mixture of temporal and other modalities which makes no assumption that all other (e.g., alethic) modalities are reducible to temporal ones. Within an approach of this kind, restrictions on the accessibility relation derive from more primitive conditions on transition probabilities and the dynamic behavior of possible worlds. Whichever reconstruction one chooses there is an immediate gain in intuitive acceptability. Further, such an increase in the power of the informal motivation of the formalism is due, in large part, to the transition interpretation of the probability functions. Other interpretations are often employed, principally some version of the "degree of rational belief" view. On some accounts, however, explaining modal concepts in terms of doxastic ones does not genuinely increase our understanding of the former even if the probability formalism does support a belief interpretation. On the latter point, we note only that some writers dispute this claim and that the program which seeks to explicate necessity in terms of belief must pause to enter this debate.

Finally we examined the question of the relative strength of the two concepts "probability" and "accessibility". It was shown that if we employ access relations of a more general sort than those used in the standard modal semantics then transition probabilities can be defined. From these may be constructed classical probability and conditional probability functions. By using van Fraassen's construction one may even represent Popper probability functions.

As for the conceptual or formal priority of probability theory over frame theory, this must remain a matter of individual taste.

NOTES

1. See especially [11] for a convincing proof of this.
2. Our account is predicated on our philosophical conviction that physical determinism is false: from the fact that a physical system is in a certain state its future states cannot all be predicted with certainty. In response to the helpful comments of the referee, we also note that certain states which are logically possible must go unrecognized. Thus while it might be logically possible for the pointer on a meter to come to rest exactly at 2, on our approach such events cannot be physically meaningful, even on a very idealized view of measurement. It follows that we cannot admit such states into the codomain of $\underline{A}$, although we are perfectly comfortable with a state in which the pointer comes to rest in a certain (measurable) interval.
3. This, of course, assumes that we are dealing only with finite sets. If infinite sets are allowed then we must employ analytic methods.
4. See also [1] where "similarity spheres" are used instead of alternativeness relations in order to define probabilities.
5. See [7] and [8] where a generalization of this sort is used to analyze deontic logics which permit "conflict of obligation" and to characterize a connection between modal logics and inference relations.

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