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# Careful Choices—A Last Word on Borel Selectors

To the memory of C. D. Papakyriakopoulos

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*Selector theory* as surveyed in [13] and [14] deals with the following problem (instances of which arise in control theory, probability, mathematical economics, operator theory, etc.): We are given a *multifunction* $F$ between reasonable spaces $T$ and $X$ (a map assigning each $t \in T$ a nonempty $F(t) \subseteq X$) and seek an ordinary function $f$ from $T$ to $X$ with acceptable measurability properties constituting a *selector* for $F$ (satisfying $f(t) \in F(t)$ for all $t$). Of course, the Axiom of Choice says that a selector exists; but to get a measurable one, we need to impose hypotheses on $F$ and choose "carefully". The past few years have seen much progress (cf. [10], [11], [14]) on the Borel case of the selector problem. In this case we assume $X$ is a *Polish* topological space (one admitting a countable basis and a complete metric) and $T$ at least a *Suslin* space (homeomorph of an analytic subspace of a Polish space). Our goal is to find weak hypotheses on $F$ guaranteeing the existence of a *Borel-measurable* selector $f$ (one for which $f^{-1}[U]$ is Borel in $T$ whenever $U$ is open in $X$).

The present paper* shows that substantial improvements of existing results on the Borel selector problem can be achieved through application of ideas developed by Vaught in his prize-winning studies [12] on the model theory of infinitary logic. The precise statement of the result obtained is given in Section 3 below. Thanks to certain counterexamples, we can say that this result is in many ways "best possible". Selector theory is thus a relatively down-to-earth area of mathematics where methods from modern logical research can be fruitfully applied.

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*1 Tools* We assemble here useful facts from the literature.

**Set Theory**: $\omega$ will denote the set $\{0, 1, 2, \ldots\}$ of natural numbers. For $m, n \in \omega$, $\langle m, n\rangle$ denotes $2^m(2n + 1) - 1$. $\omega^{<\omega}$ is the set of finite sequences from $\omega$, including the empty one $\phi$; while $\omega^\omega$ is the set of infinite sequences, or in other words of functions from $\omega$ to itself. $\omega^\omega$ can be made a Polish space by giving it the topology having as basis the sets $W_s = \{\xi : \xi \text{ extends } s\}$ for $s \in \omega^{<\omega}$. For $\xi \in \omega^\omega$ and $n \in \omega$, $\xi|n$ denotes $(\xi(0), \xi(1), \ldots, \xi(n-1))$, while $(\xi)_n$ denotes the element of $\omega^\omega$ given by $(\xi)_n(m) = \xi(\langle m, n\rangle)$. $\Omega$ is the least uncountable ordinal.

**Operation** $\mathcal{A}$: For information on the classical fusion operation $\mathcal{A}$ of M. Suslin, and for topology in general, see [6]. Operation $\mathcal{A}$, it will be recalled, acts on a family $\underset{\sim}{M}(s)$ of sets indexed by elements $s \in \omega^{<\omega}$, and produces the set:

$$\mathcal{A}(\underset{\sim}{M}) = \bigcup_{\xi\in\omega^\omega} \bigcap_{n\in\omega} \underset{\sim}{M}(\xi|n).$$

We will require three facts from the classical theory of $\mathcal{A}$.

First ([6], Section 39 II), the analytic subsets of a Polish space may be characterized as those obtainable by $\mathcal{A}$ from Borel sets. We take "obtainable by $\mathcal{A}$ from Borel sets" as *defining* "analytic" for non-Polish spaces.

Second ([6], Section 3 XIV), $\mathcal{A}$ possesses an inductive analysis. For a given family $\underset{\sim}{M}$ and for $\alpha \leqslant \Omega$ we define inductively:

$$\underset{\sim}{M}_0(k_1, \ldots, k_n) = \bigcap_{m\leqslant n} \underset{\sim}{M}(k_1, \ldots, k_m)$$

$$\underset{\sim}{M}_{\beta+1}(k_1, \ldots, k_n) = \bigcup_{k\in\omega} \underset{\sim}{M}_\beta(k_1, \ldots, k_n, k)$$

$$\underset{\sim}{M}_\alpha(k_1, \ldots, k_n) = \bigcap_{\beta<\alpha} \underset{\sim}{M}_\beta(k_1, \ldots, k_n) \text{ at limits.}$$

Then $\mathcal{A}(\underset{\sim}{M}) = \underset{\sim}{M}_\Omega(\phi)$.

Third, a Borel subset of a Polish space can be represented in the form $\mathcal{A}(\underset{\sim}{M})$ with each $\underset{\sim}{M}(s)$ closed and each $\underset{\sim}{M}_\Omega(s)$ Borel. For the class of sets so representable trivially contains the closed sets, and can be shown to be stable under countable union and intersection. Indeed, if for each $i \in \omega$ we have such a representation $A_i = \mathcal{A}(\underset{\sim}{M}^i)$, then by taking $\underset{\sim}{M}(i, k_1, \ldots, k_n) = \underset{\sim}{M}^i(k_1, \ldots, k_n)$ we obtain such a representation for $\bigcup A_i$; while by taking $\underset{\sim}{M}(\xi|n) = \bigcap_i \underset{\sim}{M}((\xi)_i|j)$ where $j$ is greatest with $\langle j, i\rangle \leqslant n$, we obtain such a representation for $\bigcap A_i$.

**Operation** $\mathcal{G}$: The theory of the closed-game-operation $\mathcal{G}$ was developed by Moschovakis in a series of papers culminating in [9]. $\mathcal{G}$ acts on families $\underset{\sim}{N}$ indexed by elements of $\omega^{<\omega}$ of even length. $\mathcal{G}$, like $\mathcal{A}$, admits an inductive analysis, *which for present purposes will be taken as its definition:*

$$\underset{\sim}{N}_0(k_1, \ldots, k_n) = \underset{\sim}{N}(k_1, \ldots, k_n)$$
$$\underset{\sim}{N}_{\beta+1}(k_1, \ldots, k_n) = \bigcap_{i\in\omega} \bigcup_{j\in\omega} \underset{\sim}{N}_\beta(k_1, \ldots, k_n, i, j) \cap \underset{\sim}{N}_\beta(k_1, \ldots, k_n)$$

$$\underset{\sim}{N}_\alpha(k_1, \ldots, k_n) = \bigcap_{\beta<\alpha} \underset{\sim}{N}_\beta(k_1, \ldots, k_n) \text{ at limits}$$
$$\mathscr{B}(\underset{\sim}{N}) = \underset{\sim}{N}_\Omega(\phi).$$

One further fact will be required:

**1.1 Theorem** (Moschovakis) *Let $Z$ be a Polish space, $E \subseteq Z$ an analytic set, $\underset{\sim}{N}$ a suitably indexed family of Borel sets. If $E$ is disjoint from $\mathscr{B}(\underset{\sim}{N})$, then $E$ is disjoint from $\underset{\sim}{N}_\alpha(\phi)$ for some $\alpha < \Omega$.*

*Topologizing the Collection of Closed Subsets of a Polish Space:* Let $Z$ be a Polish space, $K(Z)$ the collection of its nonempty closed subsets. For open $U \subseteq Z$, let $U^+ = \{L \in K(Z): L \cap U \neq \phi\}$. The *local* topology on $K(Z)$ is that having as subbasis the sets $U^+$ for open $U \subseteq Z$. $2^Z$ denotes $K(Z)$ equipped with this topology. $2^Z$ is not Polish, but such theorems as 1.1 above still apply to it in virtue of the following result of E. G. Effros (for which see [2]):

**1.2 Theorem** (Effros) *There exists a Polish topology on the collection of nonempty closed subsets of a Polish space having the same Borel σ-field as the local topology.* (Indeed, if $\mathscr{W}$ is a countable basis for the Polish space, the topology having as subbasis the $U^+$ *and their complements* for $U \in \mathscr{W}$ is one such.)

**Category Transforms**: We use the modern terminology "meager, nonmeager, comeager" in place of "1st category, 2nd category, residual". A subset $A$ of our Polish space $Z$ has the Baire property if for some meager $H$ and open $U$ we have $(A - U) \cup (U - A) \subseteq H$. And $A$ will be called *regular* if $A \cap L$ has the Baire property with respect to the subspace topology on $L$ for all $L \in K(Z)$. The regular sets form a σ-field which, by a classical theorem ([6], Section 11 VII), contains the analytic sets. Below we tacitly assume all sets mentioned are regular. For such sets $A$ and for open $U$ we define two *transforms* (omitting to write $U$ when $U = Z$):

$$A^\# U = \{L \in U^+: A \cap U \cap L \text{ is nonmeager in } U \cap L\}$$
$$A^* U = \{L \in U^+: A \cap U \cap L \text{ is comeager in } U \cap L\}.$$

Vaught [12] has studied transforms so similar to our $^\#$ and $^*$ that his proofs apply to our situation without essential modification. Before summarizing his results, we need some machinery. Fix a countable basis $\mathscr{W}$ and a complete metric $\rho$ for $Z$, with $\phi \notin \mathscr{W}$ and $\rho\text{-diam}(Z) \leqslant 1$. Let $Z(\phi) = Z$, and if $s \in \omega^{<\omega}$ and $Z(s)$ is defined, let the $Z(s, m)$ for $m \in \omega$ enumerate all $U \in \mathscr{W}$ such that: (1) closure $U \subseteq Z(s)$, and (2) $\rho\text{-diam}(U) \leqslant \frac{1}{2}\ \rho\text{-diam}(Z(s))$. Thus for any $\xi \in \omega^\omega$ the intersection of the $Z(\xi|n)$ is a singleton. Denoting complementation in $Z$ and $K(Z)$ by $-$ we have:

**1.3 Theorem** (Vaught)

(a) $A^\# Z(s) = \bigcup_{m\in\omega} A^* Z(s, m)$

(b) $A \in K(Z) \rightarrow A^* Z(s) = -(Z(s) - A)^+$

(c) $\left(\bigcap_{n\in\omega} A_n\right)^* Z(s) = \bigcap_{n\in\omega} (A_n{}^* Z(s))$

(d) $\left(\bigcup_{n\epsilon\omega} A_n\right)^* Z(s) = Z(s)^+ \cap \bigcap_{i\epsilon\omega}\left[-Z(s,i)^+ \cup \bigcup_{n\epsilon\omega}\bigcup_{j\epsilon\omega}(A_n{}^*Z(s,i,j))\right]$

(e) $(\mathcal{A}(\underset{\sim}{M}))^*Z(s) = \not{b}(\underset{\sim}{N})$ *where* $\underset{\sim}{N}(i_1, \langle j_1, k_1\rangle, \ldots, i_n, \langle j_n, k_n\rangle) = -Z(s, i_1, j_1, \ldots, j_{n-1}, i_n)^+ \cup \underset{\sim}{M}_0(k_1, \ldots, k_n)^*Z(s, i_1, j_1, \ldots, i_n, j_n)$

(f) *$A$ Borel in $Z \to A^{\#}$, $A^*$ Borel in $2^Z$*

(g) *$A$ analytic in $Z \to A^{\#}$, $A^*$ analytic in $2^Z$.*

As Vaught remarks, a result like 1.3(f) was obtained in classical times by P. S. Novikov. A result like 1.3(g) was obtained independently of Vaught by Kechris [5].

*2 **Key lemmas*** We retain the notation of the preceding section.

**2.1 Proposition** *Let $C \subsetneq Z$ be co-analytic and $E \subsetneq 2^Z$ analytic. If $E \subsetneq C^{\#}$, then $E \subsetneq B^{\#}$ for some Borel $B \subsetneq C$.*

*Proof:* Let $A = Z - C$, and fix a representation $A = \mathcal{A}(\underset{\sim}{M})$ of $A$ as obtained by $\mathcal{A}$ from Borel sets. Apply 1.3(e) with $s = \phi$ to obtain a representation $A^* = \not{b}(\underset{\sim}{N})$. We claim that for $\alpha < \Omega$, $\underset{\sim}{M}_\alpha(\phi)^* = \underset{\sim}{N}_\alpha(\phi)$, and indeed more generally:

$$\underset{\sim}{N}_\alpha(i_1, \langle j_1, k_1\rangle, \ldots, i_n, \langle j_n, k_n\rangle) \cap Z(i_1, j_1, \ldots, i_n, j_n)^+ = \underset{\sim}{M}_\alpha(k_1, \ldots, k_n)^*Z(i_1, \ldots, j_n).$$

This identity (alluded to in [12], p. 276) is readily established by induction, using the definition of $\underset{\sim}{N}$ for $\alpha = 0$, 1.3(d) for $\alpha = \beta + 1$, and 1.3(c) at $\alpha$ a limit.

Now our assumption is that $\not{b}(\underset{\sim}{N}) = A^* = -C^{\#}$ is disjoint from the analytic set $E$. By 1.1 it follows $E$ is disjoint from some $\underset{\sim}{N}_\alpha(\phi) = \underset{\sim}{M}_\alpha(\phi)^* = -(-\underset{\sim}{M}_\alpha(\phi))^{\#}$, $\alpha < \Omega$. It suffices to set $B = -\underset{\sim}{M}_\alpha(\phi)$.

**2.2 Proposition** *Let $B \subsetneq Z$ be Borel. There exists a Borel-measurable $h: B^{\#} \to B$ satisfying $h(L) \in L$ for all $L \in B^{\#}$.*

*Proof:* Fix a representation $B = \mathcal{A}(\underset{\sim}{M})$ with each $\underset{\sim}{M}(s)$ closed and each $\underset{\sim}{M}_\Omega(s)$ Borel. Think of $\omega$ and $\omega^{<\omega}$ as countable discrete (and hence Polish) spaces. Then by 1.3(f) in the nice space $2^Z \times \omega^{<\omega} \times \omega^{<\omega}$ the set $C = \{(L, s, t): L \in \underset{\sim}{M}_\Omega(s)^*Z(t)\}$ is Borel. As auxiliaries to the definition of $h$ we define functions $f, g: C \to \omega$. Let $f(L, s, t)$ be the least $i$ with $L \in Z(t, i)^+$, and let $g(L, s, t)$ be the least $\langle j, k\rangle$ with $(L, (s, k), (t, i, j)) \in C$. (Such exists by 1.3(d) since $\underset{\sim}{M}_\Omega(s) = \bigcup_k \underset{\sim}{M}_\Omega(s, k)$.) Both $f$ and $g$ are Borel-measurable by 1.3(f).

Now given $L \in B^{\#}$, let $m$ be least with $L \in B^*Z(m)$. (Such exists by 1.3(a).) Set $s_0 = \phi$, $t_0 = (m)$. If $s_n, t_n$ have been defined, let $i_{n+1} = f(L, s_n, t_n)$, $\langle j_{n+1}, k_{n+1}\rangle = g(L, s_n, t_n)$ and set $s_{n+1} = (s_n, k_{n+1})$, $t_{n+1} = (t_n, i_{n+1}, j_{n+1})$. Finally, let $h(L)$ be the unique element of the intersection of the $Z(t_n)$. The function $h: B^{\#} \to Z$ is Borel-measurable since $f$ and $g$ are. Moreover, reviewing the construction we see that for each $s_n = (k_1, \ldots, k_n)$ and $t_n = (m, i_1, j_1, \ldots, i_n, j_n)$ we have $L \in \underset{\sim}{M}_\Omega(s_n)^*Z(t_n) \subsetneq \underset{\sim}{M}(s_n)^*Z(t_n)$. Then by 1.3(b) we conclude that $\phi \neq L \cap Z(t_n) \subsetneq \underset{\sim}{M}(s_n)$. It follows that $h(L) \in L$ and $h(L) \in \bigcap_n \underset{\sim}{M}(s_n) \subsetneq \mathcal{A}(\underset{\sim}{M}) = B$, as required to complete the proof.

The case $B = Z$ ($B^{\#} = 2^Z$) of 2.2 seems to be due to Christensen [2]; its utility for selector theory was pointed out by Dellacherie [3]. The utility of "Vaught Thought" was demonstrated by Sarbardhikari [10] and Miller [8].

*3 The theorem* Let $F$ be a multifunction from a Suslin space $T$ to a Polish space $X$. In order to obtain a Borel-measurable selector for $F$ there are three sorts of hypotheses we might impose:

1. *Values.* We might require each value $F(t)$ of $F$ to be a set of some special kind: closed, $K_\sigma$ (= $\sigma$-compact), $F_\sigma$-and-$G_\delta$, $G_\delta$, nonmeager, or relatively nonmeager. Here $A \subseteq X$ is relatively nonmeager (relatively comeager) if it is nonmeager (comeager) in the subspace topology on its closure. $G_\delta$ sets are relatively comeager by elementary topology.

2. *Measurability.* We might require $F$ to be *$\mathscr{N}$-measurable* for some $\sigma$-field $\mathscr{N}$ of subsets of $T$. This means that for each open $U \subseteq X$, the set $F^{-}[U] = \{t \in T\colon F(t) \cap U \neq \phi\}$ belongs to $\mathscr{N}$. In the cases where $\mathscr{N}$ is $\{T, \phi\}$, the Borel $\sigma$-field of $T$, or the $\sigma$-field generated by the analytic sets, we speak of *trivial, Borel,* and *analytic* measurability.

3. *Graph.* We might require the *graph* $\mathrm{Gr}(F) = \{(t, x) \in T \times X\colon x \in F(t)\}$ of $F$ to be, say, Borel or co-analytic in $T \times X$.

**3.1 Theorem** *Let $F$ be a multifunction from the Suslin space $T$ to the Polish space $X$ and assume that:* (a) *each value of $F$ is relatively nonmeager,* (b) *$F$ is Borel-measurable, and* (c) *the graph of $F$ is co-analytic. Then $F$ admits a Borel-measurable selector.*

*Proof:* Fix a Polish space $Y$ having $T$ as an analytic subspace. By (c) there is a co-analytic $C \subseteq Y \times X$ with $\mathrm{Gr}(F) = C \cap (T \times X)$.

Define $G\colon T \to 2^{Y\times X}$ by $G(t) = \{t\} \times$ closure $F(t)$, and let $E$ = range $G$. It follows from (b) that $G$ is Borel-measurable, and hence $E$ analytic.

Unpacking the definitions, we see from (a) that $E \subseteq C^{\#}$.

Apply 2.1 to obtain a Borel $B \subseteq C$ with $E \subseteq B^{\#}$, and 2.2 to obtain a Borel-measurable $h\colon B^{\#} \to B$ satisfying $h(L) \in L$. Unpacking the definitions we see that for each $t \in T$, $h(G(t))$ is an element of $\mathrm{Gr}(F)$ of form $(t, x)$. Letting $f(t)$ be this $x$, we obtain a Borel-measurable selector.

The following answers a question of D. H. Wagner:

**3.2 Corollary** *Let $F$ be a multifunction from the Suslin space $T$ to the Polish space $X$ and assume that:* (a′) *each value of $F$ is relatively comeager,* (b) *$F$ is Borel-measurable, and* (c) *the graph of $F$ is co-analytic. Then there exist Borel-measurable selectors $f_n$, $n \in \omega$, for $F$ such that for each $t \in T$ the set $\{f_n(t)\colon n \in \omega\}$ is dense in $F(t)$.*

*Proof:* Fix a countable basis $U_i$, $i \in \omega$, for $X$. Since each $F^{-}[U_i]$ is Borel, we can define a Borel-measurable $g\colon \omega \times T \to \omega$ by $g(n, t)$ = the $n^{\text{th}}$ $i \in \omega$ with $t \in F^{-}[U_i]$. For each $i$ the multifunction $F_i$ from $F^{-}[U_i]$ to $X$ given by $F_i(t) = F(t) \cap U_i$ satisfies the hypotheses of 3.1 and so admits a Borel-measurable selector $h_i$. It suffices to set $f_n(t) = h_{g(n,t)}(t)$.

The table compares Theorem 3.1 above, which we shall call (0), with other selection results. (0) was immediately inspired by: (1) an unpublished result of G. Debs (reported in [14], Section 6), and (2) Srivastava's Theorem 4.2 in [11], and Sarbardhikari's Theorem 2 in [10]. Since writing the bulk of this

Table

| Theorem | Value Hypothesis | Measurability Hypothesis | Graph Hypothesis |
|---|---|---|---|
| (0) | relatively nonmeager | Borel | co-analytic |
| (1) | $F_\sigma$-and-$G_\delta$ | Borel | co-analytic |
| (2) | $G_\delta$ | Borel | Borel |
| (3) | nonmeager | ------ | co-analytic |
| (4) | closed | Borel | ------------- |
| (5) | $K_\sigma$ | ------ | Borel |
| Counterexample | | | |
| (6) | open | trivial | analytic |
| (7) | closed | ------- | closed |
| (8) | $F_\sigma$ | trivial | $F_\sigma$ |
| (9) | countable | trivial | co-analytic |

paper the author came to learn of (3), a strengthening of Sarbardhikari's result due to D. Cenzer and R. D. Mauldin [1]. Whereas (1) and (2) are immediate from (0), the derivation of (3) requires a little work, and is given as Corollary 3.3 below. Also included for comparison is (4), the special case for Borel sets of the very general Fundamental Selection Theorem of K. Kuratowski and C. Ryll-Nardzewski/Ch. Castaing (cf. [13]). Note that though no graph hypothesis is made in (4), it follows almost trivially from the other hypotheses that the graph is in fact Borel. So (0) succeeds in unifying (1)-(4). The most important Borel selector result not subsumed under (0) an unpublished result of Srivastava, strengthening (5) an older theorem of Shchegolkov. Very recently A. Maitra (unpublished) has produced an "effective" result unifying (0) and the old P. G. Hinman/S. K. Thomason result that a nonmeager $\Pi^1_1$ lightface set of reals has a hyperarithmetic element. (For the latter result, see [5].)

3.3 **Corollary** (Cenzer and Mauldin) *Let F be a multifunction from the Suslin space T to the Polish space X and assume that:* (a) *each value of F is nonmeager, and (b) the graph of F is co-analytic. Then F admits a Borel-measurable selector f.*

*Proof:* Fix a Polish space $Y$ having $T$ as an analytic subspace, and a co-analytic $C \subseteq Y \times X$ with $\mathrm{Gr}(F) = C \cap (T \times X)$. Fix a countable basis $U_i$, $i \in \omega$, for $X$. Let $P_0 = Y - T$, $P_{i+1} = \{t \in T: \{x \in U_i: (t, x) \in C\}$ is comeager in $U_i\}$. By Vaught's work [12] (cf. 1.3(g)), the $P_i$ are all co-analytic. Moreover, $\bigcup_i P_i = Y$. So by the classical Reduction Principle (see [6]) there exist pairwise disjoint Borel sets $B_i$ with $B_i \subseteq P_i$ and $\bigcup_i B_i = Y$. Let $F_i$ be the multifunction from $T \cap B_{i+1}$ to $U_i$ given by $F_i(t) = F(t) \cap U_i$. $F_i$ is comeager-valued, hence trivially measurable, and so falls under 3.1 and admits a Borel-measurable selector $f_i$. The required $f$ can be obtained by combining the $f_i$.

The Table also indicates the properties of several counterexamples, multifunctions from $T = X = \omega^\omega$ to itself admitting no Borel-measurable selectors. (6) is due to V. V. Srivatsa (reported in [14], Section 6); (7) to Maitra [7]; (8) to Kallman and Mauldin [4]. (9) is our 3.4 below. Srivastava has an example (reported in [14], Section 6) to show that the conclusion of 3.2 above does not follow from the hypotheses of 3.1 above.

**3.4 Example**: There exists a multifunction $F$ from $\omega^\omega$ to itself each of whose values is a countable dense set, and whose graph is co-analytic, but which does not admit a Borel-measurable selector.

**Construction**: We give a bare outline, leaving details to the interested reader. Let $P$ be a universal co-analytic set. So $P \subseteq \omega^\omega \times \omega^\omega$ is co-analytic and for every co-analytic $E \subseteq \omega^\omega$ there is an $x$ with $E = \{y\colon (x, y) \in P\}$. Let $P^i = \{(x, y)\colon ((x)_i, y) \in P\}$, $C = \{x\colon \forall y \forall n \exists i ((x)_n, y) \in P^i\}$. By the Reduction Principle there exist pairwise disjoint co-analytic $Q^i \subseteq P^i$ with $\bigcup Q^i = \bigcup P^i$. Define $g\colon C \times \omega^\omega \to \omega^\omega$ by letting $(g(x, y))(n) = i$ iff $((x)_n, y) \in Q^i$. For any Borel-measurable $f\colon \omega^\omega \to \omega^\omega$ there is an $x$ with $f = g(x, \cdot)$. Express the complement of the co-analytic set $C$ as the projection of a closed $B \subseteq \omega^\omega \times \omega^\omega$, and for $x \notin C$, let $h(x)$ be the lexicographically least $y$ with $(x, y) \in B$. Let $f(x) = g(x, x)$ for $x \in C$, and $= h(x)$ for $x \notin C$. Let $F(x)$ be the set of all elements of $\omega^\omega$ of form $(i_0, \ldots, i_{n-1}, 0, (f(x))(n+1) + 1, (f(x))(n+2) + 1, (f(x))(n+3) + 1, \ldots)$ for $(i_0, \ldots, i_{n-1}) \in \omega^{<\omega}$. Tedious but routine computations establish that $\mathrm{Gr}(F)$ is co-analytic. But for any Borel-measurable $e$ there is an $x \in C$ with $e(x) = g(x, x) \notin F(x)$.

The many papers we have cited contain more positive and negative results than have been quoted here. The present paper is *the author's* last word on the subject, but not *the* last word!

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