Notre Dame Journal of Formal Logic
Volume 22, Number 2, April 1981

Material Implication in Orthomodular (and Boolean) Lattices

GARY M. HARDEGREE

1 Introduction A number of mathematical structures have been investigated in the quantum logic approach to the foundations of quantum mechanics (see, e.g., [21]), but orthomodular lattices (OMLs) have received the majority of attention from logicians.* When interpreting such structures from the viewpoint of logic, it is customary to interpret the lattice elements as propositions, the meet operation as conjunction, the join operation as disjunction, and the orthocomplement operation as negation. It is also customary to interpret the partial order relation as the *relation* of implication.

In classical logic, if we regard propositions as sets of possible worlds, then we have the following analogs: The set of all subsets of the set W of possible worlds is a Boolean (ortho)lattice (BL), where meet (conjunction) is set-intersection, join (disjunction) is set-union, and orthocomplementation (negation) is set-complementation. On the other hand, the partial-order relation is the set-inclusion relation, which represents the *relation* of implication among propositions.

Implication in this sense is quite different from the logical connectives. For example, whereas the conjunction (intersection) of two propositions A and B is another proposition just like A and B, the inclusion of A in B is not; there is no set of worlds corresponding to "A implies B", in *this* sense of 'implies'.

There is an analogy in formal logic. Recall that a formula ϕ *semantically entails* a formula ψ just in case every interpretation that satisfies ϕ also satisfies ψ. If we regard an interpretation as assigning a proposition to each formula, then we can equivalently state this as follows: ϕ semantically entails ψ just in case for every interpretation i, $i(\phi)$ implies $i(\psi)$, or more concretely, $i(\phi)$ is

*This research was partially supported by NSF Grant #Soc76-82527.

Received June 6, 1979; revised June 30, 1980

included in $i(\psi)$. Thus, implication among propositions corresponds to semantic entailment among formulas.

Now, the statement "ϕ semantically entails ψ" is not a formula like ϕ or ψ, for whereas ϕ and ψ occur in the *object language*, "ϕ semantically entails ψ" occurs in the *metalanguage*. Since statements involving semantic entailment are in the formal mode of speech, we might call semantic entailment a *formal implication*, and by analogy we might call the partial order relation (set-inclusion) among propositions a *formal implication*.

As is well-known, implicational concepts are often expressed in the material, as well as in the formal, mode of speech. Accordingly, we have a notion of *material implication*, or a notion of a *conditional* (if-then) connective. Mathematically, formal implication is represented by a two-place lattice *relation* (the partial order or set-inclusion relation), and material implication is represented by a two-place lattice *operation*, which we generically denote $\rightarrow$. From the viewpoint of formal logic, "ϕ semantically entails ψ" is a statement in the metalanguage, but $\phi \rightarrow \psi$ is a formula in the object language just like ϕ and ψ. From a metaphysical viewpoint "A implies B" ("$A \leqslant B$" or "$A \subseteq B$") is a statement about the propositions (lattice elements) A and B, and is not a proposition itself, while $A \rightarrow B$ is a proposition (lattice element) exactly on a par with A and B. If we combine these ideas via formal semantics, we have the following general principle: if i is an interpretation, and $i(\phi) = A$, and $i(\psi) = B$, then $i(\phi \rightarrow \psi) = A \rightarrow B$.

The best-known example of a material implication is the horseshoe ($\supset$) of classical logic, which is defined so that $A \supset B$ is equivalent to $\sim A \vee B$. The horseshoe is *a* material implication according to our criteria, since $\phi \supset \psi$ is a formula at exactly the same linguistic level as ϕ and ψ, but contrary to common usage it is not *the* material implication, according to our usage, since there are numerous conditional connectives, each of which provides an analysis of "if-then" or implication in its material mode. Although the horseshoe has many characteristics that make it satisfactory, it has many other characteristics that make it unsatisfactory as a material implication. Dissatisfaction with the horseshoe has led to the investigation of a large variety of alternative implications; examples include intuitionistic implication, the strict implications of C. I. Lewis [25], the relevant implications of Anderson and Belnap [1], and the counterfactual implications of Stalnaker [31] and D. Lewis [26] (among others).

This article is concerned with two questions:

1. Are there *any* plausible material implications definable in the context of orthomodular lattices?[1]

2. Supposing that the first question is answered affirmatively, is there a *privileged* material implication that plays a role in orthomodular-based logics analogous to the role of the horseshoe in Boolean-based logics?

In regard to the first question, I propose a number of criteria by which to judge the adequacy of a binary (ortho)lattice operation as a material implication (or conditional) connective. In Section 2, four criteria are proposed as absolutely minimal; in Section 3, a fifth criterion is proposed as extremely plausible. It is shown that there is an abundance of binary operations satisfying these five criteria, both in the orthomodular and in the Boolean context.

In regard to the second question, in Section 4, I show that the connective known as the Sasaki arrow[2] plays a role in orthomodular logics exactly parallel to the role played by the horseshoe in Boolean logics. In particular, I show that: (a) the Sasaki arrow is the only OML operation satisfying the proposed implicative criteria that agrees with the horseshoe for compatible pairs of elements, (b) the Sasaki arrow (horseshoe) is the least strict material implication definable on an OML (BL), and (c) every material implication definable on an OML (BL) can be defined on the basis of the Sasaki arrow (horseshoe) and a family of necessity operators. When this family has just one element, we obtain a strict implication (å la C. I. Lewis); when this family has more than one element, we obtain a variably strict implication (à la D. Lewis). Because of (b) and (c), we see that the Sasaki arrow and horseshoe play a privileged role in their respective classes of logics.

In order to render the article more nearly self-contained, an appendix on OMLs is included.

*2 **Basic implicative criteria*** As noted in the introduction, in the logical interpretation of OMLs, meet is interpreted as conjunction, join as disjunction, and orthocomplement as negation. Still to be considered is what OML operations might count as the lattice counterparts of the various material implications, or conditionals.

Since not just any binary lattice operation should qualify as a material implication, we must determine what criteria must (should) be satisfied by a lattice operation in order to be regarded as a material implication.

First, it seems plausible to require that every implication operation $\rightarrow$ be related to the implication relation (set-inclusion) in such a way that if a proposition A implies (is included in) a proposition B, then the conditional proposition $A \rightarrow B$ is universally true, and conversely. This criterion may be stated as follows, where W is the class of all possible worlds:

(e) $A \subseteq B$ iff $A \rightarrow B = W$.

Translating this into the more general lattice context, we obtain:

(E) $a \leqslant b$ iff $a \rightarrow b = 1$.

Here 1 is the lattice unit element, which corresponds to the universally true proposition.

Next, it seems plausible to require every implication connective $\rightarrow$ to satisfy the law of modus ponens: A conjoined with $A \rightarrow B$ implies B.

(mp) $A \cap (A \rightarrow B) \subseteq B$.

In classical logic, where the negation of a proposition A is simply its set-complement, $-A$, (mp) may be equivalently stated as follows:

(mt) $-B \cap (A \rightarrow B) \subseteq -A$

(ng) $A \cap -B \subseteq -(A \rightarrow B)$.

Although (mt), the law of modus tollens, and (ng), the law of negation, are equivalent to (mp) in the classical context, they are not equivalent in general. It accordingly seems plausible that we officially add (mt) and (ng) to our list of implicative criteria. Translating these conditions into lattice-theoretic notation, we obtain:

(MP) $a \wedge (a \to b) \leqslant b$
(MT) $b^{\perp} \wedge (a \to b) \leqslant a^{\perp}$
(NG) $a \wedge b^{\perp} \leqslant (a \to b)^{\perp}$.

Here, $^{\perp}$ is the lattice operation corresponding to negation, which in the particular case we are considering is the orthocomplement operation.

We refer to the above four conditions, (E), (MP), (MT), (NG) as the *minimal implicative conditions* (criteria). Whereas (E) and (MP) should be satisfied by any implication operation on a bounded lattice, (MT) and (NG) should additionally be satisfied on any lattice-with-negation. In calling these four conditions the minimal implicative conditions, we do not mean to rule out further implicative criteria as being minimal (for example, in Section 3, we consider an additional implicative criterion that we propose as minimal).

In addition to our officially so-called minimal implicative criteria, there are other restrictions one might wish to impose on implication operations. For example, a very powerful implicative principle, the law of *importation-exportation,* may be stated lattice-theoretically as:[3]

(IE) $a \wedge b \leqslant c$ iff $a \leqslant b \to c$.

Although both the classical[4] and intuitionistic conditionals satisfy (IE), no arrow operation satisfying (IE) can be defined on any non-Boolean OML, since a lattice admits an arrow operation satisfying (IE) only if it is distributive.[5] On the other hand, (IE) does not seem to be a plausible candidate as a minimal implicative criterion, since it is satisfied by neither strict nor counterfactual conditionals. As we see later, however, there is a natural weakening of (IE) that is plausible to require every implication operation to satisfy.

Next, we consider the law of *transitivity,* which may be stated lattice-theoretically as:

(T) $(a \to b) \wedge (b \to c) \leqslant a \to c$.

As an immediate consequence of (T), by substituting $a \wedge b$ for a, noting that $(a \wedge b) \to b = 1$ by (E), we obtain the following law of *weakening*:

(W) $b \to c \leqslant (a \wedge b) \to c$.

Although (T) and (W) are plausible implicative criteria, they are laws that *counterfactual* conditionals should definitely *not* satisfy, since they entail the validity of the following (invalid) argument (cf. [26]):

(A) if I *were* to drop this glass, then it *would* break; therefore, if I *were* to drop this glass *and* it *were* shatterproof, then it *would* break.

Since we accept counterfactual conditionals as legitimate implication connectives, we cannot accept (T) or (W) as *minimal* implicative criteria. The same holds for the law of *contraposition,* which is another plausible implicative criterion violated by counterfactual conditionals (cf. [26]):

(CN) $a \to b = b^{\perp} \to a^{\perp}$.

Observe that every ortholattice admits at least one implication operation satisfying (T) in addition to the minimal implicative conditions, namely the ultrastrict implication $\Rightarrow$, defined as:

(US) $a \Rightarrow b = 1$ if $a \leqslant b$; otherwise, $a \Rightarrow b = 0$.

There are two notions of strictness that apply to material implications. First, we say that one arrow operation $\Rightarrow$ on a lattice L is *stricter than* another arrow operation $\to$ on L if for all $a, b \in L$, $a \Rightarrow b \leqslant a \to b$. It is evident that the "stricter than" relation among material implications on a given lattice is a partial order relation. It is also evident that the ultrastrict implication is the least element in the ordering; that is, $\Rightarrow$ is the *strictest* material implication.[6] Whether there is a greatest element (least strict implication) in this ordering is not obvious in the most general case. However, as we see in Section 4, in the special case of OMLs and BLs, if we impose one additional plausible implicative criterion, then there is a least strict material implication as well.

In addition to the strictness-order relation, there is a qualitative notion of strictness; (qualitatively) strict implications traditionally arise in *modal logic.* In classical modal logic, either necessity (or possibility) or strict implication can be taken as primitive, the relation between them being given as:

(m1) $a \prec b = N(a \supset b)$
(m2) $N(a) = 1 \prec a$
(m3) $N(a) = a^{\perp} \prec 0$.

Here, N is the necessity operator (see Section 3), and $\prec$ is the strict implication operation.

There are general implicative principles at work in the above definitions. Given any material implication $\to$, we have the following as a result of (MP) and (MT):

(i1) $1 \to a \leqslant a$
(i2) $a \to 0 \leqslant a^{\perp}$.

Therefore, given any material implication $\to$, we can define two (quasi)modal operators:

(N) $N(a) =_{df} 1 \to a$
(I) $I(a) =_{df} a \to 0$.

Whether N and I are interdefinable is not determined a priori. For example, if $\to$ satisfies contraposition (CN), then N and I are interdefinable in the usual manner: $I(a) = N(a^{\perp})$; $N(a) = I(a^{\perp})$. However, in the case of certain counterfactual conditionals, I is a nontrivial modal operator, but N is simply the identity function. Thus, in the most general case, N and I are independent modal operators.

By virtue of (N), (I), (i1), and (i2), any material implication induces a pair of (quasi)modal operators N and I. If *either* of these operators is nontrivial, we say that the implication is a *strict implication* (in the qualitative sense); if both operators are trivial, then we say that the implication is *nonstrict.* Stating matters formally, we say that an arrow operation $\rightarrow$ on an ortholattice L is *nonstrict* if it satisfies the following conditions; otherwise, we say that it is *strict*:

(NS1) $1 \rightarrow a = a$
(NS2) $a \rightarrow 0 = a^{\perp}$.

We refer to (NS1) and (NS2) collectively as the *nonstrictness condition,* which is denoted (NS). Whereas the strict implications of modal logic, and the counterfactual implications, are strict implications in this sense, both the classical and intuitionistic implications are nonstrict.

At this point it is interesting to compare (NS) with the transitivity condition (T). Their relation may be summed up in the following theorem, which states that no material implication satisfying *both* (NS) and (T) can be defined on a *non-Boolean* OML.

Theorem 1 *Let L be an OML, and let $\rightarrow$ be a binary operation on L satisfying (E), (MP), (T), and (NS). Then L is Boolean.*

Proof: By (T), $(a \rightarrow 1) \wedge (1 \rightarrow b) \leqslant a \rightarrow b$, and $(a \rightarrow 0) \wedge (0 \rightarrow b) \leqslant a \rightarrow b$. By (NS), $1 \rightarrow b = b$, and $a \rightarrow 0 = a^{\perp}$. By (E), $a \rightarrow 1 = 1$, and $0 \rightarrow b = b$. Therefore, $b \leqslant a \rightarrow b$, and $a^{\perp} \leqslant a \rightarrow b$, from which it follows that $a^{\perp} \vee b \leqslant a \rightarrow b$. From this it follows that $a \wedge (a^{\perp} \vee b) \leqslant a \wedge (a \rightarrow b)$, but by (MP) $a \wedge (a \rightarrow b) \leqslant b$, so we have $a \wedge (a^{\perp} \vee b) \leqslant b$, for all $a, b \in L$. It is a theorem about OMLs that $a \wedge (a^{\perp} \vee b) \leqslant b$ iff aCb. We thus have that every pair of elements of L is compatible, from which it follows that L is Boolean.

A condition that subsumes (NS) requires that the arrow operation be defined in terms of the standard ortholattice operations, meet, join, and orthocomplement, which is to say that the arrow operation is an ortholattice polynomial. Formally stated, we say that a material implication is a *polynomial conditional* if it satisfies the following condition:

(P) $a \rightarrow b = p(a, b)$, for some two-place ortholattice polynomial p.

In the case of Boolean lattices, the horseshoe is the only polynomial function that satisfies the minimal implicative criteria. In particular, the Boolean ortholattice freely generated by two elements has 16 elements, so there are exactly 16 distinct two-place Boolean polynomials. Of these, only the horseshoe satisfies (E) and (M).

In the case of orthomodular lattices, we must consider the OML freely generated by two elements, which has 96 elements, being isomorphic to the direct product of the 16-element Boolean lattice and the 6-element OML. This OML turns out to be identical to the modular ortholattice freely generated by two elements, so we can appeal to the work of Kotas [23] concerning polynomial conditionals on modular ortholattices.

To begin with, the polynomial elements formed out of two OML elements that are compatible form a Boolean subOML. Therefore, since the horseshoe is the only polynomial conditional on a Boolean lattice, we have the following restriction on OML polynomial conditionals:

(LB) if aCb, then $a \to b = a^\perp \vee b$.

Kotas shows (in effect) that there are exactly six OML polynomials satisfying (LB), defined as:

(c1) $c_1(a, b) =_{df} a^\perp \vee (a \wedge b)$
(c2) $c_2(a, b) =_{df} (a^\perp \wedge b^\perp) \vee b$
(c3) $c_3(a, b) =_{df} (a \wedge b) \vee (a^\perp \wedge b) \vee (a^\perp \wedge b^\perp)$
(c4) $c_4(a, b) =_{df} (a \wedge b) \vee (a^\perp \wedge b) \vee ((a^\perp \vee b) \wedge b^\perp)$
(c5) $c_5(a, b) =_{df} (a \wedge (a^\perp \vee b)) \vee (a^\perp \wedge b) \vee (a^\perp \wedge b^\perp)$
(c6) $c_6(a, b) =_{df} a^\perp \vee b.$

It is routine to verify the following facts:

(F1) c_i satisfies (LB) for $i = 1\text{-}6$
(F2) c_i satisfies (NG) for $i = 1\text{-}6$
(F3) c_i satisfies (E) for $i = 1\text{-}5$
(F4) c_i satisfies (MP) for $i = 1\text{-}4$
(F5) c_i satisfies (MT) for $i = 1\text{-}3, 5$.

Thus, whereas the horseshoe c_6 satisfies only one of the minimal implicative criteria, there are three polynomials, c_1, c_2, c_3, that satisfy all of them.

Since (NS) follows from (LB), we see that every Kotas conditional is *nonstrict*, and so no Kotas conditional is transitive, by virtue of Theorem 1. Nevertheless, the corresponding *biconditionals* satisfy an analogous transitivity law. Given a conditional c_i, the associated biconditional b_i is defined as:

(BC) $b_i(a, b) =_{df} c_i(a, b) \wedge c_i(b, a).$

Concerning the associated Kotas biconditionals, the following facts may be routinely verified:

(F6) $b_i(a, b) = (a \wedge b) \vee (a^\perp \wedge b^\perp)$ for $i = 1\text{-}5$
(F7) $b_i(a, b) \wedge b_i(b, c) \leqslant b_i(a, c)$ for $i = 1\text{-}5$.

So in particular, although the three polynomial conditionals are different, they all give rise to the same biconditional, and this operation satisfies the corresponding transitivity law.

*3 **Residuation and implication*** In this section, we consider an additional plausible implicative criterion, which is a generalization of the classical importation-exportation law (IE). Recall that (IE) may be stated lattice-theoretically as:

(IE) $a \wedge b \leqslant c$ iff $a \leqslant b \to c$.

Allowing ourselves the benefit of higher-order conditions, as we did in the case of (P), we can generalize (IE) in the following way:

(R) $a + b \leqslant c$ iff $a \leqslant b \to c$, for some binary operation $+$.

If an arrow operation $\to$ satisfies (R), we say that it is *residual.*[7]

Let us briefly explain our terminology, which is borrowed from mathematics [2]. In the theory of partially ordered sets (posets), including lattices, a function (map) f from a poset P to a poset P^* is said to be *residuated* if the pre-image of every principal ideal on P^* is a principal ideal on P, and f is said to be *dually residuated,* or *residual,* if the pre-image of every principal filter on P^* is a principal filter on P. Residuated and residual maps come in pairs: for every residuated (resp., residual) map f from P into P^*, there is a residual (resp., residuated) map f^+ from P^* into P. This correspondence allows us to characterize residuated and residual maps as follows (note: for convenience, we specialize to the case in which $P = P^*$):

(R1) f is residuated iff: there is a function f^+ such that, for all $x, y \in L$, $f(x) \leqslant y$ iff $x \leqslant f^+(y)$.

(R2) f is residual iff: there is a function f^+ such that, for all $x, y \in L$, $x \leqslant f(y)$ iff $f^+(x) \leqslant y$.

To see how residuation relates to (R) and (IE), consider a lattice L and element $b \in L$, and define a map f_b from L into L as follows:

($\mathbf{f_b}$) $f_b(x) =_{df} x \wedge b.$

To say that f_b is residuated is to say the following holds, for some function f_b^+ on L, for all $a, c \in L$:

(R1*) $f_b(a) \leqslant c$ iff $a \leqslant f_b^+(c)$.

Writing "$a \wedge b$" in place of "$f_b(a)$", and writing "$b \rightarrow c$" in place of "$f_b^+(c)$", we obtain:

(IE) $a \wedge b \leqslant c$ iff $a \leqslant b \rightarrow c.$

Turning things around, given a lattice L, an arrow operation $\rightarrow$ on L, and an element $b \in L$, we can define the function g_b on L as:

($\mathbf{g_b}$) $g_b(x) =_{df} b \rightarrow x.$

To say that g_b is residual is to say the following holds for some function g_b^+ on L, for all $a, c \in L$:

(R2*) $a \leqslant g_b(c)$ iff $g_b^+(a) \leqslant c.$

Writing "$b \rightarrow c$" in place of "$g_b(c)$", and writing "$a + b$" in place of "$g_b^+(a)$", we obtain the converse of (R), less the quantifier expression:

(R*) $a \leqslant b \rightarrow c$ iff $a + b \leqslant c.$

Thus, (IE) may be understood as saying that the family (f_b: $b \in L$) of maps are all residuated, where (g_b: $b \in L$) are the corresponding residual maps (i.e., $f_b^+ = g_b$). Alternatively, (IE) may be understood as saying that the family (g_b: $b \in L$) of maps are all residual, where (f_b: $b \in L$) are the corresponding residuated maps (i.e., $g_b^+ = f_b$).

On the other hand, condition (R) generalizes (IE), saying that the family (g_b: $b \in L$) are all residual, without specifying the exact nature of the corresponding family (g_b^+: $b \in L$) of residuated maps. In short, a material implication $\rightarrow$ is residual (in the derivative sense) if and only if the family (g_b: $b \in L$)– where $g_b(x) = b \rightarrow x$–are all residual (in the strict sense).[8]

There is another very important class of examples of residual maps in classical logic. In classical modal logic, if we suppose that propositions are sets of worlds, and we suppose that the necessity operator N is definable in terms of an *accessibility relation* R on W, then the necessity operator may be defined as a one-place propositional operation, as follows:

(N1) $\quad w \in N(B)$ iff $w^* \in B$ for all w^* such that wRw^*.

Given the accessibility relation R on W, one can define a corresponding *accessibility function* r on the set of propositions:

(r) $\quad r(A) =_{df} \{w \in W\colon aRw$ for some $a \in A\}$.

Using (r), we can restate (N1) as:

(N2) $\quad A \subseteq N(B)$ iff $r(A) \subseteq B$.

Translating this into the lattice context, we obtain:

(N3) $\quad a \leqslant N(b)$ iff $r(a) \leqslant b$.

Comparing (N3) with (R1) and (R2), we see that N is residual, where N^+ is r, and r is residuated, where r^+ is N. It may be similarly shown that every classical possibility operator is residuated.

By a *normal* necessity operator, we mean one for which $N(a) \leqslant a$ for all $a \in L$. This is equivalent to requiring that R is reflexive, which is equivalent to requiring that $a \leqslant r(a)$ for all $a \in L$. Since we are exclusively concerned with *normal residual* necessity operators, we will simply refer to them as necessity operators.

As noted in Section 2, the strict implications of classical modal logic may be defined in terms of the horseshoe together with a (normal residual) necessity operator N. Since the horseshoe is residual, and since every classical necessity operator is residual, it follows that every classical strict implication is residual. More generally, let L be any (ortho)lattice, let $\rightarrow$ be any residual conditional on L, and let N be any (normal residual) necessity operator on L. Then the associated strict implication $⥽$, defined as follows, is also residual:[9]

($⥽$) $\quad a ⥽ b =_{df} N(a \rightarrow b)$.

We can actually state things much more generally. Let L be any (ortho)lattice, let $\rightarrow$ be any residual conditional on L, and let (N_a: $a \in L$) be any *family* of necessity operators on L, indexed by L. Then the "variably strict" implication $>$, defined as follows, is also residual:[9]

($>$) $\quad a > b =_{df} N_a(a \rightarrow b)$.

In the special case that $N_a = N_b$ for all $a, b \in L$, we have a nonvariably strict implication.

Our most general result has a converse. Specifically, given any residual implication $\rightarrow$, there is a corresponding family (N_a: $a \in L$) of necessity operators, where each N_a is defined as follows (see Section 4, Lemma 3):

($\mathrm{N_a}$) $\quad N_a(x) =_{df} x \wedge (a \rightarrow x)$.

Thus, every family of necessity operators induces a residual conditional, and every residual conditional induces a family of necessity operators. The correspondence, however, is not unique.

Variably strict implications have arisen in the investigation of counterfactual conditionals (cf. [26]). In the case of certain kinds of counterfactual conditionals, a more direct method may be used to show that they are residual. As van Fraassen [32] has shown (cf. [26]), the Stalnaker analysis and the Lewis analysis (accepting the Limit Assumption) can be subsumed under a common framework, which employs a special selection function f. Specifically, where w is a possible world, and A is a proposition (set of worlds), $f(w, A)$ is a set of worlds, which may (but need not) be thought of as the set of possible worlds in A that are "minimally distant" from w.[10]

Various restrictions may be placed on f, by which one obtains various sorts of counterfactual conditionals. However, from our viewpoint, the only relevant feature of f is the way it is utilized in characterizing the associated conditional, which we generically denote $>$. Given a selection function, f, one may characterize the associated conditional $>$ as:[11]

(CC1) $\quad a \in B > C$ iff $f(a, B) \subseteq C$.

Associated with f is a corresponding image function, also denoted f, defined as:

(f) $\quad f(A, B) =_{df} \{w \in W\colon w \in f(a, B)$ for some $a \in A\}$.

This allows us to restate (CC1) as:

(CC2) $\quad A \subseteq B > C$ iff $f(A, B) \subseteq C$.

Restating this in lattice-theoretic terms, we obtain:

(CC3) $\quad a \leqslant b > c$ iff $f(a, b) \leqslant c$.

So $>$ is residual, where $a + b = f(a, b)$. Thus, we see that any counterfactual conditional that is characterizable by reference to a selection function, as in (CC1), is residual.

In addition to the abovementioned connectives, the arrow connective of relevant logic R [5] is also residual. The + operation is defined as:

(r+) $\quad a + b =_{df} \neg(b \rightarrow \neg a)$.

Here, "$a + b$" is read "a is cotenable with b", and $\neg$ is the negation of relevant logic, which incidentally is *not* an orthocomplementation.

Having shown that many implication connectives are residual, we now ask whether there are any residual conditionals definable on general OMLs. The answer is affirmative; indeed, there is an abundance of such operations.

To begin with, as is generally well-known, the family (s_b: $b \in L$) of *Sasaki projections* on an OML L are all residuated. Each Sasaki projection s_b is defined as follows [7, 8]:

($\mathrm{s_b}$) $\quad s_b(x) =_{df} (x \vee b^{\perp}) \wedge b$.

Note that when b and x are compatible, $s_b(x) = f_b(x) = x \wedge b$. To say that s_b is residuated is to say that the following holds for some arrow operation $\rightarrow$:

(R3) $(a \vee b^\perp) \wedge b \leqslant c$ iff $a \leqslant b \rightarrow c$.

Now, condition (R3) uniquely specifies an arrow operation, namely, the *Sasaki arrow*, which is identical to polynomial c_1 from Section 2:

(SA) $b \rightarrow_s c =_{df} b^\perp \vee (b \wedge c)$.

Also, as a result of (R3), the Sasaki arrow is residual, where the associated family of residuated maps is the family of Sasaki projections; in particular, $a + b = (a \vee b^\perp) \wedge b$. Notice that $a + b = (b \rightarrow_s a^\perp)^\perp$, which is analogous to classical logic $[(b \supset a^\perp)^\perp = a \wedge b]$, as well as relevant logic [see (r+)].

As noted earlier, given any residual conditional $\rightarrow$ on a lattice L, and given any (normal residual) necessity operator N on L, the corresponding strict implication $\prec$, defined so that $a \prec b = N(a \rightarrow b)$, is also a residual conditional. We accordingly have a residual conditional on an OML L for each necessity operator on L. For example, if we define N as:[12]

(N) $N(a) = 1$ if $a = 1$; otherwise, $N(a) = 0$,

then we obtain the ultrastrict implication on L:

(US) $a \Rightarrow b = 1$ if $a \leqslant b$; otherwise, $a \Rightarrow b = 0$.

There are other special cases worth considering. If N is a residual *interior operator,* that is, a necessity operator satisfying the following additional restriction:

(In) $N(a) \leqslant N(N(a))$,

then we obtain the class of *S4* strict implications definable on L.[13] If N is a *symmetric* necessity operator, that is, a necessity operator satisfying the following additional restriction:[14]

(S) $r(a) = (N(a^\perp))^\perp$,

then we obtain the class of *B* strict implications definable on L. Finally, if N is a symmetric residual interior operator, then we obtain the class of *S5* strict implications definable on L. For example, the ultrastrict implication $\Rightarrow$ is an *S5* implication.

We also noted earlier that, given any residual conditional $\rightarrow$ on a lattice L, and given any family $(N_a\colon a \in L)$ of necessity operators on L, the corresponding variably strict implication $>$, defined so that $a > b = N_a(a \rightarrow b)$, is also a residual conditional on L. We accordingly have a residual conditional on an OML L for each family of necessity operators on L, definable on the basis of the Sasaki arrow.

There is at least one natural family of necessity operators on an OML L. For any $a \in L$, define N_a as:[15]

($\mathbf{N_a}$) $N_a(x) =_{df} (a \wedge x) \vee (a^\perp \wedge x)$.

It is routine to show that each N_a is a symmetric residual interior operator on L, and hence an $S5$ necessity operator. On the other hand, $N_a(x) = x$ iff aCx, and $aCa \to_s b$. It follows that the associated variably strict implication $>$ is identical to the Sasaki arrow. That the Sasaki arrow has a completely natural interpretation as a Stalnaker conditional in the context of conventional (Hilbert space) quantum logic is discussed elsewhere [10, 12, 13, 17].[16]

4 The fundamental character of the Sasaki arrow We have now considered five restrictions that are plausible to impose on any binary (ortho)lattice operation that is to be regarded as a material implication: the officially so-called minimal implicative conditions (E), (MP), (MT), (NG), and the residual condition (R). As we have seen, there are numerous operations, both in Boolean and in the orthomodular context, that satisfy all five restrictions. First, the Sasaki arrow (horseshoe) satisfies all five restrictions; second, if $(N_a\colon a \in L)$ is a family of (normal residual) necessity operators on L, then the variably strict implication $>$, defined so that $a > b = N_a(a \to_s b)$ $[= N_a(a \supset b)]$, also satisfies all five restrictions. In the special case that the family consists of exactly one necessity operator N, we have a (nonvariably) strict implication $⥽$, defined so that $a ⥽ b = N(a \to_s b)$ $(= N(a \supset b))$.

In the present section, we show that this representation is canonical: if $\to$ is an operation on an OML (BL) satisfying the five implicative restrictions, then there is a family $(N_a\colon a \in L)$ of necessity operators, not necessarily unique, such that $a \to b = N_a(a \to_s b)$ $[= N_a(a \supset b)]$. Toward this end we show that the Sasaki arrow (horseshoe) is the least strict material implication (satisfying the five restrictions) definable on an OML (BL). These two results strongly suggest that the role of the Sasaki arrow in orthomodular-based logics closely parallels the role of the horseshoe in Boolean-based logics.

Our first key lemma states that the Sasaki arrow (horseshoe) is the least strict operation definable on an OML (BL) satisfying the five implicative criteria. In order to prove this, we first prove an important lemma concerning residual conditionals.

Lemma 1 *Let L be a lattice, and let $\to$ be any binary operation on L satisfying (E), (MP), and (R). Then for all $a, b \in L$, $a \to (a \wedge b) = a \to b$.*

Proof: Assume the hypothesis, and let $+$ be the operation cited in (R). By (R), $(a \to b) + a \leqslant b$, since $a \to b \leqslant a \to b$. By (R) and (E), $(a \to b) + a \leqslant a$, since $a \to b \leqslant a \to a = 1$. Therefore, $(a \to b) + a \leqslant a \wedge b$, so by (R), $a \to b \leqslant a \to (a \wedge b)$. Similarly, $[a \to (a \wedge b)] + a \leqslant a \wedge b$, so $[a \to (a \wedge b)] + a \leqslant b$. Therefore, by (R), $a \to (a \wedge b) \leqslant a \to b$. Thus, $a \to (a \wedge b) = a \to b$.

Lemma 2 *Let L be any ortholattice, and let $\to$ be any operation on L satisfying (E), (MP), (NG), and (R). Then for all $a, b \in L$, $a \to b \leqslant a \to_s b$ $[= a^\perp \vee (a \wedge b)]$.*

Proof: Assume the hypothesis. Then by Lemma 1, $a \to (a \wedge b) = a \to b$. By (NG) $a \wedge c^\perp \leqslant (a \to c)^\perp$, so $a \to c \leqslant (a \wedge c^\perp)^\perp = a^\perp \vee c$, so in particular, $a \to (a \wedge b) \leqslant a^\perp \vee (a \wedge b)$. But $a \to (a \wedge b) = a \to b$, so $a \to b \leqslant a^\perp \vee (a \wedge b)$.

As noted in Section 3, every residual conditional $\rightarrow$ on a lattice L induces a family $(N_a : a \in L)$ of (normal residual) necessity operators on L, where for each $a \in L$, $N_a(x) =_{df} x \wedge (a \rightarrow x)$. We now officially state this result as Lemma 3.

Lemma 3 *Let L be any lattice, and let $\rightarrow$ be any operation on L satisfying (R). For each $a \in L$, define the function N_a on L as follows: $N_a(x) =_{df} x \wedge (a \rightarrow x)$. Then N_a is a normal residual necessity operator.*

Proof: Assume the hypothesis, and let + be the operation cited in (R). Then $x \leqslant N_a(y)$ iff $x \leqslant y \wedge (a \rightarrow y)$ iff $x \leqslant y$ and $x \leqslant a \rightarrow y$. But by (R), $x \leqslant a \rightarrow y$ iff $x + a \leqslant y$, so $x \leqslant N_a(y)$ iff $x \leqslant y$ and $x + a \leqslant y$ iff $x \vee (x + a) \leqslant y$. Thus, N_a is residual, where the associated residuated map N_a^+ is defined so that $N_a^+(x) = x \vee (x + a)$. That N_a is normal follows immediately from the definition.

With one more lemma, we can prove our major result, Theorem 2.

Lemma 4 *Let L be any OML (BL) and let $\rightarrow$ be any operation on L satisfying (E), (MP), and (R). Then $a \rightarrow b = a \rightarrow (a \rightarrow_s b)$ $[= a \rightarrow (a \supset b)]$.*

Proof: Assume the hypothesis, and let + be the operation cited in (R). Then (as in Lemma 1) $(a \rightarrow b) + a \leqslant a \wedge b$, but $a \wedge b \leqslant a^{\perp} \vee (a \wedge b) = a \rightarrow_s b$, so $(a \rightarrow b) + a \leqslant a \rightarrow_s b$. Therefore, by (R), $a \rightarrow b \leqslant a \rightarrow (a \rightarrow_s b)$. Similarly, $[a \rightarrow (a \rightarrow_s b)] + a \leqslant a \wedge (a \rightarrow_s b)$, but by (MP), $a \wedge (a \rightarrow_s b) \leqslant b$, so $[a \rightarrow (a \rightarrow_s b)] + a \leqslant b$. Therefore, by (R), $a \rightarrow (a \rightarrow_s b) \leqslant a \rightarrow b$. Thus, $a \rightarrow (a \rightarrow_s b) = a \rightarrow b$, and if L is Boolean, $a \rightarrow (a \supset b) = a \rightarrow b$.

Theorem 2 *Let L be any OML (BL), and let $\rightarrow$ be any operation on L satisfying (E), (MP), (NG), and (R). Then there is a family $(N_a : a \in L)$ of (normal residual) necessity operators on L such that for all $a, b \in L$: $a \rightarrow b = N_a(a \rightarrow_s b)$ $[= N_a(a \supset b)]$.*

Proof: Assume the hypothesis. Define N_a so that $N_a(x) =_{df} x \wedge (a \rightarrow x)$. By Lemma 3, $(N_a : a \in L)$ is a family of necessity operators on L. Therefore, $N_a(a \rightarrow_s b) =_{df} (a \rightarrow_s b) \wedge (a \rightarrow (a \rightarrow_s b)$, but by Lemma 4, $a \rightarrow (a \rightarrow_s b) = a \rightarrow b$, and by Lemma 2, $a \rightarrow b \leqslant a \rightarrow_s b$. It follows that $N_a(a \rightarrow_s b) = a \rightarrow b$. In the case L is Boolean, $N_a(a \supset b) = a \rightarrow b$.

In concluding this section, we add one more detail. As noted earlier, the the Sasaki arrow is "locally Boolean" in the sense that it agrees with the horseshoe for all compatible pairs:

(LB) $\quad a \rightarrow_s b = a \supset b = a^{\perp} \vee b$, if aCb.

There are numerous binary OML operations satisfying (LB), including for example, all the Kotas connectives (see Section 2). However, only one OML operation, the Sasaki arrow, satisfies (LB) in addition to the proposed implicative criteria. As we see in Lemma 2, if an arrow operation $\rightarrow$ is residual, then $a \rightarrow (a \wedge b) = a \rightarrow b$, for all $a, b \in L$. But $aCa \wedge b$, so by (LB), $a \rightarrow (a \wedge b) = a \supset (a \wedge b) = a^{\perp} \vee (a \wedge b)$. Thus, $a \rightarrow b = a^{\perp} \vee (a \wedge b)$. This result can be summarized by saying that the Sasaki arrow is the unique locally Boolean residual orthomodular conditional.

5 Concluding remarks We have argued that the Sasaki arrow plays a privileged role in orthomodular-based logics, parallel to the role of the horseshoe in Boolean-based logics. In particular, if we grant certain implicative criteria, then we can show that the Sasaki arrow is the least strict material implication, that every material implication can be defined in terms of the Sasaki arrow and a family of necessity operators. Although the Sasaki arrow plays a privileged role in orthomodular logics, it is by no means the *only* plausible orthomodular material implication. On the contrary, this investigation suggests that the variety of orthomodular material implications is at least as rich as the variety of Boolean material implications, including strict and variably strict (counterfactual) implications.

The present investigation suggests a number of lines of research: a more detailed investigation of the various strict and variably strict implications definable on OMLs; and the investigation of how the general results can be transferred to other non-Boolean logics, including intuitionistic logic (pseudo-Boolean lattices), and relevant logic (deMorgan lattices).

Appendix: Orthomodular lattices A *partially ordered set* (*poset*) is, by definition, a set A together with a binary relation $\leqslant$ on A satisfying the following restrictions for all $a, b, c \in A$:

(p1) $a \leqslant a$
(p2) if $a \leqslant b$, and $b \leqslant c$, then $a \leqslant c$
(p3) if $a \leqslant b$, and $b \leqslant a$, then $a = b$.

A poset $(A, \leqslant)$ is said to be a *lattice* if every pair of elements a, b has both a greatest lower bound (meet) $a \wedge b$, and a least upper bound (join) $a \vee b$, with respect to $\leqslant$. The following respectively characterize the meet and join of a and b:

(m) $x \leqslant a \wedge b$ iff $x \leqslant a$ and $x \leqslant b$
(j) $a \vee b \leqslant x$ iff $a \leqslant x$ and $b \leqslant x$.

A lattice may also be equationally characterized as a set A together with two binary operations, meet $\wedge$, and join $\vee$, satisfying the following restrictions for all $a, b, c \in A$:

(L1) $a \wedge b = b \wedge a$
(L2) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
(L3) $a \wedge (a \vee b) = a$
(L4) $a \vee b = b \vee a$
(L5) $a \vee (b \vee c) = (a \vee b) \vee c$
(L6) $a \vee (a \wedge b) = a$.

Given a structure satisfying (L1)-(L6), the associated partial order relation $\leqslant$ may be defined in either of the following ways:

(df1) $a \leqslant b$ iff$_{df}$ $a \wedge b = a$
(df2) $a \leqslant b$ iff$_{df}$ $a \vee b = b$.

Given a lattice L, a subset F of L is said to be a *filter* on L if it satisfies the following restrictions for all $a, b \in L$:

(f1) if $a \in F$, and $a \leqslant b$, then $b \in F$
(f2) if $a \in F$, and $b \in F$, then $a \wedge b \in F$.

A filter F is said to be a *principal filter* if there is an element $a \in L$ such that the following holds for every $x \in L$:

(pf) $x \in F$ iff $a \leqslant x$.

Note that this notion can also be applied to general posets; a subset F of a poset P is a principal filter on P if and only if there is an $a \in P$ satisfying (pf).

Given a lattice L, a subset I of L is said to be an *ideal* on L if it satisfies the following restrictions for all $a, b \in L$:

(i1) if $a \in I$, and $b \leqslant a$, then $b \in I$
(i2) if $a \in I$, and $b \in I$, then $a \vee b \in I$.

An ideal I is said to be a *principal ideal* if there is an element $a \in L$ such that the following holds for every $x \in L$:

(pi) $x \in I$ iff $x \leqslant a$.

The notion of principal ideal also applies to general posets.

A poset (lattice) P is said to be *bounded* if there are distinguished elements $0, 1 \in P$ satisfying the following restrictions for all $b \in P$:

(b1) $0 \leqslant b$
(b2) $b \leqslant 1$.

By virtue of (p3), 0 and 1 are unique.

Given a bounded lattice L, and an element $b \in L$, an element $c \in L$ is said to be a *complement* of b if the following conditions obtain:

(c1) $c \wedge b = 0$
(c2) $c \vee b = 1$.

A lattice L is said to be *complemented* if L is bounded and every element of L has at least one complement, and L is said to be *uniquely complemented* if L is bounded and every element of L has exactly one complement.

Given a complemented lattice L, an *orthocomplementation* on L is, by definition, any function o from L into L satisfying the following conditions for all $a, b \in L$:

(o1) $o(a)$ is a complement of a
(o2) $o(o(a)) = a$
(o3) if $a \leqslant b$, then $o(b) \leqslant o(a)$.

An *ortholattice* is, by definition, a complemented lattice L together with an orthocomplementation function o on L. Every ortholattice satisfies deMorgan's laws:

(dM1) $o(a \wedge b) = o(a) \vee o(b)$
(dM2) $o(a \vee b) = o(a) \wedge o(b)$.

Henceforth, we write "$x^{\perp}$" in place of "$o(x)$".

Given an ortholattice L, one can define an *orthogonality relation* $\perp$ on L as follows:

(df3) $a \perp b$ iff$_{df}$ $a \leqslant b^{\perp}$.

It is easy to show that $\perp$ satisfies the following:

(F1) $a \perp b$ iff $b \perp a$
(F2) $a \perp a$ only if $a = 0$.

One can also define a *compatibility relation* C on L as:

(df4) aCb iff$_{df}$ $a = (a \wedge b) \vee (a \wedge b^{\perp})$.

In spite of its name, the C relation is not automatically *symmetric* (aCb iff bCa). Indeed, C is symmetric on an ortholattice L if and only if L is an *orthomodular lattice.* We could use the symmetry of C as the definition of orthomodular lattices, but this is not the customary route.

Given a lattice L, a (nonordered) triple $\{a, b, c\}$ of elements of L is said to be a *distributive triple* if every ordered triple of its elements satisfies the following equations:

(d1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
(d2) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

A lattice L is said to be *distributive* if every triple of elements of L is distributive. A distributive complemented lattice is called a *Boolean lattice,* and a distributive ortholattice is called a *Boolean ortholattice.* Obviously every Boolean ortholattice is a Boolean lattice. On the other hand, every Boolean lattice induces a unique ortholattice, since every Boolean lattice is uniquely complemented. Thus Boolean lattices and Boolean ortholattices are coextensive.

Given a lattice L, an ordered pair (a, b) of elements of L is said to be *modular* if the following holds for all $c \in L$:

(m1) if $c \leqslant b$, then $\{a, b, c\}$ is distributive.

A lattice L is said to be *modular* if every ordered pair of elements of L is modular; alternately, L is modular if it satisfies either of the following conditions:

(m2) if $a \leqslant c$, then $a \vee (b \wedge c) = (a \vee b) \wedge c$
(m3) $a \vee (b \wedge (a \vee c)) = (a \vee b) \wedge (a \vee c)$.

An ortholattice L is said to be *orthomodular* if every orthogonal pair is modular: $a \perp b$ only if (a, b) is modular. Alternately, an orthomodular (ortho)lattice is an ortholattice L satisfying either of the following conditions for all $a, b \in L$:

(om1) if $a \leqslant b$, then $b = a \vee (a^{\perp} \wedge b)$
(om2) $a \vee (a^{\perp} \wedge (a \vee b)) = a \vee b$.

The key calculational theorem of orthomodular lattice theory is the Foulis-Holland Theorem [9, 20], which states that in order to apply the distributive laws to a triple of elements of an orthomodular lattice, it is sufficient that any one of the elements be compatible with the remaining two:

(FH) if aCb, and aCc, then $\{a, b, c\}$ is distributive.

This theorem is quite useful in connection with the following easily proved facts about compatibility:

(C1) aCa
(C2) aCb iff bCa
(C3) if $a \leqslant b$, then aCb
(C4) if $a \perp b$, then aCb
(C5) if aCb, then $aCb^{\perp}$
(C6) if aCb, and aCc, then $aCb \wedge c$
(C7) if aCb, and aCc, then $aCb \vee c$.

NOTES

1. Concern over this question was expressed as early as 1940 by Weyl [33]. More recent work on this subject includes: Clark [4], Finch [6], Hardegree [10]-[17], Herman, et al. [18], Herman and Piziak [19], Kalmbach [22], Kotas [23], Kunsemüller [24], Mittelstaedt [27], [28], Piziak [30], Zeman [34].

2. This term is borrowed from Herman, et al. [18], who refer to it as the "Sasaki hook"; the justification for this terminology is given in Section 3.

3. Note that (IE) amounts to saying that the conditional proposition $a \rightarrow b$ is the weakest proposition for which modus ponens holds: $a \rightarrow b = \sup\{x\colon a \wedge x \leqslant b\}$.

4. By classical conditional, I mean the horseshoe.

5. Let $r = (a \wedge c) \vee (b \wedge c)$. Then $a \wedge c \leqslant r$, so by (IE), $a \leqslant c \rightarrow r$. Also, $b \wedge c \leqslant r$, so by (IE), $b \leqslant c \rightarrow r$. Therefore, $a \vee b \leqslant c \rightarrow r$, so by (IE), $(a \vee b) \wedge c \leqslant r = (a \wedge c) \vee (b \wedge c)$. From this it follows that the lattice in question is distributive.

6. Let $\Rightarrow$ be the ultrastrict implication, and let $\rightarrow$ be any operation satisfying (E). Consider two elements a, b. There are two cases: $a \leqslant b$; in this case $a \rightarrow b = 1$, so $a \Rightarrow b \leqslant a \rightarrow b$; not $a \leqslant b$; in this case $a \Rightarrow b = 0$, so $a \Rightarrow b \leqslant a \rightarrow b$.

7. Note very carefully that the binary operation + *need not* be syntactically expressible in the object language. In particular, propositions a, b may correspond to formulas without $a + b$ corresponding to a formula in the object language. On the other hand, in many actual cases, + is in fact syntactically expressible. For example, in the case of the classical horseshoe, + corresponds to conjunction.

8. It might be useful to consider what residuation amounts to in the specialized (though common) case of a *complete lattice*—for example, the set $\mathcal{P}(W)$ of *all* subsets of a set W. A function f on a complete lattice L is residual if and only if it meets the following requirement for any subset $\{b_i\colon i \in I\}$ of L:

   (RC) $$f\left(\bigwedge_i [b_i]\right) = \bigwedge_i [f(b_i)].$$

   Accordingly, an arrow operation $\rightarrow$ is residual (in the derivative sense) if and only if it meets the following requirement for any element a of L and any subset $\{b_i\colon i \in I\}$ of L:

   (RC*) $$a \rightarrow \left[\bigwedge_i b_i\right] = \bigwedge_i [a \rightarrow b_i].$$

Thus, in the special case of complete lattices, the question whether a conditional is residual amounts to the question whether it distributes over *infinite conjunction.* Distribution over finite conjunction $[a \to (b \wedge c) = (a \to b) \wedge (a \to c)]$ is nonproblematic, and can be captured syntactically. Distribution over infinite conjunction cannot be captured syntactically (without an infinitary language), but it can be captured semantically (algebraically), provided the propositions (as opposed to formulas) are closed under infinite conjunction. If we regard every subset of possible worlds to be a proposition, then infinite propositional conjunction corresponds to infinite intersection. More generally, if the propositions form a complete lattice, then infinite conjunction corresponds to infinite infimum (meet).

9. Let $\to$ be residual, where + is the associated operation, and let $(N_a\colon a \in L)$ be a family of residual functions, where $(r_a\colon a \in L)$ are the associated residuated functions. Define $\Rightarrow$ so that $a \Rightarrow b =_{df} N_a(a \to b)$. Then $a \leqslant b \Rightarrow c$ iff $a \leqslant N_b(b \to c)$ iff $r_b(a) \leqslant b \to c$ iff $r_b(a) + b \leqslant c$. Thus, $\Rightarrow$ is residual, where the associated operation $+^*$ is defined so that $a +^* b =_{df} r_b(a) + b$.

10. Note very carefully that the selection function f is defined on *propositions, not formulas,* as in some treatments of counterfactual conditionals. The conditional is then defined as a binary operation on propositions, which is the algebraic counterpart of the syntactic conditional connective.

11. A more general treatment of selection function semantics for counterfactual conditionals is given by Nute in [29]. Note, however, that in his treatment the selection function is defined on formulas, rather than propositions. Nevertheless, it appears that this semantics can be algebraically reformulated, so that the selection function is defined on propositions, rather than formulas. In such a reformulation, (CC1) would be a central postulate.

12. $N^+(= r)$ is defined so that $r(x) = 0$ if $x = 0$, and otherwise $r(x) = 1$.

13. This might be compared with the class of "S4" strict implications investigated by Herman and Piziak [19]. They do not require the interior operators to be residual; in fact, they do not even require them to be topological interior operators. Whereas the former preserve all existing meets, the latter preserve only finite meets.

14. The term 'symmetric' is adapted from Blyth and Janowitz [2]. Note that symmetric necessity operators are associated with symmetric accessibility relations.

15. As noted in Lemma 3 (Section 4), each residual conditional $\to$ induces a family $(N_a\colon a \in L)$ of normal residual necessity operators, where $N_a(x) =_{df} x \wedge (a \to x)$. The family $(N_a\colon a \in L)$, where $N_a(x) =_{df} (a \wedge x) \vee (a^\perp \wedge x)$, is the family of necessity operators so induced by the Sasaki arrow.

16. The thesis that the Sasaki arrow may be interpreted as *a* Stalnaker conditional has been disputed by Bugajski [3]. In [17], I disarm his criticisms, uncovering a number of mistaken assumptions, and clarifying the precise nature of our disagreement.

REFERENCES

[1] Anderson, A. R., and N. D. Belnap, *Entailment: The Logic of Relevance and Necessity,* Princeton University Press, Princeton, New Jersey, 1975.

[2] Blyth, T. S., and M. F. Janowitz, *Residuation Theory,* Pergamon Press, Oxford, 1972.

[3] Bugajski, S., "Probability implication in the logics of classical and quantum mechanics," *Journal of Philosophical Logic,* vol. 7 (1978), pp. 95-106.

[4] Clark, I. D., "An axiomatization of quantum logic," *The Journal of Symbolic Logic,* vol. 38 (1973), pp. 389-392.

[5] Dunn, J. M., *The Algebra of Intensional Logics,* Doctoral Dissertation, Department of Philosophy, University of Pittsburgh, 1966.

[6] Finch, P. D., "Quantum logic as an implication algebra," *Bulletin of the Australian Mathematical Society,* vol. 2 (1970), pp. 101-106.

[7] Finch, P. D., "Sasaki projections on orthocomplemented posets," *Bulletin of the Australian Mathematical Society,* vol. 1 (1969), pp. 319-324.

[8] Foulis, D. J., "Baer *-semigroups," *Proceedings of the American Mathematical Society,* vol. 11 (1960), pp. 648-654.

[9] Foulis, D. J., "A note on orthomodular lattices," *Portugaliae Mathematica,* vol. 21 (1962), pp. 65-72.

[10] Hardegree, G. M., "The conditional in quantum logic," *Synthese,* vol. 29 (1974), pp. 63-80; reprinted in *Logic and Probability in Quantum Mechanics,* ed., P. A. Suppes, D. Reidel, Dordrecht, 1976, pp. 55-72.

[11] Hardegree, G. M., "Quasi-implicative lattices and the logic of quantum mechanics," *Zeitschrift für Naturforschung,* vol. 30a (1975), pp. 1347-1360.

[12] Hardegree, G. M., "Stalnaker conditionals and quantum logic," *Journal of Philosophical Logic,* vol. 4 (1975), pp. 399-421.

[13] Hardegree, G. M., "The conditional in abstract and concrete quantum logic," *The Logico-Algebraic Approach to Quantum Mechanics, Vol. 2,* ed., C. A. Hooker, D. Reidel, Dordrecht, 1979.

[14] Hardegree G. M., "Quasi-implication algebras; Part 1: Elementary theory, Part 2: Structure theory," *Algebra Universalis,* in press (1981).

[15] Hardegree, G. M., "An axiom system for orthomodular quantum logic," *Studia Logica,* in press (1981).

[16] Hardegree, G. M., "The positive and implicative fragments of orthomodular quantum logic," to appear.

[17] Hardegree, G. M., "Two views on quantum counterfactuals," to appear.

[18] Herman, L., E. Marsden, and R. Piziak, "Implication connectives in orthomodular lattices," *Notre Dame Journal of Formal Logic,* vol. 16 (1975), pp. 305-328.

[19] Herman, L. and R. Piziak, "Modal propositional logic on an orthomodular basis.I," *The Journal of Symbolic Logic,* vol. 39 (1974), pp. 478-488.

[20] Holland, S. S., "The current interest in orthomodular lattices," pp. 41-126 in *Trends in Lattice Theory,* ed., J. C. Abbott, Van Nostrand Reinhold, Co., New York, 1970.

[21] Hooker, C. A., ed., *The Logico-Algebraic Approach to Quantum Mechanics* (two volumes), D. Reidel, Dordrecht, 1975 and 1979.

[22] Kalmbach, G., "Orthomodular logic," *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, vol. 20 (1974), pp. 395-406.

[23] Kotas, J., "An axiom system for the modular logic," *Studia Logica,* vol. 21 (1967), pp. 17-38.

[24] Kunsemüller, H., "Zur axiomatik der quantenlogik," *Philosophia Naturalis,* vol. 8 (1964), pp. 363-376.

[25] Lewis, C. I. and C. H. Langford, *A Survey of Symbolic Logic,* Dover Publications, New York, 1932 (2nd ed., 1959).

[26] Lewis, D., *Counterfactuals,* Harvard University Press, Cambridge, Massachusetts, 1973.

[27] Mittelstaedt, P., "On the interpretation of the lattice of subspaces of Hilbert space as a propositional calculus," *Zeitschrift für Naturforschung,* vol. 27a (1972), pp. 1358-1362.

[28] Mittelstaedt, P., *Philosophical Problems of Modern Physics,* D. Reidel, Dordrecht, 1976.

[29] Nute, D., "Simplification and substitution of counterfactual antecedents," *Philosophia,* vol. 7 (1978), pp. 317-326.

[30] Piziak, R., "Orthomodular lattices as implication algebras," *Journal of Philosophical Logic,* vol. 3 (1974), pp. 413-418.

[31] Stalnaker, R. C., "A theory of conditionals," in *Studies in Logical Theory,* ed., N. Rescher, Blackwell, Oxford, 1968.

[32] van Fraassen, B. C., "Hidden variables in conditional logic," *Theoria,* vol. 40 (1974), pp. 176-190.

[33] Weyl, H., "The Ghost of Modality," pp. 278-303 in *Philosophical Essays in Memory of Edmund Husserl,* ed., M. Farber, Harvard University Press, Cambridge, Massachusetts, 1940.

[34] Zeman, J. J., "Quantum logic with implication," *Notre Dame Journal of Formal Logic,* vol. 20, no. 4 (1979), pp. 723-728.

Department of Philosophy
University of Massachusetts
Amherst, Massachusetts 01003