Semantic Paradox of Material Implication

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1   The classical paradoxes of material implication* are valid formulas of the truth-functional calculus which, like $p \rightarrow (q \rightarrow p)$ and $\sim p \rightarrow (p \rightarrow q)$ suggest that the material conditional is not an adequate rendering of the English "if . . . then . . .", nor perhaps of *any* sense of "implies".[1] The philosophical difficulty of assessing the significance of these formulas is aggravated by the formal fact that they all involve either the embedding of one conditional in another, or require the use of some further connective besides the conditional. The nesting of conditionals is a construction which is rare enough in natural languages that our intuitions about when such compounds are true are not reliable.[2] Where another connective is involved, it is clear that only the *joint* behavior of the conditional and the connective can be impugned. It is the purpose of this note to point out that the objectionable features of the truth-functional conditional are reflected in semantic features of the set of pure first-order conditionals (that is, sentences of the form $p \rightarrow q$ for primitive p and q), which involve no embedding or further connectives. In particular, any consistent assignment of truth values to those sentences determines the truth values of all of the primitive sentences. This is absurd, because no set of purely *hypothetical* facts should determine all of the *categorical* facts.

    Consider a language with primitive propositional variables p, q, r, etc., and whose sole connective is the conditional $\rightarrow$. We suppose that it is partitioned into two sets Ct and Cf, the first consisting of all first-order conditionals which are taken as true, and the second consisting of the rest, which are taken to be false. The question is whether such a partition of the set C of all sentences of the form $p \rightarrow q$ determines truth values for all the propositional variables according to a given interpretation of the conditional. If it does, we will say that the partition $\langle Ct, Cf \rangle$ *spans* the language.

We use the fact that a material conditional is false just in case its antecedent is true and its consequent false. This feature of the material conditional results from taking Frege's semantic principle that inference is warranted only if it does not lead from true premises to a false conclusion to express a *sufficient* as well as a necessary condition for appropriate inference. The counterintuitive consequence of such an interpretation is that every partition of the material conditionals which is nontrivial in the sense that *Cf* is nonempty turns out to span the language.[3] To see this, consider some conditional $p \rightarrow q$ in *Cf*. Now construct the set $T = \{r: p \rightarrow r \ \epsilon \ Ct\}$, and $F = \{r: p \rightarrow r \ \epsilon \ Cf\}$. Clearly T consists of true primitive propositions, and F consists of false ones. It also follows immediately that T union F includes all the propositional variables of the language, and thus that the partition $\langle Ct, \ Cf \rangle$ spans that language.

It is clear that modal versions of the material conditional, paradigmatically Lewis' strict implication, will not have this unfortunate property of determining the semantic values of the primitive propositions which appear as their antecedents and consequents. For example, if there are just two possible worlds, which are accessible to each other, and only two primitive propositions p and q, if $\Box(p \rightarrow q)$ is true in both worlds and $\Box(q \rightarrow p)$ is false in both, it may be *either* that while q is true in both worlds p is false in both *or* that p is false in one and true in the other. So knowing the truth values of all strict conditionals in all worlds generally won't enable one to settle the truth values of all the primitive propositions in any world. It is also obviously not the case that in multivalued logics in general knowing which conditionals take designated values (and even knowing *which* designated values) is sufficient for determining which primitive propositions take designated values, let alone what their multivalues are.

2   The previous argument adopted the unusual strategy of making assignments of truth values directly to classes of logically compound propositions, and considering the consequences according to standard rules for compatible assignments to the primitive propositions which are their components. One of the main reasons the reverse procedure is standard is that every partition of the primitive propositions can consistently be regarded as dividing that class into true propositions on the one hand, and false ones on the other. But not every partition of a set of logically compound propositions can consistently be treated as a division of that class (e.g., the first-order conditionals) into those which are true and those which are false. For instance, a partition of the material conditionals into classes A and B in which $p \rightarrow q$, $q \rightarrow p$, and $q \rightarrow r$ are members of A and $p \rightarrow r$ is a member of B is simply not consistent. That is, there is no way to assign truth values to p, q, and r so as to make all the elements of A true and those of B false or vice versa.

We simply assumed that the partitions of conditionals we dealt with in Section 1 were consistent in this sense, by specifying that one element of the partition was to consist of true conditionals, and the other of false ones. Were it inconsistent, the division would require that some primitive proposition be both true and false, and this in turn would entail that some conditional appear in both sets, contradicting the assumption that we were dealing with a partition. But can the assumption of a consistent partition of conditionals be

spelled out at the level of conditionals *without* assuming that we already know the truth values of all of the primitive propositions? If not, then we had already presupposed what we set out to derive, since the only way we could know that our partition of conditionals consistently divided them into a set of possible falsehoods would be to know what possible assignment of values to primitive propositions the partition had been derived from.

The following conditions for assignments of truth values to conditionals both ensure consistency and hold of every consistent partition:

Condition 1   *No proposition appears both as the antecedent of some conditional in Cf (the element of the partition which is to correspond to false conditionals) and as the consequent of some conditional in Cf.*

Condition 2   *If $p \to q$ is in Ct and $q \to r$ is in Ct, then $p \to r$ is in Ct.*

The necessity of these conditions is obvious. To show their sufficiency for a consistent partition, we proceed as follows. We may indicate the antecedent of a conditional c by $^{*}c$, and its consequent by c^{*}. Our earlier argument motivates these definitions: A *T-chain* of the partition $\langle Ct, Cf \rangle$ is a sequence of conditionals $c_1, c_2, \ldots, c_n$ such that

    i.   There is a c_0 in Cf such that $^{*}c_0 = {}^{*}c_1$
    ii. For all $n > 0$ $c_n^{*} = {}^{*}c_{n+1}$ and c_n is in Ct.

Clearly all the primitive propositions appearing in a *T*-chain are true. Similarly, an *F*-chain is a sequence such that

    iii. There is a c_0 in Cf such that $c_0^{*} = c_1^{*}$
    iv. For all $n > 0$ $^{*}c_n = c_{n+1}^{*}$ and c_n is in Ct.

Again, all the primitive propositions which appear as elements of conditionals in an *F*-chain must be false.

The proof is indirect. Suppose $\langle Ct, Cf \rangle$ is a partition satisfying Conditions 1 and 2 but which is *in*consistent. Then there must be some proposition p such that there are a *T*-chain and an *F*-chain in which p appears as an antecedent or a consequent. More specifically, there is some p such that

    a. There is some $c_0 = r \to s$ which is in Cf such that there are $c_1, \ldots, c_k$ all in Ct such that $^{*}c_1 = r$ and for all $n > 0$ $c_n^{*} = {}^{*}c_{n+1}$ and $c_k^{*} = p$.
    b. There is some $c_0' = r' \to s'$ such that there are $c_1', \ldots, c_m'$ in Ct such that $c_1'^{*} = s'$ and for all $n > 0$ $^{*}c_n' = c_{n+1}'^{*}$ and $^{*}c_m' = p$.

But now consider the conjoined sequence of conditionals $c_1, \ldots, c_k, c_m'$, $c_{m-1}', \ldots, c_1'$. All of these conditionals will be elements of Ct, by a and b above. Further, by construction that sequence of true conditionals begins with a conditional whose antecedent is r, ends with a conditional whose consequent is s', and for all adjacent intervening conditionals the antecedent of the second is the consequent of the first. It follows by the Transitivity Condition 2 that $r \to s'$ is in Ct. But since both $r \to s$ and $r' \to s'$ are in Cf, r must be true and s' must be false, and so $r \to s'$ will be in Cf as well. But this contradicts our assumption that $\langle Ct, Cf \rangle$ is a partition. Thus given Condition 1, we conclude that the set of primitive propositions appearing as antecedents or consequents

in conditionals in *T*-chains and the similar set associated with *F*-chains are disjoint. This suffices for the desired result: that the partition of conditionals is truth-functionally consistent. We showed in Section 1 that any partition of the first-order material conditionals of a language which could consistently be regarded as a division of them into true ones and false ones determines the truth values of all of the primitive propositions. We have now shown how to state this consistency condition on partitions of conditionals without in any way appealing to truth value assignments to primitive propositions.

NOTES

1. See for instance, Sections 1.1 and 5.1 of [1].

2. This point is argued for in Chapter 13 of [2].

3. This condition must be imposed since the trivial case in which all the first-order conditionals of the language are true is consistent both with the situation in which all the primitive propositions are true and with that in which they are all false.

REFERENCES

[1] Anderson, A. R. and N. D. Belnap, *Entailment,* Princeton University Press, Princeton, New Jersey, 1975.

[2] Dummet, M., *Frege: Philosophy of Language,* Harper and Row, New York, 1973.

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