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Consequences, Consistency, and Independence in Boolean Algebras

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Introduction In this paper we work within an arbitrary but fixed Boolean algebra $(B, +, \cdot, ', 0, 1)$ and with vectors $x = (x_1, \ldots, x_n) \in B^n, y = (y_1, \ldots, y_m) \in B^m$, where n and m are two arbitrary but fixed positive integers.* A Boolean function $f: B^n \to B$ is characterized by the Boole expansion theorem [1], [2][1]

$$(1) \qquad f(x_1, \ldots, x_n) = \sum_{(\alpha_1, \ldots, \alpha_n) \in \{0,1\}^n} f(\alpha_1, \ldots, \alpha_n) x_1^{\alpha_1} \ldots x_n^{\alpha_n} ,$$

where $\sum$ denotes iterated sum (disjunction) when the vector $(\alpha_1, \ldots, \alpha_n)$ runs over $\{0, 1\}^n$ and $x^0 = x'$, $x^1 = x$. If each $(\alpha_1, \ldots, \alpha_n) \in \{0, 1\}^n$ is interpreted as a number $i \in \{0, \ldots, 2^n - 1\}$ written in basis 2 and the corresponding minterm $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ is denoted by $m_i(x)$, formula (1) becomes

$$(2) \qquad f(x) = \sum_{i=0}^{2^n-1} f(i) m_i(x) .$$

In particular a Boolean function $r: B^{n+m} \to B$ admits the expansions

$$(3) \qquad \begin{aligned} r(x, y) &= \sum_{i=0}^{2^n-1} r(i, y) m_i(x) \\ &= \sum_{j=0}^{2^m-1} r(x, j) m_j(y) \\ &= \sum_{i=0}^{2^n-1} \sum_{j=0}^{2^m-1} r(i, j) m_i(x) m_j(y) \end{aligned}$$

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As is well known, every system of Boolean equations and/or inequalities in the variables x, y is equivalent to a single equation of the form

$$r(x, y) = 1 \quad . \tag{4}$$

We will refer to the Boolean function r as the *resolvent* of the system. For every $y \in B^m$, Equation 4 is consistent with respect to x if and only if

$$(e_x r)(y) = 1 \tag{5}$$

where the *eliminant* $e_x r$ is defined by

$$(e_x r)(y) = \sum_{i=0}^{2^n-1} r(i, y) \quad . \tag{6}$$

Poretski [20]-[23] has been concerned with the problem of determining all y-consequences of Equation 4. More precisely, a *consequence of* (4) *independent of* x, or briefly, a *y-consequence of* (4), is a Boolean equation $h(y) = 1$ such that

$$\forall x \forall y (r(x, y) = 1 \Rightarrow h(y) = 1) \quad . \tag{7}$$

By an abuse of terminology we will refer also to the Boolean function h as a *y-consequence* of Equation 4 or, simply, of the function r. Poretski made a remark which, in modern terminology, can be restated to the effect that the set of all y-consequences of r is the principal filter generated by the eliminant $e_x r$. As a matter of fact, there is a one-to-one correspondence between the y-consequences $h(y) = 1$ of $r(x, y) = 1$ and the Boolean functions h such that for all x and y, $r(x, y) \leqslant h(y)$. Thus not only is $(e_x r)(y) = 1$ *the least consequence of* $r(x, y) = 1$ *independent of* x, but $e_x r$ *is the least upper bound of* r *independent of* x and the mapping $r \mapsto e_x r$ is a *closure operator.* This last remark has turned out to be an efficient tool for solving many problems in Boolean algebra, as shown by Lapscher and his school ([10], [4], [5], [11], [3]). As a matter of fact, the mapping e_x is more than a closure operator: it is a *quantifier* in the sense of Halmos [8] (cf. [7], [10]). Propositions 1 and 2 below state in a more formal way a part of the results recalled in this paragraph; more details can be found in [24].

The aim of this paper is to study the important particular case when r is the resolvent of a system of equations of the form

$$y = f(x) \quad , \tag{8}$$

where $f = (f_1, \ldots, f_m)$ is a vector of Boolean functions $f_1, \ldots, f_m$ each from B^n to B. The y-consequences of (8) are the relations of functional dependence which exist among $f_1, \ldots, f_m$; Equation 5 characterizes in this case the range of the map f: $B^n \to B^m$ and we describe the eliminant $e_x r$ in terms of the functions $f_1, \ldots, f_m$ (Theorem 1). Thus in particular the consistency of the system (8) is equivalent to the functional independence of the functions $f_1, \ldots, f_m$ and also to the surjectivity of the map f; we give a direct characterization of this situation in terms of the functions $f_1, \ldots, f_m$ (Theorem 2). Moreover, we obtain the eliminant of a subsystem of $y = f(x)$ from the eliminant of the whole system (Theorem 3). Finally (Propositions 3, 4, and Theorems 4, 5), we construct a parallel theory in which we confine our attention to y-consequences h

which are *simple Boolean functions,* i.e., built up from variables and the basic operations $+, \cdot, '$ without making use of constants from B; or equivalently, Boolean functions h such that $h(i) \in \{0, 1\}$ for all $i = 0, \ldots, 2^n - 1$ (cf. [24] which lays stress on the distinction between Boolean functions and simple Boolean functions). To construct this theory, we introduce a new closure operator s (in fact, a quantifier) which associates with every Boolean function ψ the least simple Boolean function $s\psi$ which includes ψ (Lemmas 4 and 5) and we work with the closure operator se_x (Lemma 6) instead of e_x. The most significant result (Theorem 5) seems to be the fact that the role of functional independence in the former theory is played in the latter theory by the weaker independence introduced by Moore [18] as early as 1906,[2] which turns out to coincide with Marczewski's general concept of independence in abstract algebras applied to Boolean algebras (cf. [6], [14]-[17], and especially [15]).

1 We begin with an equivalent definition for y-consequences:

Lemma 1 *The relation* (7) *holds if and only if*

$$\forall y((\exists x\, r(x, y) = 1) \Rightarrow h(y) = 1) \ . \tag{9}$$

Proof: (7) $\Rightarrow$ (9): Take $y \in B^m$. If $r(x, y) = 1$ for that y and some $x \in B^n$, then $h(y) = 1$ by (7).
(9) $\Rightarrow$ (7): Take $y \in B^m$ and $x \in B^n$. If $r(x, y) = 1$, then $\exists x\, r(x, y) = 1$, hence $h(y) = 1$ by (9).

Thus condition (i) in Proposition 1 below is a stronger variant of definition (9) of y-consequences and, in fact, Proposition 1 suggests various ways of saying that e_x is the least y-consequence of r.

In the sequel, notations like $h = 1$ will always denote identities: for all y, $h(y) = 1$.

Proposition 1 *Suppose the Boolean equation* $r(x, y) = 1$ *is consistent with respect to* (x, y)*. Then the following conditions are equivalent for a Boolean function* $g: B^m \to B$:

(i) $\forall y((\exists x\, r(x, y) = 1) \Longleftrightarrow g(y) = 1)$
(ii) *A Boolean equation* $h(y) = 1$ *is a* y*-consequence of* $r(x, y) = 1$ *if and only if*

$$\forall y(g(y) = 1 \Rightarrow h(y) = 1) \tag{10}$$

(iii) *A Boolean equation* $h(y) = 1$ *is a* y*-consequence of* $r(x, y) = 1$ *if and only if* $g \leqslant h$
(iv) *A Boolean function* $h: B^m \to B$ *fulfills*

$$(\forall x \in B^n)(\forall y \in B^m)(r(x, y) \leqslant h(y)) \tag{11}$$

if and only if $g \leqslant h$
(v) *The function* g *is the eliminant* $g = e_x r$.

Proof: (v) $\Rightarrow$ (i): Take an arbitrary but fixed y and apply the Boole-Schröder Theorem ([1], [2], [26], (vol. 1, Section 22), (cf. the dual of Theorem 2.3 in [24])).

(i) $\Rightarrow$ (ii): Suppose (9) and take $y \in B^m$. If $g(y) = 1$, then $\exists x(r(x, y) = 1)$ by (i), hence $h(y) = 1$ by (9). Conversely, suppose (10) and take again $y \in B^m$. If $\exists x(r(x, y) = 1)$, then $g(y) = 1$ by (i), hence $h(y) = 1$ by (10).
(ii) $\Rightarrow$ (iii): Taking $h = g$ in (10) we see that g is a y-consequence of r by (ii). Thus

$$\forall x \forall y (r(x, y) = 1 \Rightarrow g(y) = 1) \tag{12}$$

by (7) and since (4) is consistent in (x, y) by hypothesis, it follows from (12) that $g(y) = 1$ is consistent, too. Now we make use of the second part of the so-called Verification Theorem [19], [12] (cf. the dual of Theorem 2.14 in [24]), which states precisely that if the equation $g(y) = 1$ is consistent, then (10) is equivalent to $g \leqslant h$.
(iii) $\Rightarrow$ (iv): Suppose (11). Then h is a y-consequence of r, hence $g \leqslant h$ by (iii). Conversely, suppose $g \leqslant h$. Then (7) holds by (iii). But (4), regarded as an equation in (x, y), is consistent by hypothesis, therefore (7) implies (11) via the Verification Theorem.
(iv) $\Rightarrow$ (v): As e_x is a closure operator, $r \leqslant e_x r$, hence $g \leqslant e_x r$ by (iv). On the other hand, taking $h = g$ in (iv) we deduce $r \leqslant g$. But e_x leaves unchanged the functions independent of x, hence $r \leqslant g$ implies $e_x r \leqslant e_x g = g$.

Remark 1 Suppose $r(x, y) = 1$ is inconsistent. Then $(e_x r)(y) = 1$ has no solutions, hence properties (i) and (ii) above are still valid for $g = e_x r$. In other words, every Boolean equation in y is a consequence of the (false!) equation $r(x, y) = 1$. Thus property (iii) fails for $g = e_x r$. However properties (i), (ii), and (iii) hold for the constant function $g = 0$.

Remark 2 If r is expressed in a *disjunctive form,* i.e., if r is given as a sum of terms, then $e_x r$ is obtained simply by deleting all the x-variables, with the convention that if a term contains only x-variables, the result of the deletion is 1.

This known property follows from (6) by observing that each $r(i, y)$ is obtained by deleting the x-variables from some (possibly none) terms of r and by cancelling the remaining terms (if any) and conversely, the y-part of each term of r is included in some $r(i, y)$.

Example 1 Let

$$\begin{aligned} r(x_1, x_2, y_1, y_2, y_3, y_4) = {} & x_1 x_2 y_1 y_2 y_3' y_4' + x_1 x_2' y_1 y_2' y_3' y_4 + x_1' x_2 y_1' y_2 y_3 y_4' \\ & + x_1' x_2' y_1' y_2' y_3 y_4 \ . \end{aligned}$$

Then

$$(e_x r)(y) = y_1 y_2 y_3' y_4' + y_1 y_2' y_3' y_4 + y_1' y_2 y_3 y_4' + y_1' y_2' y_3 y_4 \ .$$

Example 2 Let

$$\begin{aligned} r(x_1, x_2, y_1, y_2) = {} & abx_1 x_2 y_1 y_2 + ab' x_1 y_1 y_2' + a x_1 x_2' y_1 y_2' + a' b x_2 y_1' y_2 \\ & + b x_1' x_2 y_1' y_2 + a' b' y_1' y_2' + a' x_2' y_1' y_2' + b' x_1' y_1' y_2' + x_1' x_2' y_1' y_2' \ . \end{aligned}$$

Then

$$
\begin{aligned}
(e_{\boldsymbol{x}}r)(\boldsymbol{y}) &= aby_1y_2 + ab'y_1y_2' + ay_1y_2' + a'by_1'y_2 + by_1'y_2 + a'b'y_1'y_2' \\
&\quad + a'y_1'y_2' + b'y_1'y_2' + y_1'y_2' \\
&= y_1'y_2' + by_1' + ay_2' + ab \ .
\end{aligned}
$$

If we specialize Proposition 1 to the case when the eliminant is the constant function 1, we recapture the Boole-Schröder Theorem plus the property that in this case there are no nontrivial y-consequences of r:

Proposition 2 *The following conditions are equivalent:*

(i) $\forall \boldsymbol{y}\, \exists \boldsymbol{x}\ r(\boldsymbol{x}, \boldsymbol{y}) = 1$
(ii) *A Boolean equation $h(\boldsymbol{y}) = 1$ is a $\boldsymbol{y}$-consequence of $r(\boldsymbol{x}, \boldsymbol{y}) = 1$ if and only if h is the constant function $h = 1$*
(iii) *The eliminant is the constant function $e_{\boldsymbol{x}}r = 1$.*

Proof: If for all $\boldsymbol{x}$ and $\boldsymbol{y}$, $r(\boldsymbol{x}, \boldsymbol{y}) \neq 1$, any Boolean equation $h(\boldsymbol{y}) = 1$ is a consequence of the false premise (4) and properties (i), (ii), and (iii) fail. If (4) is consistent, apply Proposition 1 with $g = 1$.

In the remainder of this section we confine our attention to the particular case when r is the eliminant of a system of equations of the form

$$
\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}) \ . \tag{8}
$$

In this case we refer to $e_{\boldsymbol{x}}r$ as the eliminant of the system (8) and to the $\boldsymbol{y}$-consequences of (4) as $\boldsymbol{y}$-consequences of (8).

Lemma 2 *A Boolean equation $h(\boldsymbol{y}) = 1$ is a $\boldsymbol{y}$-consequence of the system $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$ if and only if*

$$
(\forall \boldsymbol{x} \in B^n)(h(\boldsymbol{f}(\boldsymbol{x})) = 1) \ . \tag{13}
$$

Proof: Suppose (9) and take $\boldsymbol{x} \in B^n$. Setting $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$, it follows that $\exists \boldsymbol{x}(\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}))$; hence $h(\boldsymbol{y}) = 1$, that is, $h(\boldsymbol{f}(\boldsymbol{x})) = 1$.

Conversely, suppose (13). Take $\boldsymbol{y} \in B^m$ such that $\exists \boldsymbol{x}(\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}))$. Then $h(\boldsymbol{y}) = h(\boldsymbol{f}(\boldsymbol{x})) = 1$.

Remark 3 Lemma 2 may be reworded to the effect that *the $\boldsymbol{y}$-consequences of the system of Equations* (8) *coincide with the relations of Boolean functional dependence* (13) connecting the functions $f_1, \ldots, f_m$. Notice also that (13) can be written in the compact form

$$
h \circ \boldsymbol{f} = 1 \ . \tag{14}
$$

In view of Remark 3 we see that Theorem 1 below solves the following equivalent problems: (α) find the consistency conditions for (8) with respect to $\boldsymbol{x}$; (β) determine the range of the map $\boldsymbol{f}$; (γ) determine all the relations of Boolean functional dependence connecting the functions $f_1, \ldots, f_m$.

Theorem 1 *The following conditions are equivalent for a Boolean function $g: B^m \to B$:*

(i) $\forall y((\exists x(y = f(x))) \Longleftrightarrow g(y) = 1)$

(ii) *The range of the map* $f\colon B^n \to B^m$ *is*

$$\{y \in B^m | g(y) = 1\} \tag{15}$$

(iii) *A Boolean function* $h\colon B^m \to B$ *fulfills the identity* $h \circ f = 1$ *if and only if*

$$\forall y(g(y) = 1 \Rightarrow h(y) = 1) \tag{10}$$

(iv) *A Boolean function* $h\colon B^m \to B$ *fulfills the identity* $h \circ f = 1$ *if and only if* $g \leqslant h$.

(v) *The function g is given by*

$$g(y) = \sum_{j=0}^{2^m-1} \left[\sum_{i=0}^{2^n-1} m_j(f(i))\right] m_j(y) \tag{16}$$

(vi) *g is the eliminant* $g = e_{\boldsymbol{x}} r$ *of the resolvent r of the system* $y = f(x)$.

Comment Formula (16) is a compact version of

$$g(y_1, \ldots, y_m) = \sum_{\gamma_1,\ldots,\gamma_m \in \{0,1\}} \left[\sum_{i=0}^{2^n-1} (f_1(i))^{\gamma_1} \ldots (f_m(i))^{\gamma_m}\right] y_1^{\gamma_1} \ldots y_m^{\gamma_m} \tag{17}$$

Proof: (i) $\Longleftrightarrow$ (ii) : Trivial paraphrase.
(i) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (vi) : As the system (8) is consistent with respect to (x, y), we can apply the equivalences (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (v) in Proposition 1 via Remark 3.
(ii) $\Longleftrightarrow$ (v): By Corollary 2 to Theorem 1 in [25].

Corollary 1 *The following identity holds:*

$$(\forall x \in B^n)((e_{\boldsymbol{x}} r)(f(x)) = 1) \ . \tag{18}$$

Corollary 2 *The equation* $(e_{\boldsymbol{x}} r)(y) = 1$ *is consistent.*

Example 3 Consider the system

$$y_1 = ax_1,\ y_2 = bx_2$$

where a and b are constants from B. Then

$$r(x, y) = (y_1 a x_1 + y_1'(a' + x_1'))(y_2 b x_2 + y_2'(b' + x_2'))$$

so that r is the function given in Example 2, where we have determined $e_{\boldsymbol{x}} r$. Now $e_{\boldsymbol{x}} r$ can be computed directly using formula (16):

$$\begin{aligned}(e_{\boldsymbol{x}} r)(y) &= \left(\sum_{i=0}^{3} f_1'(i) f_2'(i)\right) y_1' y_2' + \left(\sum_{i=0}^{3} f_1'(i) f_2(i)\right) y_1' y_2 \\ &\quad + \left(\sum_{i=0}^{3} f_1(i) f_2'(i)\right) y_1 y_2' + \left(\sum_{i=0}^{3} f_1(i) f_2(i)\right) y_1 y_2 \\ &= (1\cdot 1 + \ldots) y_1' y_2' + (1\cdot 0 + 1\cdot b + a'\cdot 0 + a'b) y_1' y_2 + (0\cdot 1 + 0\cdot b' \\ &\quad + a\cdot 1 + a\cdot b') y_1 y_2' + (0\cdot 0 + 0\cdot b + a\cdot 0 + a\cdot b) y_1 y_2 \\ &= y_1' y_2' + b y_1' + a y_2' + ab \ .\end{aligned}$$

Example 4 Consider the system

$$y_1 = x_1,\ y_2 = x_2,\ y_3 = x_1',\ y_4 = x_2' \ .$$

Then

$$r(x, y) = (y_1x_1 + y_1'x_1')(y_2x_2 + y_2'x_2')(y_3x_1' + y_3'x_1)(y_4x_2' + y_4'x_2)$$
$$= (y_1y_3'x_1 + y_1'y_3x_1')(y_2y_4'x_2 + y_2'y_4x_2')$$

so that r is the function given in Example 1 where we have determined $e_{\mathbf{x}}r$. Now $e_{\mathbf{x}}r$ can be computed directly using formula (16), which involves 16 $\mathbf{y}$-minterms. This computation is left to the reader.

Theorem 2 below specializes Theorem 1 to the case when $g = 1$. The equivalences (iii) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (v) in Theorem 2 were first established by Whitehead [27] and Löwenheim [13] (cf. Theorem 8.3 in [24]; see also Corollary 3 to Theorem 1 in [25]).

Theorem 2 *The following conditions are equivalent:*

(i) $\forall \mathbf{y} \exists \mathbf{x}(\mathbf{y} = \mathbf{f}(\mathbf{x}))$
(ii) $\forall \mathbf{y} \in \{0, 1\}^m \exists \mathbf{x}(\mathbf{y} = \mathbf{f}(\mathbf{x}))$
(iii) *The map* $\mathbf{f}\colon B^n \to B^m$ *is surjective*
(iv) *The only Boolean function* $h\colon B^m \to B$ *such that*

$$(\forall \mathbf{x} \in B^n)(h(\mathbf{f}(\mathbf{x})) = 1) \tag{13}$$

is the constant function $h = 1$.

(v) $\sum_{i=0}^{2^n-1} m_j(\mathbf{f}(i)) = 1 \qquad (j = 0, \ldots, 2^m - 1)$

(vi) *The eliminant of the resolvent* r *of the system* $\mathbf{y} = \mathbf{f}(\mathbf{x})$ *is the constant function* $e_{\mathbf{x}}r = 1$.

Proof: (i) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (v) $\Longleftrightarrow$ (vi): From Theorem 1 with $g = 1$.
(i) $\Rightarrow$ (ii): Trivial.
(ii) $\Rightarrow$ (i): It follows from (ii) and Theorem 1 that $(e_{\mathbf{x}}r)(j) = 1$ for every $j \in \{0, 1\}^m$; hence $(e_{\mathbf{x}}r)(\mathbf{y}) = 1$ for every $\mathbf{y} \in B^m$ in view of the first part of the Verification Theorem [19], [12] (cf. Theorem 2.13 in [24]), which states precisely that a Boolean equation $\phi(\mathbf{z}) = \psi(\mathbf{z})$ is verified for every $\mathbf{z}$ if and only if it is fulfilled for all the vectors $\mathbf{z}$ made up of 0's and 1's. Finally $\forall \mathbf{y}((e_{\mathbf{x}}r)(\mathbf{y}) = 1)$ implies (i) again by Theorem 1.

At this point we state formally an idea which, in fact, has already appeared in Theorems 1 and 2. We will say that the family $\{f_1, \ldots, f_m\}$ is *functionally dependent* provided there is a nonconstant function h such that the identity (13) is fulfilled; otherwise the family $\{f_1, \ldots, f_m\}$ is *functionally independent.*

Theorem 2 yields two corollaries.

Corollary 1 *If* $m > n$ *the family* $\{f_1, \ldots, f_m\}$ *is functionally dependent.*

Proof: Theorem 2 characterizes the functional independence of $\{f_1, \ldots, f_m\}$ by conditions (v), which can be written successively in the following equivalent forms:

$$\prod_{j=0}^{2^m-1} \sum_{i=0}^{2^n-1} m_j(\mathbf{f}(i)) = 1 \tag{19}$$

$$\sum_{\phi} \prod_{j=0}^{2^m-1} m_j(f(\phi(j))) = 1 \tag{20}$$

where ϕ runs over all maps $\{0, \ldots, 2^m - 1\} \to \{0, \ldots, 2^n - 1\}$. But $m > n$ implies that for each map ϕ we can find $j_1, j_2 \in \{0, \ldots, 2^m - 1\}$ such that $j_1 \neq j_2$ and $\phi(j_1) = \phi(j_2)$, therefore

$$\prod_{j=0}^{2^m-1} m_j(f(\phi(j))) \leqslant m_{j_1}(f(\phi(j_1)))m_{j_2}(f(\phi(j_1))) = 0 \ .$$

Thus the left side of (20) vanishes, so that (20) fails.

Corollary 2 *Let f: $B \to B$ be a Boolean function. The singleton $\{f\}$ is functionally independent if and only if f is of the form*

$$f(x) = ax' + a'x = a \oplus x \ . \tag{21}$$

Proof: Formula (20) becomes

$$m_0(f(0))m_1(f(1)) + m_0(f(1))m_1(f(0)) = 1.$$

That is, $f'(0)f(1) + f'(1)f(0) = 1$, which is equivalent to $f(0) = f'(1)$.

Functional independence is a *hereditary property,* i.e., every subfamily of an independent family is also independent. This follows from the obvious fact that functional dependence is *co-hereditary,* i.e., every superfamily of a dependent family is also dependent. However no conclusion can be drawn a priori on the independence of two disjoint nonempty families, the union of which is dependent. Take, e.g., $x = (x_1, x_2)$ and $f_1(x) = x_1$, $f_2(x) = x_2$, $f_3(x) = x'_1$, $f_4(x) = x'_2$. Then $\{f_1, f_2, f_3, f_4\}$, $\{f_1, f_2, f_3\}$, $\{f_1, f_3\}$, $\{f_2, f_4\}$ are dependent families, whereas $\{f_1, f_2\}$, $\{f_3, f_4\}$, $\{f_4\}$ are independent; cf. Examples 5-7 below.

The next theorem relates the dependence of a family to the dependence of a subfamily. For the sake of convenience we denote a vector and the corresponding unordered family of components by the same letter.

Theorem 3 *Consider a partition $\boldsymbol{f} = (\boldsymbol{f}^0, \boldsymbol{f}^1)$ of the family $\boldsymbol{f}$ of Boolean functions. Let $\boldsymbol{y} = (\boldsymbol{y}^0, \boldsymbol{y}^1)$ be the corresponding partition of the variable vector and let r, t, and v be the resolvents of the systems $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$, $\boldsymbol{y}^0 = \boldsymbol{f}^0(\boldsymbol{x})$ and $\boldsymbol{y}^1 = \boldsymbol{f}^1(\boldsymbol{x})$, respectively. Then*

$$r(\boldsymbol{x}, \boldsymbol{y}) = t(\boldsymbol{x}, \boldsymbol{y}^0)v(\boldsymbol{x}, \boldsymbol{y}^1) \tag{22}$$

$$e_{\boldsymbol{x}}t = e_{\boldsymbol{y}^1}e_{\boldsymbol{x}}r \tag{23}$$

$$e_{\boldsymbol{x}}v = e_{\boldsymbol{y}^0}e_{\boldsymbol{x}}r \ . \tag{24}$$

Proof: As the identity (22) is immediate, it suffices to prove (23). But (22) implies $r(\boldsymbol{x}, \boldsymbol{y}) \leqslant t(\boldsymbol{x}, \boldsymbol{y}^0)$, hence $(e_{\boldsymbol{x}}r)(\boldsymbol{y}) \leqslant (e_{\boldsymbol{x}}t)(\boldsymbol{y}^0)$, therefore

$$(e_{\boldsymbol{y}^1}e_{\boldsymbol{x}}r)(\boldsymbol{y}^0) = e_{\boldsymbol{y}^1}(e_{\boldsymbol{x}}r)(\boldsymbol{y}) \leqslant e_{\boldsymbol{y}^1}(e_{\boldsymbol{x}}t)(\boldsymbol{y}^0) = (e_{\boldsymbol{x}}t)(\boldsymbol{y}^0) \ ,$$

so that $e_{\boldsymbol{y}^1}e_{\boldsymbol{x}}r \leqslant e_{\boldsymbol{x}}t$ and it remains to prove $(e_{\boldsymbol{x}}t)(\boldsymbol{y}^0) \leqslant (e_{\boldsymbol{y}^1}e_{\boldsymbol{x}}r)(\boldsymbol{y}^0)$. By reason of Proposition 1 it suffices to show that $(e_{\boldsymbol{y}^1}e_{\boldsymbol{x}}r)(\boldsymbol{y}^0) = 1$ is a consequence of $t(\boldsymbol{x}, \boldsymbol{y}^0) = 1$. Take $\boldsymbol{x}$, $\boldsymbol{y}^0$ such that $t(\boldsymbol{x}, \boldsymbol{y}^0) = 1$. Then $\boldsymbol{y}^0 = \boldsymbol{f}^0(\boldsymbol{x})$ and setting $\boldsymbol{y}^1 = \boldsymbol{f}^1(\boldsymbol{x})$, $\boldsymbol{y} = (\boldsymbol{y}^0, \boldsymbol{y}^1)$, it follows that $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$, consequently $r(\boldsymbol{x}, \boldsymbol{y}) = 1$, hence there is an x such that $r(x, y) = 1$, therefore $(e_{\boldsymbol{x}}r)(y) = 1$ by Proposition 1; further there is a $\boldsymbol{y}^1$ such that $(e_{\boldsymbol{x}}r)(\boldsymbol{y}^0, \boldsymbol{y}^1) = 1$ and this implies $e_{\boldsymbol{y}^1}(e_{\boldsymbol{x}}r)(\boldsymbol{y}^0, \boldsymbol{y}^1) = 1$, again by Proposition 1; finally $(e_{\boldsymbol{y}^1}(e_{\boldsymbol{x}}r))(\boldsymbol{y}^0) = 1$, as desired.

Example 5 Take the system in Example 4 and $\boldsymbol{y}^0 = (y_1, y_2), \boldsymbol{y}^1 = (y_3, y_4)$. Then

$$\begin{aligned} t(\boldsymbol{x}, \boldsymbol{y}^0) &= (y_1x_1 + y_1'x_1')(y_2x_2 + y_2'x_2') \\ &= x_1x_2y_1y_2 + x_1x_2'y_1y_2' + x_1'x_2y_1'y_2 + x_1'x_2'y_1'y_2' \\ v(\boldsymbol{x}, \boldsymbol{y}^1) &= (y_3x_1' + y_3'x_1)(y_4x_2' + y_4'x_2) \\ &= x_1'x_2'y_3y_4 + x_1'x_2y_3y_4' + x_1x_2'y_3'y_4 + x_1x_2y_3'y_4' \end{aligned}$$

and $r = tv$ is the function in Examples 1 and 4. From $e_{\boldsymbol{x}}r$ which has been computed in Example 1 we obtain $e_{\boldsymbol{x}}t$ and $e_{\boldsymbol{x}}v$ by Theorem 3:

$$(e_{\boldsymbol{x}}t)(\boldsymbol{y}^0) = e_{\boldsymbol{y}^1}((e_{\boldsymbol{x}}r)(\boldsymbol{y}) = y_1y_2 + y_1y_2' + y_1'y_2 + y_1'y_2' = 1$$

and similarly $e_{\boldsymbol{x}}v = 1$.

Example 6 Take again the system in Example 4 with $\boldsymbol{y}^0 = (y_1, y_3), \boldsymbol{y}^1 = (y_2, y_4)$. Then

$$(e_{\boldsymbol{x}}t)(\boldsymbol{y}^0) = e_{\boldsymbol{y}^1}((e_{\boldsymbol{x}}r)(\boldsymbol{y}) = y_1y_3' + y_1y_3' + y_1'y_3 + y_1'y_3 = y_1y_3' + y_1'y_3$$

and similarly $(e_{\boldsymbol{x}}v)(\boldsymbol{y}^1) = y_2y_4' + y_2'y_4$. The functional relations $(e_{\boldsymbol{x}}t)(\boldsymbol{f}^0) = 1$ and $(e_{\boldsymbol{x}}v)(\boldsymbol{f}^1) = 1$ (cf. Corollary 1 to Theorem 1) mean, of course, $f_3 = f_1'$ and $f_4 = f_2'$.

Example 7 Now take the system in Example 4 with $\boldsymbol{y}^0 = (y_1, y_2, y_3), \boldsymbol{y}^1 = (y_4)$. Then

$$\begin{aligned} (e_{\boldsymbol{x}}t)(\boldsymbol{y}^0) &= e_{\boldsymbol{y}^1}((e_{\boldsymbol{x}}r)(\boldsymbol{y})) = y_1y_2y_3' + y_1y_2'y_3' + y_1'y_2y_3 + y_1'y_2'y_3 \\ &= y_1y_3' + y_1'y_3 \ , \\ (e_{\boldsymbol{x}}v)(\boldsymbol{y}^1) &= e_{\boldsymbol{y}^0}((e_{\boldsymbol{x}}r)(\boldsymbol{y})) = y_4' + y_4 + y_4' + y_4 = 1 \ . \end{aligned}$$

The latter result follows also from Corollary 2 to Theorem 1.

Remark 4 The above theory of functional dependence cannot be generalized by taking elements of an arbitrary Boolean algebra instead of Boolean functions. For 0 fulfills $0' = 1$ and every element $a \neq 0$ fulfills $f(a) = 1$, where $f \neq 1$ is the Boolean function $f(x) = a' + x$, for all $x \in B$. Thus every nonempty set would be dependent.

2 In this section we confine our attention to those y-consequences $h(y) = 1$ of $r(x, y) = 1$ which are expressed by simple Boolean functions h and we construct a theory parallel to the one in the previous section.

Recall that for every Boolean function $\psi: B^q \to B$ one can find a constant vector $\boldsymbol{a} \in B^p$ and a simple Boolean function $\psi^*: B^{p+q} \to B$ such that

$$(\forall z \in B^q)(\psi(z) = \psi^*(\boldsymbol{a}, z)) \ . \tag{25}$$

The nonnegative integer p, the vector $\boldsymbol{a}$ and the function ψ^* are not unique (cf. Theorem 1.10 in [24]). It will be useful to remark that when dealing with a finite number of Boolean functions, we may suppose without loss of generality that the above vector $\boldsymbol{a}$ is the same for all the functions. The following property will be also needed: the Boolean function ψ is simple if and only if $\psi(k) \in \{0, 1\}$ for every $k \in \{0, 1\}^q$; cf. Theorem 1.7 in [24].

In the sequel we will write simply

$$(\forall z \in B^q)(\psi(z) = \psi(a;z)) \tag{26}$$

instead of (25).

The next lemma provides a better insight into the representation (26):

Lemma 3 *Let $\psi: B^q \to B$ be a Boolean function and* (26) *a representation of it as a simple Boolean function, with $a \in B^p$. Set*

$$H = \{h \in \{0, 1\}^p | m_h(a) \neq 0\} . \tag{27}$$

Then:

α) *The representation* (26) *can be written in the form*

$$\psi(z) = \textstyle\sum_{h \in H} \psi(h;z) m_h(a) \tag{28}$$

and the coefficients $\psi(h;k)$ are given by

$$\psi(h;k) = \begin{cases} 1 & \text{if } m_h(a) \leqslant \psi(k) \\ 0 & \text{otherwise}. \end{cases} \tag{29}$$

β) *The following conditions are equivalent:*

(i) $(\forall h \in H)(\forall z \in B^q)(\psi(z) = \psi(h;z))$
(ii) $(\forall h \in H)(\forall k \in \{0, 1\}^q)(\psi(k) = \psi(h;k))$
(iii) ψ *is a simple Boolean function.*

Proof α) The expansion (28) is immediate from (3). To prove (29), note that each $\psi(k)$ is a sum of elements $m_h(a)$ with $h \in H$, say

$$\psi(k) = \textstyle\sum_{h \in H_k} m_h(a) \ . \tag{30}$$

It follows from (30) that $h \in H_k$ implies $m_h(a) \leqslant \psi(k)$, while if $h \in H - H_k$ then $m_h(a)\psi(k) = 0 \neq m_h(a)$; therefore

$$H_k = \{h \in H | m_h(a) \leqslant \psi(k)\} \ . \tag{31}$$

But (30) can be also written in the form

$$\textstyle\sum_{h \in H} \psi(h;k) m_h(a) = \sum_{h \in H_k} m_h(a) \ .$$

Hence by suitable multiplication we get

$$h \in H_k \Rightarrow \psi(h;k) m_h(a) = m_h(a) \Rightarrow 0 \neq m_h(a) \leqslant \psi(h;k) \Rightarrow \psi(h;k) \neq 0 \Rightarrow \psi(h;k) = 1$$

while

$$h \in H - H_k \Rightarrow \psi(h;k) m_h(a) = 0 \Rightarrow \psi(h;k) \neq 1 \Rightarrow \psi(h;k) = 0 \ .$$

That is, $\psi(h;k) = 1 \Longleftrightarrow h \in H_k$ and this is precisely (29) on account of (31). Note that (29)-(31) are also valid in the case $H_k = \phi$, which is equivalent to $\psi(k) = 0$.

β) We make use of Theorem 1.7 in [24], quoted above.

(i) ⇒ (iii): For every $k \in \{0, 1\}^p$ we have $\psi(k) = \psi(h;k) \in \{0, 1\}$ because $\psi(w;z)$ is a simple Boolean function. Hence ψ is simple.
(iii) ⇒ (ii): Take $k \in \{0, 1\}^q$ and $h_0 \in H$. But

$$\psi(k) = \textstyle\sum_{h \in H} \psi(h;k) m_h(\boldsymbol{a}) \tag{32}$$

by (28). If $\psi(k) = 0$, then (32) implies in particular $0 = \psi(h_0;k) m_{h_0}(\boldsymbol{a})$ with $m_{h_0}(\boldsymbol{a}) \neq 0$, hence $\psi(h_0;k) \neq 1$, therefore $\psi(h_0;k) = 0 = \psi(k)$. If $\psi(k) = 1$, multiply (32) by $m_{h_0}(\boldsymbol{a})$, thus obtaining $m_{h_0}(\boldsymbol{a}) = \psi(h_0;k) m_{h_0}(\boldsymbol{a})$, hence $0 \neq m_{h_0}(\boldsymbol{a}) \leqslant \psi(h_0;k)$, therefore $\psi(h_0;k) \neq 0$, consequently $\psi(h_0;k) = 1 = \psi(k)$.
(ii) ⇒ (i): By the Verification Theorem.

Corollary *The constant vector* $\boldsymbol{a}$ *determines uniquely the simple Boolean function* ψ^* *associated with* ψ *in* (25).

At this point we associate with every Boolean function $\psi\colon B^q \to B$, the function $s\psi\colon B^q \to B$ such that

$$(\forall z \in B^q)\left((s\psi)(z) = \sum_{\psi(k) \neq 0} m_k(z)\right) . \tag{33}$$

Lemma 4 *$s\psi$ is a simple Boolean function and*

$$(\forall k \in \{0, 1\}^q)\left((s\psi)(k) = \begin{cases} 1 & \text{if } \psi(k) \neq 0 \\ 0 & \text{if } \psi(k) = 0 \end{cases}\right) \tag{34}$$

$$(\forall z \in B^q)\left((s\psi)(z) = \textstyle\sum_{h \in H} \psi(h;z)\right) . \tag{35}$$

Proof: Take $k_0 \in \{0, 1\}^q$. The index k_0 appears or not in the right-hand side of (33) according as $\psi(k_0) \neq 0$ or $\psi(k_0) = 0$. Now (33) shows that in the former case $(s\psi)(k_0) \geqslant m_{k_0}(k_0) = 1$, while in the latter case all $m_k(k_0) = 0$, so that $(s\psi)(k_0) = 0$. This proves (34).

From (33) and (34) we get

$$(s\psi)(z) = \sum_{\psi(k) \neq 0} 1 \cdot m_k(z) + \sum_{\psi(k) = 0} 0 \cdot m_k(z) = \textstyle\sum_{k=0}^{2^q - 1} (s\psi)(k) m_k(z)$$

and, moreover, all $(s\psi)(k) \in \{0, 1\}$, so that $s\psi$ is a simple Boolean function.

Finally, using (29) and (31), then (30) and (34), we see that

$$\textstyle\sum_{h \in H} \psi(h;k) = 0 \iff \psi(h;k) = 0 \ (\forall h) \iff H_k = \phi \iff \psi(k) = 0 \iff (s\psi)(k) = 0$$

so that (35) holds for $z = k \in \{0, 1\}$; hence it is generally valid in view of the Verification Theorem.

Remark 5 The expansions (28) and (35) show that $s\psi$ can be obtained from ψ by deleting the $\boldsymbol{a}$-letters; cf. Remark 2.

Example 8 Let

$$r(\boldsymbol{x}, \boldsymbol{y}) = (y_1 a x_1 + y_1'(a' + x_1'))(y_2 b x_2 + y_2'(b' + x_2'))$$

be the function in Examples 2 and 3. Here $\boldsymbol{a} = (a, b)$; hence

$$\begin{aligned}(sr)(\boldsymbol{x}, \boldsymbol{y}) &= y_1'y_2' + y_1'y_2x_2 + y_1x_1y_2' + y_1x_1y_2x_2 \\ &= y_1'y_2' + x_1y_2' + x_2y_1' + x_1x_2\end{aligned}$$

while for the function

$$(e_x r)(y) = y_1' y_2' + b y_1' + a y_2' + ab$$

determined in Examples 2, 3, we get $s e_x r = 1$ because $(e_x r)(1, 1; y) = 1$.

Lemma 5 (i) *$s\psi$ is the least simple Boolean function which includes ψ.* (ii) *The map $\psi \mapsto s\psi$ is a closure operator.*

Proof: Write the Boole expansion of ψ in the form

$$\psi(z) = \sum_{\psi(k) \neq 0} \psi(k) m_k(z)$$

and compare to (33): it follows that $\psi \leqslant s\psi$.

Further if $\psi \leqslant \phi$ then $\psi(k) \neq 0 \Rightarrow \phi(k) \neq 0$, whence $s\psi \leqslant s\phi$ follows by (33).

Notice that if σ is a simple Boolean function, then $s\sigma = \sigma$. For (34) becomes for all k, $(s\sigma)(k) = \sigma(k)$, hence $s\sigma = \sigma$ by the Verification Theorem.

Taking in particular $\sigma = s\psi$, it follows that $ss\psi = s\psi$.

Finally if σ is a simple Boolean function and $\psi \leqslant \sigma$, then $s\psi \leqslant s\sigma = \sigma$.

Remark 6 As a matter of fact, s is even a quantifier, i.e., it fulfills the stronger set of conditions $s0 = 0$, $\psi \leqslant s\psi$, and $s(\psi \cdot s\chi) = s\psi \cdot s\chi$.

Example 9 The functions r and $e_x r$ in Examples 1 and 4-7 being simple, we have $sr = r$ and $s e_x r = e_x r$.

The next lemma relates the closure operators e_x and s.

Lemma 6 α) *The closure operators e_x and s commute: $s e_x = e_x s$.*
β) *$s e_x$ is a closure operator.*
γ) *$s e_x r$ is the least simple Boolean function independent of x which includes r and*

$$(\forall y \in B^m)\left((s e_x r)(y) = \sum_{h \in H} \sum_{i=0}^{2^n-1} r(h; i, y)\right) \quad . \tag{36}$$

Proof: The expansion (36) follows immediately from (6) and (35); it implies

$$\begin{aligned}(s e_x r)(y) &= \sum_{i=0}^{2^n-1} \sum_{h \in H} r(h; i, y) \\ &= \sum_{i=0}^{2^n-1} (sr)(i, y) = (e_x s r)(y) \quad .\end{aligned}$$

That is, α.

The implication $\alpha \Rightarrow \beta$ is well known and easy to prove; let us check, e.g., the idempotency:

$$(s e_x)(s e_x) = s(e_x s) e_x = s(s e_x) e_x = (ss)(e_x e_x) = s e_x \quad .$$

γ) is immediate from the properties already established of the operators e_x and s.

Example 10 For the function

$$(sr)(x, y) = y_1' y_2' + x_2 y_1' + x_1 y_2' + x_1 x_2 y_1 y_2$$

in Example 8 we have

$$(e_{\mathbf{x}}sr)(\mathbf{y}) = y_1'y_2' + y_1' + y_2' + y_1y_2 = 1$$

and also $se_{\mathbf{x}}r = 1$ as found in Example 8. The same result may be obtained by applying formula (36) to the function r in Examples 2, 3, and 8.

Notice that the commutativity $se_{\mathbf{x}}r = e_{\mathbf{x}}sr$ is trivial in the case of a simple Boolean function r.

At this point we are in a position to begin the realization of our program. In the sequel we refer to $h(\mathbf{y}) = 1$ as a *simple Boolean equation* provided h is a simple Boolean function.

Proposition 3 *Suppose the Boolean equation $r(\mathbf{x}, \mathbf{y}) = 1$ is consistent with respect to $(\mathbf{x}, \mathbf{y})$. Then the following conditions are equivalent for a simple Boolean function g_o: $B^m \to B$:*

(i) *A simple Boolean equation $h(\mathbf{y}) = 1$ is a $\mathbf{y}$-consequence of $r(\mathbf{x}, \mathbf{y}) = 1$ if and only if*

$$\forall \mathbf{y}(g_o(\mathbf{y}) = 1 \Rightarrow h(\mathbf{y}) = 1) \tag{37}$$

(ii) *A simple Boolean equation $h(\mathbf{y}) = 1$ is a $\mathbf{y}$-consequence of $r(\mathbf{x}, \mathbf{y}) = 1$ if and only if $g_o \leqslant h$*

(iii) *A simple Boolean function h: $B^m \to B$ fulfills*

$$(\forall \mathbf{x} \in B^n)(\forall \mathbf{y} \in B^m)(r(\mathbf{x}, \mathbf{y}) \leqslant h(\mathbf{y})) \tag{38}$$

if and only if $g_o \leqslant h$

(iv) *The function g_o is $g_o = se_{\mathbf{x}}r$.*

Proof: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv): Similar to the proof of the corresponding implications (ii) $\Rightarrow \ldots \Rightarrow$ (v) in Proposition 1, via Lemma 6.
(iv) $\Rightarrow$ (i): Let $h(\mathbf{y}) = 1$ be a simple Boolean equation. Suppose (37) holds. If $r(\mathbf{x}, \mathbf{y}) = 1$, then a fortiori $(se_{\mathbf{x}}r)(\mathbf{y}) = 1$; therefore $h(\mathbf{y}) = 1$. Conversely, suppose h is a $\mathbf{y}$-consequence of r. Then $e_{\mathbf{x}}r \leqslant h$ by Proposition 1, hence $se_{\mathbf{x}}r \leqslant sh = h$ by Lemma 4, and from $se_{\mathbf{x}}r \leqslant h$ we infer (37).

Proposition 4 *The following conditions are equivalent:*

(i) *A simple Boolean equation $h(\mathbf{y}) = 1$ is a $\mathbf{y}$-consequence of $r(\mathbf{x}, \mathbf{y}) = 1$ if and only if h is the constant function $h = 1$.*

(ii) *The identity $se_{\mathbf{x}}r = 1$ holds.*

Proof: Similar to the proof of Proposition 2.

Theorem 4 *The following conditions are equivalent for a simple Boolean function g_o: $B^m \to B$:*

(i) *A simple Boolean function h: $B^m \to B$ fulfills the identity $h \circ f = 1$ if and only if*

$$\forall \mathbf{y}(g_o(\mathbf{y}) = 1 \Rightarrow h(\mathbf{y}) = 1) \tag{39}$$

(ii) *A simple Boolean function h: $B^m \to B$ fulfills the identity $h \circ f = 1$ if and only if $g_o \leqslant h$*

(iii) *The function g_o is given by*

(40) $$g_o(y) = \prod_{\substack{\alpha_1,\ldots,\alpha_m \in \{0,1\} \\ f_1^{\alpha_1}+\ldots+f_m^{\alpha_m}=1}} (y_1^{\alpha_1} + \ldots + y_m^{\alpha_m}) \ ,$$

(iv) *The function* g_o *is* $g_o = se_{\mathbf{x}}r$, *where* r *is the resolvent of the system* $y = f(x)$.

Proof: (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iv): As the system $y = f(x)$ is consistent with respect to (x, y), we can apply the corresponding equivalences in Proposition 3, via Remark 3.

(iii) $\Longleftrightarrow$ (iv): Let g_o be the function defined by (40). The Verification Theorem shows that in order to prove $g_o = se_{\mathbf{x}}r$, it suffices to show that $g_o(j) = (se_{\mathbf{x}}r)(j)$ for every $j \in \{0, 1\}^m$. Thus fix an arbitrary $j = (\beta_1, \ldots, \beta_m) \in \{0, 1\}^m$. Then

$$\begin{aligned} g_o(j) = 0 &\Longleftrightarrow (\exists \alpha_1, \ldots, \alpha_m \in \{0, 1\})(f_1^{\alpha_1} + \ldots + f_m^{\alpha_m} = 1 \ \& \ \beta_1^{\alpha_1} + \ldots + \beta_m^{\alpha_m} = 0) \\ &\Longleftrightarrow (\exists \alpha_1, \ldots, \alpha_m \in \{0, 1\})(f_1^{\alpha_1} + \ldots + f_m^{\alpha_m} = 1 \ \& \ \alpha_1 = \beta_1' \ \& \ldots \& \ \alpha_m = \beta_m') \\ &\Longleftrightarrow f_1^{\beta_1'} + \ldots + f_m^{\beta_m'} = 1 \Longleftrightarrow f_1^{\beta_1} \ldots f_m^{\beta_m} = 0 \\ &\Longleftrightarrow (\forall x)([f_1(x)\beta_1 + f_1'(x)\beta_1'] \ldots [f_m(x)\beta_m + f_m'(x)\beta_m'] = 0) \\ &\Longleftrightarrow (\forall x)(r(x, j) = 0) \Longleftrightarrow (\forall i \in \{0, 1\}^n)(r(i, j) = 0) \\ &\Longleftrightarrow \sum_{i=0}^{2^n-1} r(i, j) = 0 \Longleftrightarrow (e_{\mathbf{x}}r)(j) = 0 \Longleftrightarrow (se_{\mathbf{x}}r)(j) = 0 \ , \end{aligned}$$

by (34) in Lemma 4.

Example 11 Take again the system

$$f_1(x_1, x_2) = x_1 \, , \quad f_2(x_1, x_2) = x_2 \, , \quad f_3(x_1, x_2) = x_1' \, , \quad f_4(x_1, x_2) = x_2'$$

in Examples 4-7 and 10. Then

$$f_1^{\alpha_1} + f_2^{\alpha_2} + f_3^{\alpha_3} + f_4^{\alpha_4} = 1 \Longleftrightarrow \alpha_1 = \alpha_3 \text{ or } \alpha_2 = \alpha_4 \ ,$$

hence

$$(se_{\mathbf{x}}r)(y) = \left[\prod_{\alpha_1,\alpha_2,\alpha_4 \in \{0,1\}} (y_1^{\alpha_1} + y_2^{\alpha_2} + y_3^{\alpha_1} + y_4^{\alpha_4})\right] \left[\prod_{\alpha_1,\alpha_2,\alpha_3 \in \{0,1\}} (y_1^{\alpha_1} + y_2^{\alpha_2} + y_3^{\alpha_3} + y_4^{\alpha_2})\right] .$$

But the first factor equals

$$\begin{aligned} &\prod_{\alpha_1 \in \{0,1\}} \ \prod_{\alpha_2,\alpha_4 \in \{0,1\}} (y_1^{\alpha_1} + y_3^{\alpha_1} + y_2^{\alpha_2} + y_4^{\alpha_4}) \\ &= \prod_{\alpha_1 \in \{0,1\}} \left(y_1^{\alpha_1} + y_3^{\alpha_3} + \prod_{\alpha_2,\alpha_4 \in \{0,1\}} (y_2^{\alpha_2} + y_4^{\alpha_4})\right) \\ &= \prod_{\alpha_1 \in \{0,1\}} (y_1^{\alpha_1} + y_3^{\alpha_3}) = (y_1' + y_3')(y_1 + y_3) = y_1'y_3 + y_3'y_1 \end{aligned}$$

and similarly for the second factor; therefore

$$(se_{\mathbf{x}}r)(y) = (y_1'y_3 + y_3'y_1)(y_2'y_4 + y_4y_2')$$

which coincides with $(e_{\mathbf{x}}r)(y)$ found in Example 1. This should happen because $e_{\mathbf{x}}r$ is a simple Boolean function.

As stated in the Introduction, the main result of this section establishes the equivalence between the identity $se_{\boldsymbol{x}}r = 1$ and the Moore-Marczewski independence of the family $\{f_1, \ldots, f_m\}$.

We recall first that the origin of the concept of Moore independence is the following remark: a system $\{p_1, \ldots, p_m\}$ of axioms is independent if and only if none of the propositions

$$p_i' \prod_{\substack{j=1 \\ j \neq i}}^{m} p_j \qquad (i = 1, \ldots, m)$$

is identically false. Moore [18] has introduced the concept of a *completely independent* system of propositions: this means a system $\{p_1, \ldots, p_m\}$ such that none of the minterms

$$p_1^{\alpha_1} \ldots p_m^{\alpha_m} \qquad (\alpha_1, \ldots, \alpha_m \in \{0, 1\})$$

is an identically false proposition. On the other hand, Marczewski [14] has used the framework of universal algebra to obtain a common generalization of many concepts called "independence" in various fields of mathematics. A system $a_1, \ldots, a_m$ of elements of a general algebra A is said to be *independent* if every map from $\{a_1, \ldots, a_m\}$ to A can be extended to a homomorphism from the subalgebra generated by $a_1, \ldots, a_m$ to A. It turns out [15] that in the case of Boolean algebras this concept reduces to Moore's complete independence.

Theorem 5 *The following conditions are equivalent:*

(i) *The only simple Boolean function $h: B^m \to B$ such that*

$$(\forall \boldsymbol{x} \in B^n)(h(\boldsymbol{f}(\boldsymbol{x})) = 1) \tag{13}$$

is the constant function $h = 1$.

(ii) $(\forall \alpha_1, \ldots, \alpha_m \in \{0, 1\})(f_1^{\alpha_1} + \ldots + f_m^{\alpha_m} \neq 1)$

(iii) $(\forall \alpha_1, \ldots, \alpha_m \in \{0, 1\})(f_1^{\alpha_1} \ldots f_m^{\alpha_m} \neq 0)$

(iv) $se_{\boldsymbol{x}}r = 1$, *where r is the resolvent of the system $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$.*

Proof: (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iv): From the equivalences (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv) in Theorem 4 with $g_o = 1$.

(ii) $\Longleftrightarrow$ (iii): Observe that if $(\alpha_1, \ldots, \alpha_m)$ runs over $\{0, 1\}^m$, so does $(\alpha_1', \ldots, \alpha_m')$; therefore both (ii) and (iii) are equivalent to

$$(\forall (\alpha_1', \ldots, \alpha_m') \in \{0, 1\}^m)(f_1^{\alpha_1'} \ldots f_m^{\alpha_m'} \neq 0) \ .$$

Corollary 1 *Let f: $B \to B$ be a Boolean function. The singleton $\{f\}$ is Moore-Marczewski independent if and only if $0 \neq f \neq 1$.*

Corollary 2 *The concept of functional independence is actually stronger than that of Moore-Marczewski independence.*

Proof: Trivially a functionally independent family is also Moore-Marczewski independent. The converse does not hold as shown, e.g., by the above Corollary 1.

Another example proving that the two concepts of independence are not equivalent is provided by Examples 2, 3, 8, and 10, in which $e_x r \neq 1$ but $se_x r = 1$.

Example 12 Let us study the Moore-Marczewski independence of the system $\{f_1, f_2\}$ where $f_1(x) = ax$ and $f_2(x) = bx$, with a, b constants from B. The condition (iii) in Theorem 5 can be written successively in the following equivalent forms:

$$\exists x\, abx \neq 0 \;\&\; \exists x\, ab'x \neq 0 \;\&\; \exists x\, a'bx \neq 0 \;\&\; \exists x\, a'b' + x' \neq 0,$$
$$ab \neq 0 \;\&\; ab' \neq 0 \;\&\; a'b \neq 0 \;\&\; a'b' \neq 0.$$

Example 12 shows in particular that Corollary 1 to Theorem 2 is no longer valid for Moore-Marczewski independence. On the other hand, in the comments preceding Theorem 3 we have shown that no conclusion can be drawn on the independence of a subfamily of a family $\{f_1, \ldots, f_m\}$ from the simple fact that the latter is dependent. Examples 5-7 show that these comments remain valid for Moore-Marczewski independence.

Finally notice that in strong contrast to Remark 4, the definition of Moore-Marczewski independence makes sense without any change for elements of an arbitrary Boolean algebra instead of Boolean functions.

Addendum

We have discovered after completion of the manuscript that the equivalence of conditions (i) and (ii) in our Theorem 2 has already been remarked by J. Kuntzmann [9] and taken by him as a definition of independence in the case of simple functions (in our terminology). Kuntzmann establishes our Corollary 1 and studies in some detail the case $m = n$, as well as other properties.

NOTES

1. Often referred to by switching theorists as "the Shannon expansion theorem".
2. Marczewski attributes this concept to Fichtenholz and Kantorovitch in a paper published in 1934.

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