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General Models of Set Theory

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Quine explains the concept of set by considering an open sentence and observing that

> The notion of class is such that there is supposed to be, in addition to the various things of which that sentence is true, also a further thing which is the *class* having each of those things and no others as member. It is the class determined by the open sentence. ([4], p. 1)

The task of set theory is to formulate this idea rigorously in a way that blocks the paradoxes. In addressing the problem, technical investigations of set theory have been confined to what might be called its "internal structure", in which the discussion is framed in an underlying logic containing only those primitive predicates essential for formulating set theory itself. However, the intuitive notion of class is one in which sets can be specified by any clear extra-set theoretical condition. For example, informal asides in expositions of set theory frequently illustrate basic ideas using sets specified by open sentences such as "x is blue", "x is a man", "x is a parent of y", and so on. Further, the notion of an interpretation in model-theoretic semantics appears to require that sets in the domain of the model be specified by predicates of the object theory that are no part of the set theory in terms of which the model is formulated. Most importantly, however, if the notion of class is to be used in formalizing everyday or scientific discourse, for example, to explicate the application of mathematics to empirical subject matter, specification of sets by extra-set theoretical formulas is inescapable.*

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Thus, the notion of class that set theory is intended to capture is one that is essentially indifferent to the subject matter to which it is applied, much as the logic of identity applies to any and all predicates in a first-order language, no matter how they may be interpreted. Such a liberal conception might seem to reopen the door to the paradoxes, but since the familiar difficulties arise only within the internal structure of set theory itself, allowing sets to be specified by extra-set theoretical predicates need not conflict with moves adopted to avoid internal inconsistency. From an intuitive point of view, then, set theory appears as a logic that, while requiring careful attention to its internal structure to block the paradoxes, can be applied without restriction to any extra-set theoretical content whatever.

In sum, to represent adequately the intuitive notion of class, a formal set theory must be general in the sense of providing for the specification of classes by predicates in addition to those of the set theory itself, no matter how these extra-set theoretical predicates may be interpreted substantively. This requirement can be formulated as a demand that an acceptable set theory have a general model, that is, a model in which the extra-set theoretical primitive predicate symbols of the language can be interpreted within the domain of the model in any fashion without altering the status of the interpretation of the set theoretical predicates as a model of the set theory. It is important to recognize that a stipulation that a set theory have a general model is a semantic one, for it concerns the interpretation of the theory rather than its logical structure. Consequently, this requirement cannot be formulated within the theory itself but instead must be stated in a metalanguage.

The requirement that an acceptable formalization of the intuitive notion of class be general seems natural and compelling. It is appropriate, therefore, to investigate in a preliminary way the availability and properties of general models of set theory. The surprising result is that the concept of a general model is extremely strong. By rather simple methods we show that the only set theories in a very wide class, which includes Quine's *NF* [4] and Ackermann's theory [1], having general models are the Zermelo-Fraenkel theory *ZF* and its close relatives. Moreover, by a straightforward argument it follows that if *ZF* has a general model, then the Morse theory *M* ([2], pp. 138-146) is consistent and has a general model. The ease with which these results are obtained suggests that the approach of considering not merely the internal structure of a formal theory but also its intended interpretation may prove to be a powerful technique in sorting out issues that are indeterminate when considered from within the formal structure of the theory alone.

1 ***General models*** For the object language $\mathcal{L}$ consider a first-order predicate calculus with primitive predicates ε, $P_1^1, P_2^1, \ldots, P_1^2, P_2^2, \ldots, \ldots$, where ε is a dyadic predicate for set membership and P_j^i is a predicate taking i arguments. We introduce identity by assuming

(1) Extensionality Schema $\forall z(z \;\varepsilon\; x \longleftrightarrow z \;\varepsilon\; y) \wedge Ax \;.\!\rightarrow Ay$, *where* Ax *is any atomic formula of* $\mathcal{L}$.

Note that Ax may have further free variables in addition to x, and that for typographical simplicity we reduce the use of parentheses by employing dots in a manner similar to Quine [5]. We then define identity, $x \equiv y$, as an abbreviation for $\forall z(z \,\varepsilon\, x \longleftrightarrow z \,\varepsilon\, y)$, from which reflexiveness, $x \equiv x$, is immediate, and substitutivity, $x \equiv y \wedge Ax \,.\!\rightarrow Ay$, is just (1) written in abbreviated notation ([3], pp. 159-161, [5], [6], pp. 62-63). We shall say that an occurrence of ε in a formula of $\mathcal{L}$ is *inessential* if and only if it is in a context that can be abbreviated as identity. We can accommodate atoms by treating them as degenerate sets that are their own sole members ([4], [5]) provided axioms of regularity, if assumed, are limited to sets having no atoms as members, but for simplicity we shall assume explicitly that there are no atoms.

Let D be a nonempty set and $E \subseteq D \times D$ an interpretation of ε in D. By an *interpretation* Φ *of the* P_j^i in D we mean a doubly indexed sequence Φ_j^i, $i = 1, 2, \ldots, j = 1, 2, \ldots$, such that Φ_j^i is an i-adic relation on D. An E, Φ-*formula* in a model $\langle D, E, \Phi\rangle$, then, is a formula built up from E and the Φ_j^i by truth-functional composition and quantification restricted to D. If S is a set theory having (1) together with some specified additional axioms, then $\langle D, E, \Phi\rangle$ will, in general, be a model of S only for certain interpretations Φ of the P_j^i in D, but not for others. Consequently, we adopt:

Definition 1 $\langle D, E\rangle$ is a *general model* of S if and only if $\langle D, E, \Phi\rangle$ is a model of S for every interpretation Φ of the P_j^i in D.

We prove here a simple property of general models. A model is said to be *normal* when the relation interpreting identity within the model coincides with actual identity between objects in the domain of the model.

Theorem 1 *A general model is normal.*

Proof: Let $\langle D, E\rangle$ be a general model of a set theory S with ε interpreted as E in which (1) holds, and let $I = \{\langle x, y\rangle\colon \forall z(\langle z, x\rangle \in E \longleftrightarrow \langle z, y\rangle \in E)\}$ be the resulting interpretation of $\equiv$ in D. Consider an interpretation Φ of the P_j^i in D such that P_1^2 is interpreted as $\Phi_1^2 = \{\langle x, y\rangle\colon x, y \in D \wedge x = y\}$. Since $\langle D, E\rangle$ is general, the properties of identity continue to hold:

(i) $\langle x, x\rangle \in I$
(ii) $\langle x, y\rangle \in I \wedge Ax \,.\!\rightarrow Ay$, where Ax is any atomic E, Φ-formula.

And since, for $x, y \in D$, $\langle x, y\rangle \in \Phi_1^2 \longleftrightarrow x = y$, we have

(i′) $\langle x, x\rangle \in \Phi_1^2$
(ii′) $\langle x, y\rangle \in \Phi_1^2 \wedge Ax \,.\!\rightarrow Ay$, where Ax is any atomic E, Φ-formula.

Now take Ax in (ii) as $\langle u, x\rangle \in \Phi_1^2$ to get $\langle x, y\rangle \in I \wedge \langle u, x\rangle \in \Phi_1^2 \,.\!\rightarrow \langle u, y\rangle \in \Phi_1^2$. But this holds for all u, and in particular when u is x. Hence, recalling (i′), we have that $\langle x, y\rangle \in I \rightarrow \langle x, y\rangle \in \Phi_1^2$. The converse follows by an exactly parallel argument.

Consequently we may confine attention to normal models.

2 Existence of general models for protostandard set theory Set theory has been developed from a variety of different and not entirely intertranslatable points of view. However, despite the lack of a canonical formulation, certain basic constructions hold in most reasonable proposals for axiomatic set theory. In particular, every plausible formalization of the intuitive notion of class that does not involve proper classes assumes extensionality, allows the formation of pairs, permits the construction of new sets from old by forming sum-sets and power-sets, assumes some version of a replacement principle, and asserts the existence of sets outright in some form of assumed or derived axiom of infinity. Set theories of this sort we shall call "standard". More formally,

Definition 2 A set theory without atoms or proper classes is *standard* if and only if (1) and (2)-(7) below are axioms or theorems.

(2) **Pairs** $\exists z \forall w(w \,\varepsilon\, z \longleftrightarrow. w \equiv u \vee w \equiv v)$.

(3) **Sum Sets** $\exists z \forall w(w \,\varepsilon\, z \longleftrightarrow \exists x(w \,\varepsilon\, x \wedge x \,\varepsilon\, v))$.

(4) **Power Sets** $\exists z \forall w(w \,\varepsilon\, z \longleftrightarrow \forall x(x \,\varepsilon\, w \rightarrow x \,\varepsilon\, v))$.

(5) **Replacement Schema** $\forall w \forall x \forall x'(Gwx \wedge Gwx' .\rightarrow x' \equiv x) \rightarrow \exists y \forall x(x \in y \longleftrightarrow \exists w(w \in v \wedge Gwx))$, *provided* Gwx *is an* ε*-allowable formula of* $\mathcal{L}$ *with no free occurrences of* y, *where* ε*-allowability is some explicit but here unspecified constraint that can be decided in a finite number of steps and is such that any formula of* $\mathcal{L}$ *having only inessential occurrences of* ε *is* ε*-allowable.*

(6) **Infinity** $\exists y \forall x(x \equiv \Lambda \rightarrow x \,\varepsilon\, y .\wedge. x \,\varepsilon\, y \rightarrow x^+ \,\varepsilon\, y)$, *where* Λ *is the empty set and* x^+ *is the successor of* x.

(7) **Pure Sets** *There are no atoms.*

Although the main concern in this paper is with standard set theories, it will be useful first to consider general models for theories with less structure. Specifically, we weaken (5) to

(5′) **Aussonderung Schema** $\exists y \forall x(x \,\varepsilon\, y \longleftrightarrow. x \,\varepsilon\, v \wedge Fx)$, *provided* Fx *is an* ε*-allowable formula of* $\mathcal{L}$ *with no free occurrences of* y.

Then we may adopt:

Definition 3 A set theory without atoms or proper classes is *protostandard* if and only if (1)-(4), (5′), (6), and (7) are axioms or theorems.

Since (5′) follows from (5) by taking Gwx in (5) as $w \equiv x \wedge Fx$, every standard set theory is protostandard.

Protostandard set theories can be classified in various ways, but two types will be of particular importance here. First, note that the successor function in (6) can be defined in a number of ways. In particular, following Frege, we can take x^+ to be the set consisting of all those objects w' obtained by adding a new element to some member w of x. In such *Frege theories*, as we may call them, it follows from (6) and (3) that there is a universal set V such that $x \,\varepsilon\, V$ for all x. Thus, in Frege theories, (5′) reduces to a version of the classical comprehension schema

> $\exists y \forall x(x \,\varepsilon\, y \longleftrightarrow Fx)$, provided Fx is an ε-allowable formula of $\mathcal{L}$ with no free occurrences of y.

Clearly, the ε-allowability constraint here is essential, since otherwise Russell's paradox is forthcoming at once. The most familiar example of a Frege theory is Quine's *NF* [4], in which ε-allowable formulas are those that are stratifiable. Second, as an alternative, we can follow Zermelo and take every formula of $\mathcal{L}$ as ε-allowable. In this case, x^+ must be construed in a more limited way, such as Zermelo's own or von Neumann's. The prototypical example of a Zermelo theory is Zermelo's original set theory with *Aussonderung* but not replacement, and lacking the axioms of choice and regularity. Thus, let $Z_{(-)}$ be the set theory given by (1)-(4), (5′), (6), and (7), where in (5′) all formulas are ε-allowable; and let Z be $Z_{(-)}$ augmented with

(8) Regularity $\quad v \not\equiv \Lambda \rightarrow \exists u(u \,\varepsilon\, v \wedge \forall t(t \,\varepsilon\, v \rightarrow t \not\varepsilon\, u))$.

The distinction between Frege and Zermelo theories is fundamental: Frege theories cannot have general models, but apparently Zermelo theories can.

Theorem 2 *A general model of a protostandard set theory is a general model of* $Z_{(-)}$, *and if the membership relation in the model is well-founded, the model is isomorphic to a natural model of Z. Moreover, a natural model of Z is general.*

Proof: Let $\langle D, E\rangle$ be a general model for a protostandard set theory S, where ε-allowability has some fixed specification. What is required first is to show that (5′) holds in the model for any formula. Let the free variables in an E, Φ-formula $Fx_1 \ldots x_i$ be just $x_1, \ldots, x_i$. Since $\langle D, E\rangle$ is a general model, interpret P_1^i as $\{\langle x_1, \ldots, x_i\rangle : \langle x_1, \ldots, x_i\rangle \in D \wedge Fx_1 \ldots x_i\}$. Then, since $P_1^i x_1 \ldots x_i$ is ε-allowable, (5′) yields

$$\exists y \forall x_1(x_1 \,\varepsilon\, y \longleftrightarrow . \; x_1 \,\varepsilon\, v \wedge P_1^i x_1 \ldots x_i).$$

Thus, since P_1^i is equivalent in D to F, (5′) holds in $\langle D, E\rangle$ without restriction, and so $\langle D, E\rangle$ is a general model of $Z_{(-)}$.

Next if a general model for S has a well-founded membership relation, it is isomorphic to a model $\langle D, E\rangle$ such that $\langle x, y\rangle \in E$ if and only if $x, y \in D$ and $x \in y$ ([7], p. 171). From this and Theorem 1 it follows that D is transitive. And in turn from these and (3) we have

(i) $\quad x \in D \rightarrow \cup x \in D.$

Further, suppose $w \subseteq x$ and $x \in D$. Invoking generality, interpret P_1^1 as w; then, by (5′) $w = w \cap x$ belongs to D. From this and (4) it then follows that

(ii) $\quad x \in D \rightarrow \mathcal{P}x \in D.$

So, by (i) and (ii), $R_\alpha \in D$ for every ordinal $\alpha \in D$, where $R_\alpha = \{x: \text{rank } x < \alpha\}$. Hence $D = R_\alpha$ for the least ordinal $\theta \notin D$, and so $\langle D, E\rangle$ is a natural model of Z.

Finally let R_θ be a natural model of Z, where $\theta > \omega$, and let Φ be any interpretation of the P_j^i in R_θ. Obviously (1)-(4), (6) and (7) hold. Consider any E, Φ-formula Fx. If $v \in R_\theta$, then $v \subseteq R_\theta$, and so $v \cap \{x: Fx\} \in R_\theta$. Hence (5′) holds. But Φ is arbitrary, and consequently $\langle D, E\rangle$ is a general model.

Consequently, if we require a set theory without proper classes both to be protostandard and to have a general model, we are confined essentially to extensions of $Z_{(-)}$. In particular,

Corollary *No Frege theory has a general model.*

Observe, moreover, that these results hold if the universe of a protostandard set theory is supplemented with proper classes that are defined as objects having members but which cannot themselves be members of other objects.

3 Ackermann-type theories Before taking up standard set theories, we consider one species of nonprotostandard theory, those for which the prototype is Ackermann's theory [1]. In this section, we annex an additional monadic primitive predicate M to the object language $\mathcal{L}$ and, in addition to (1), assume (2)-(4) relativized to M, as well as

(9) $Mx \wedge w \varepsilon x \,.\!\rightarrow Mw$
(10) $Mx \wedge \forall u(u \varepsilon w \rightarrow u \varepsilon x) \,.\!\rightarrow Mw$
(11) $\exists x \neg Mx$.

And, in place of (5) or (5′) we adopt

(12) *if the free variables of* Fx *other than* x *are restricted to* M, *then* $\exists y(My \wedge \forall x(x \varepsilon y \longleftrightarrow Fx))$, *provided* Fx *is an* ε*-allowable formula of* $\mathcal{L}$,

where the concept of ε-allowability is constrained as in (5) together with the condition

Fx is ε-allowable only if it has no occurrences of M, and $Fx \rightarrow Mx$.

Theorem 3 *No model of an Ackermann-type theory is general.*

Proof: Let $\langle D, E, S\rangle$ be a model for an Ackermann-type theory, where E is the interpretation of ε, and S is the interpretation of M. Let $\emptyset$ be the interpretation in D of Λ, and, recalling (11), let $\gamma \in D - S$. Define a one-one mapping

$$\psi(x) = \begin{cases} x & \text{if } x \neq \emptyset, \gamma \\ \gamma & \text{if } x = \emptyset \\ \emptyset & \text{if } x = \gamma. \end{cases}$$

Clearly, ψ induces a structure $\langle D, E^*, S^*\rangle$ on D isomorphic to $\langle D, E, S\rangle$. Hence, by construction,

$$x \in S \longleftrightarrow: x = \emptyset \,.\!\vee\!.\, x \neq \gamma \wedge x \in S^*.$$

Next, assuming that $\langle D, E, S\rangle$ is a general model, interpret P_1^1 as S^* and P_1^2 as E^*. Note that γ is the interpretation in D of the object Λ^* such that, for all x, $\neg P_1^2 x \Lambda^*$, since, for all x, $\langle x, \gamma\rangle \notin E^*$. Hence,

$$Mx \longleftrightarrow: x \equiv \Lambda \,.\!\vee\!.\, x \not\equiv \Lambda^* \wedge P_1^1 x.$$

Abbreviating the right side as Qx, we have that Qx is equivalent to Mx but contains no occurrences of M. Hence $Qx \wedge x \not\varepsilon x$ is ε-allowable, and so by (12) there is an R such that MR, and $x \in R \longleftrightarrow.\ MR \wedge x \not\varepsilon x$, from which Russell's paradox follows.

4 ***Remark*** What appears to distinguish set theories having no general models from those that do is reliance on syntactical constraints, such as stratification requirements or preclusion of certain predicates, in existence schemas to block the paradoxes. This seems to be effective so long as the only concern is with the internal structure of the theory, but when provision is made for extra-set theoretical predicates, the syntactical barriers to inconsistency apparently are subverted. For example, consider the set theory NF, given by (1) and the schema $\exists y \forall x(x \,\varepsilon\, y \longleftrightarrow Fx)$ for any ε-stratified formula of $\mathcal{L}$, where Fx is ε-stratified if and only if the variables occurring in it can be assigned to types in such a way that (a) in any occurrence $u \,\varepsilon\, v$ of ε, v is one type higher than u, and (b) in any occurrence of $P_j^i v_1 \ldots v_i$, the v_t are all of the same type. Now suppose that the axioms governing ε continue to hold but that P_1^2 satisfies the schemas $\forall z(P_1^2 zx \longleftrightarrow P_1^2 zy) \rightarrow x \equiv y$ and $\exists y \forall x(P_1^2 xy \longleftrightarrow Fx)$ for any P_1^2-stratified formula F, where P_1^2-stratification is defined by interchanging P_1^2 and ε in the definition of ε-stratification. Then $x \not\varepsilon\, x$ is P_1^2-stratified and so there is a Q such that $P_1^2 xQ \longleftrightarrow x \not\varepsilon\, x$. But $P_1^2 xQ$ is ε-stratified, so there is an R such that $x \,\varepsilon\, R \longleftrightarrow P_1^2 xQ$. Now take x as R, whence $R \,\varepsilon\, R \longleftrightarrow P_1^2 RQ$. But $P_1^2 RQ \longleftrightarrow R \not\varepsilon\, R$. In contrast, the limitation of size doctrine as a means of avoiding inconsistency [2] does not appear to be vulnerable in the same way.

5 ***General models of standard set theories*** A standard set theory is obtained from a protostandard one by assuming the replacement schema (5) instead of the *Aussonderung* schema (5′). Let $ZF_{(-)}$ be the standard set theory obtained in this way from $Z_{(-)}$. First note that, parallel to the first part of Theorem 2, we have

Theorem 4 *A general model of a standard set theory is a general model of* $ZF_{(-)}$.

Proof: Suppose $\langle D, E\rangle$ is a general model of a standard set theory S, let Gwx be any formula of $\mathcal{L}$, and interpret P_1^2 as the relation $g = \{\langle x, y\rangle : \langle x, y\rangle \in D \wedge Gxy\}$ on D. Then, since $P_1^2 wx$ is ε-allowable,

$$\forall w \forall x \forall x'(P_1^2 wx \wedge P_1^2 wx' \,.\!\rightarrow x' \equiv x) \rightarrow \exists y \forall x(x \,\varepsilon\, y \longleftrightarrow \exists w(w \in v \wedge P_1^2 wx))$$

holds, and so (5) holds since $P_1^2 wx$ is equivalent in D to Gwx. Consequently, S can be extended to a theory in which every formula of $\mathcal{L}$ is ε-allowable and for which $\langle D, E\rangle$ remains a model, and hence $\langle D, E\rangle$ is a general model of $ZF_{(-)}$.

Next, instead of pursuing the full parallel to Theorem 2, we add the axiom of regularity to obtain a considerably stronger result. Thus, let ZF be the set theory with axioms (1)-(8) where in (4) all formulas of $\mathcal{L}$ are ε-allowable.

Theorem 5 *A model of ZF is general if and only if it is isomorphic to a natural model* R_θ *where* θ *is strongly inaccessible.*

Proof: Observe that if $\langle D, E\rangle$ is a general model of ZF, then E is well-founded. For, suppose

$$x \neq \phi \wedge \forall w(w \in x \rightarrow \exists v(v \in x \wedge \langle v, w\rangle \in E)).$$

Interpret P_1^1 as x, so that

$$\exists w P_1^1 w \wedge \forall w(P_1^1 w \rightarrow \exists v(P_1^1 v \wedge v \,\varepsilon\, w)),$$

which is impossible in view of (8).

Now a general model $\langle D, E\rangle$ of ZF is a general model of Z, and so we can appeal to Theorem 2 and take $D = R_\theta$ for some ordinal θ. To prove that θ is strongly inaccessible, it suffices to show that $R_{\theta+1}$ is a natural model of the Morse theory M, that is, the impredicative extension of NBG. Since by assumption R_θ provides a model of ZF and the impredicative comprehension scheme for proper classes is obviously satisfied in $R_{\theta+1}$ the only question concerns replacement, which can be formulated as a single axiom: if $f \subseteq R_\theta \times R_\theta$ is a function,

$$\text{(i)}\quad v \in R_\theta \rightarrow \exists y(y \in R_\theta \wedge \forall x(x \in y \longleftrightarrow \exists w(w \in v \wedge \langle w, x\rangle \in f)));$$

i.e., the image of $v \in R_\theta$ under F is also in R_θ. Now, since $f \subseteq R_\theta \times R_\theta$, we can, by generality, interpret P_1^2 as F, whence (i) follows as a consequence of the replacement schema of ZF proper. Thus, θ is strongly inaccessible.

To prove the converse, assume θ is strongly inaccessible and let Φ be any interpretation of the P_j^i in R_θ. The axioms of extensionality, power set, sum set, infinity, and regularity obviously are satisfied. Consider any E, Φ-formula Fwx such that $\forall w \forall x \forall x'(Fwx \wedge Fwx' \ .\!\rightarrow x' = x)$. Observe that $f = \{\langle w, x\rangle: Fwx\} \in R_{\theta+1}$. Now, because θ is strongly inaccessible and f is a function, (i) holds. But then, substituting Fwx for $\langle w, x\rangle \in f$, we get the replacement schema for ZF, as required, and the theorem is established.

An immediate consequence of this result is

Corollary *A general model of ZF can be enlarged to a general model of M.*

6 ***Remarks*** Although existence of a general model for ZF is a powerful assumption, general models for weaker set theories are easily seen to exist within ZF. Thus, if the axiom of infinity is dropped from ZF, the remaining axioms are satisfied in R_ω. Consequently, since ω has all the properties of a strongly inaccessible number except that it is not nondenumerable, the proof of Theorem 5 can be adapted to show that the theory of finite pure sets has a general model. Or if ZF is weakened by substituting *Aussonderung* for the replacement schema, the resulting theory Z has $R_{\omega+\omega}$ as a natural model, which, by Theorem 2, is general.

We note also that atoms can be accommodated in the foregoing results by replacing (7) with an axiom asserting the existence of a set containing all atoms and prefacing (8) with a stipulation that v contain no atoms.

7 ***Conclusion*** Obviously the requirement that a set theory have a general model is extremely strong. By imposing it we narrow the otherwise embarrassingly large array of alternative axiomatizations of set theory to essentially two: extensions of ZF and, if proper classes are admitted, extensions of M. Moreover, Theorem 5 establishes a close connection between the concepts of generality and strong inaccessibility, with the consequence that all theorems about sets provable in M hold also in a general model of ZF. Thus, in a not unreasonable sense, the theorems concerning sets provable in M constitute a canonical formalization of the intuitive notion of a set, which lends some interest to the question of whether these theorems can be axiomatized without using proper classes or additional set-theoretic primitive predicates, and without assuming more about sets than can be proved in M. This relation of strong inaccessibility to generality is somewhat surprising in view of the different concerns that motivate them. The concept of strong inaccessibility arises primarily in relation to considerations that can be formulated entirely in terms of set theory itself and without reference to extra-set theoretical subject matter, such as the conception of sets as generated iteratively and the numerous important set-theoretical consequences of the existence of strongly inaccessible numbers. In contrast, the notion of generality expresses a semantic condition concerning the relation of set theory to extra-set theoretical subject matter, and as we noted earlier, the primary motivation for the assumption of generality is philosophical: without it we are in a poor position to justify interest in set theory as a medium for explicating such matters as the use of mathematics in natural science.

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