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Functional Completeness and Non-Łukasiewiczian Truth Functions

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A *three-valued truth function* is a function from $\{T, I, F\}$ to $\{T, I, F\}$. We define a *Łukasiewiczian function* as a three-valued truth function that can be defined by composition from $\neg$ and $\supset$, where:

$\supset$	T	I	F	$\neg$
T	T	I	F	F
I	T	T	I	I
F	T	T	T	T .

It is well known that $\{\neg, \supset\}$ is functionally incomplete, i.e., that not all three-valued truth functions are Łukasiewiczian. (For example, it is easily verified that no function having I as its value when its arguments are classical is Łukasiewiczian.) It is also known that the addition of Słupecki's function **T**

	T
T	I
I	I
F	I

to $\{\neg, \supset\}$ results in a set that is functionally complete [2]. The question arises whether this is an accidental feature of **T**. The purpose of this note is to show that it is not.*

Theorem 1 *For every non-Łukasiewiczian function f, the set $\{\neg, \supset, f\}$ is functionally complete.*

*The author is indebted to the editor for the observation that $Ł_7$ provides a counterexample to the generalization of Theorem 1 and for several other improvements.

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A *pure* function is a function that always assumes a classical value when each of its arguments is classical. Inspection of the tables for $\neg$ and $\supset$ makes it evident that all Łukasiewiczian functions are pure. It is less evident that the converse is also true.

Lemma *All pure functions are Łukasiewiczian.*

In addition to $\neg$ and $\supset$ the proof will appeal to the familiar Łukasiewiczian functions & and v and to the less familiar Łukasiewiczian functions f_T, f_I, f_F, and f_+ where $f_T(p) = \neg(p \supset \neg p)$, $f_I(p) = [(p \supset \neg p) \& (\neg p \supset p)]$, $f_F(p) = \neg(\neg p \supset p)$, and $f_+(p) = (p \& \neg p)$. Thus:

	f_T	f_I	f_F	f_+
T	T	F	F	F
I	F	T	F	I
F	F	F	T	F .

Let f be any pure three-valued truth function of degree n, and consider an arbitrary row i from the table that defines f.

$p_1 \ldots p_n$	$f(p_1, \ldots, p_n)$	
$\vdots \quad \vdots$	$\vdots$	
$\alpha_1 \quad \alpha_n$	β	(row i)
$\vdots \quad \vdots$	$\vdots$	

We can write a *representative* formula R_i for row i where R_i has the value β on row i and the value F on every other row:

Case 1. $\beta = T$. Let $R_i = (V(p_1) \& \ldots \& V(p_n))$, where $V(p_j)$ is $f_T(p_j)$, $f_I(p_j)$, or $f_F(p_j)$ according as p_j is T, I, or F.

Case 2. $\beta = I$. From the assumption that f is pure it follows that at least one of $\alpha_1, \ldots, \alpha_n$ is I. So let $R_i = (V(p_1) \& \ldots \& V(p_n))$, where $V(p_j)$ is $f_T(p_j)$, $f_+(p_j)$ or $f_F(p_j)$ according as p_j is T, I, or F.

Case 3. $\beta = F$. Let $R_i = \neg(p_1 \supset p_1)$.

It is now clear that f can be defined as $(R_1 \vee \ldots \vee R_m)$ where $R_1, \ldots, R_m$ are the representative formulas for the $m(= 3^n)$ rows of the table that defines f. Thus f is Łukasiewiczian.

We are now in a position to prove the theorem. Let f be any non-Łukasiewiczian function of degree n. We have just seen that f must be impure. That is, there are classical values $\alpha_1, \ldots, \alpha_n$ such that the value of $f(p_1, \ldots, p_n)$ is I where the values of $p_1, \ldots, p_n$ are respectively $\alpha_1, \ldots, \alpha_n$. Then, for each j let p_j^* be $(p \supset p)$ or $\neg(p \supset p)$ according as α_j is T or F. It is clear that the value of $f(p_1^*, \ldots, p_n^*)$ is uniformly I and, thus, that Słupecki's **T** can be defined in terms of the extended set $\{\neg, \supset, f\}$ by $\mathbf{T}(p) = f(p_1^*, \ldots, p_n^*)$. But, as remarked earlier, $\{\neg, \supset, T\}$ is functionally complete. So the theorem is established.

It was noted earlier that inspection of the tables for $\neg$ and $\supset$ makes it evident that the converse of the lemma also holds. So:

Theorem 2 *A function is Łukasiewiczian if and only if it is pure.*

Thus there is an easy test for deciding whether a function is definable from $\neg$ and $\supset$.

These results cannot be generalized to the n-valued systems $Ł_n$ of Łukasiewicz. The truth values of $Ł_n$ are $1, \ldots, n$ and the $Ł_n$-functions are those that can be defined by composition from $\neg$ and $\supset$ where

$$\neg i = (n - i) + 1$$

and

$$(i \supset j) = \max[1, (j - i) + 1].$$

Counterexamples to the lemma and therewith the second theorem can be found in any $Ł_n$ where n is odd and greater than 3. For it is easily verified that under these conditions $\{1, (n + 1)/2, n\}$ is closed under $\neg$ and $\supset$. Thus no function having, for example, the value 2 when its arguments are from $\{1, (n + 1)/2, n\}$ is definable in $Ł_n$. But some of these functions are pure. So neither the lemma nor the second theorem holds for $Ł_n$.

$Ł_7$ provides a counterexample to the first theorem. For the addition of $f_3(i) = 3$ together with $f_4(i) = 4$ to $\{\neg, \supset\}$ yields a functionally complete set while the addition of either one alone does not. This is an immediate consequence of a theorem proved by Clay [1].

REFERENCES

[1] Clay, Robert E., "Note on Słupecki *T*-functions," *The Journal of Symbolic Logic*, vol. 27 (1962), pp. 53-54.

[2] Słupecki, Jerzy, "The full three-valued propositional calculus" in *Polish Logic: 1920-1939*, ed., S. McCall, Oxford University Press, Oxford, England (1967), pp. 335-337.

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