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A Completeness-Proof Method for Extensions of the Implicational Fragment of the Propositional Calculus

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The traditional proof that the classical propositional calculus (*PC*) is strongly complete (i.e., if $\alpha \vDash A$, then $\alpha \vdash A$) is based on the notion of a maximal consistent set of formulas, and hence on certain properties of strong (i.e., *PC*-)negation. In this paper* I present a completeness-proof method which does not refer to maximal consistent sets, but only to sets which are: (i) nontrivial (not all formulas are members), (ii) deductively closed (all syntactical consequences are members), and (iii) implication saturated (for all B, $A \supset B$ is a member if A is not a member). If this proof method is applied to logics that contain strong negation, the sets turn out to be consistent with respect to strong negation. I shall first apply the proof method to a specific extension of the implicational fragment of *PC*, and next show that it also applies to the implicational fragment itself and to a large number of logics that are extensions of the implicational fragment. If such a logic is characterized by a semantics, the articulation of an axiomatic system is straightforward (in view of the proof method) and *vice versa*.

The completeness-proof method is especially fit for paraconsistent logics that are based on material implication (see [1]-[6]).[1] Paraconsistent logics are logics according to which at least some inconsistent theories are nontrivial (some sentences of the language are not derivable from the axioms of the

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theory). In view of the traditional conception of the relation between derivability and implication, a logic is paraconsistent if and only if $p \supset (\sim p \supset q)$ is not a theorem. On the other hand $p \vee \sim p$ (or $(p \supset \sim p) \supset \sim p$ if disjunction is absent) is a theorem of most but not all paraconsistent logics. Some paraconsistent logics contain a weak negation, which I shall denote by '$\sim$', as well as strong negation, which I shall denote by '$\neg$'; both $p \vee \neg p$ and $p \supset (\neg p \supset q)$ are then theorems. In such cases it is preferable to say that the logic is paraconsistent with respect to one negation ($\sim$) and not paraconsistent with respect to the other ($\neg$). In some logics that are paraconsistent with respect to $\sim$, strong negation is *definable*; e.g., if $(p \,\&\, q) \supset (\sim(p \,\&\, q) \supset r)$ is a theorem, then $\neg p$ may be defined as $\sim(p \,\&\, p)$. Strong negation cannot be defined in terms of weak negation in *strictly* paraconsistent logics, i.e., logics in which no formula of the form $A \supset (\sim A \supset B)$ is a theorem, except in case A and B share a variable. Notice, incidentally, that $\sim(p \,\&\, \sim p)$ is a theorem of some (even strictly) paraconsistent logics, e.g., of the system S described below. There are quite intuitive semantic characterizations of several paraconsistent logics based on material implication. The basic idea is that $v(\sim A) = 1$ if $v(A) = 0$, but *not conversely,* whereas, if strong negation is present, $v(\neg A) = 1$ if and only if $v(A) = 0$.

I use small Latin letters ($p, q, r, \ldots$) for propositional variables, large Latin letters ($A, B, C, \ldots$) for formulas, small Greek letters for sets of formulas, and large Greek letters for sets of sets of formulas. The set of all formulas is denoted by $\mathscr{F}$.

Let me now apply the completeness-proof method to a specific paraconsistent logic.[2] The axiomatic system is:

Axioms:

I.1	$(p \supset q) \supset ((q \supset r) \supset (p \supset r))$
I.2	$((p \supset q) \supset p) \supset p$
I.3	$p \supset (q \supset p)$
II.1	$(p \,\&\, q) \supset p$
II.2	$(p \,\&\, q) \supset q$
II.3	$p \supset (q \supset (p \,\&\, q))$
III.1	$p \supset (p \vee q)$
III.2	$q \supset (p \vee q)$
III.3	$(p \supset r) \supset ((q \supset r) \supset ((p \vee q) \supset r))$
IV.1	$p \supset \sim\sim p$
IV.2	$\sim\sim p \supset p$
V.1	$\sim(p \supset q) \supset (p \,\&\, \sim q)$
V.2	$p \supset (\sim q \supset \sim(p \supset q))$
VI.1	$(\sim p \vee \sim q) \supset \sim(p \,\&\, q)$
VI.2	$(\sim p \supset r) \supset ((\sim q \supset r) \supset (\sim(p \,\&\, q) \supset r))$
VII.1	$\sim(p \vee q) \supset (\sim p \,\&\, \sim q)$
VII.2	$\sim p \supset (\sim q \supset \sim(p \vee q))$
VIII.1	$p \vee \sim p$

Rules. Detachment and Uniform Substitution.

The semantics is:

0. v: $\mathcal{F} \to \{0, 1\}$
1. $v(A \supset B) = 1$ iff $v(A) = 0$ or $v(B) = 1$
2. $v(A \& B) = 1$ iff $v(A) = v(B) = 1$
3. $v(A \vee B) = 1$ iff $v(A) = 1$ or $v(B) = 1$
4. $v({\sim}{\sim}A) = v(A)$
5. $v({\sim}(A \supset B)) = v(A \& {\sim}B)$
6. $v({\sim}(A \& B)) = v({\sim}A \vee {\sim}B)$
7. $v({\sim}(A \vee B)) = v({\sim}A \& {\sim}B)$
8. If $v(A) = 0$, then $v({\sim}A) = 1$.

Implication, conjunction, and disjunction behave classically (clauses 1-3), but negation does not in that both a proposition and its negation may be true (the converse of clause 8 does not hold). Still, the weak negation of S does share several properties with the strong negation of the propositional calculus: (i) either a proposition or its negation is true (clause 8), and (ii) the traditional "laws of thought" concerning the negation of complex formulas are retained: the law of double negation (clause 4) and the laws that allow us to drive negations through implications, conjunctions, and disjunctions (clauses 5-7). It is provable that this logic is *strictly* paraconsistent and that it is *maximally* so in that any of its extensions is either the propositional calculus or trivial (all formulas are theorems).

Theorem 1 *If* $\alpha \vdash A$, *then* $\alpha \vDash A$.

Proof as for *PC*.

Corollary 1 *If* $\vdash A$, *then* $\vDash A$.

In order to prove the converse of Theorem 1, I shall proceed in two steps. I first prove that $\alpha \vdash A$ if A is a member of all nontrivial, deductively closed, implication-saturated extensions of α (Lemma 7), and next that A is a member of each of these extensions of α if $\alpha \vDash A$ (Lemma 10). For the first step we need the following definitions.

Definition α is trivial iff $\alpha = \mathcal{F}$.
Definition $Cn(\alpha)$ is the set of all A such that $\alpha \vdash A$.
Definition α is deductively closed iff $\alpha = Cn(\alpha)$.
Definition ξ_A is the set of all C such that, for some B, $C = A \supset B$.
Definition α is implication-saturated iff $\xi_A \in \alpha$ whenever $A \notin \alpha$.

In other words, if A is not a member of the implication-saturated set α, then all formulas of the form $A \supset B$ are members of α.

Definition Γ is the set of all nontrivial, deductively closed, implication-saturated sets of formulas.

In other words, any member of Γ contains all of its own consequences; it contains, for all B, $A \supset B$ whenever it does not contain A, and it does not contain all formulas. In the following completeness proof the members of Γ play the same role as maximal consistent sets play in the traditional completeness proof for *PC*, and ξ_A functions with respect to members of Γ exactly as $\neg A$ functions with respect to maximal consistent sets. The members of Γ are maximally nontrivial in that, for any $\gamma \in \Gamma$, $Cn(\gamma \cup \{A\})$ is trivial if $A \notin \gamma$. With respect to systems containing both strong negation and material implication (e.g., da Costa's systems $C_n (1 \leqslant n < \omega)$; see [4], p. 500) it is provable for any $\gamma \in \Gamma$, that $\neg A \in \gamma$ iff $\xi_A \in \gamma$, and hence that Γ is identical with the set of all sets that are maximally consistent with respect to strong negation (but some of which are inconsistent with respect to weak negation).

Definition Γ_α is the set of all $\gamma \in \Gamma$ such that $\alpha \subseteq \gamma$.

That is, the set of all members of Γ that are extensions of α.

The proofs of Lemmas 1 and 2 are obvious and left to the reader.

Lemma 1 *$B_1, \ldots, B_n \vdash A$ iff $B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_n \vdash (B_i \supset A)$.*
Lemma 2 *If $B_1, \ldots, B_n \vdash A$, then $B_1, \ldots, B_n, (A \supset C) \vdash C$.*
Lemma 3 *If $\alpha \cup \beta \vdash A$ and, for any $B \in \beta$, $((B \supset A) \supset A) \in \gamma$, then $\alpha \cup \gamma \vdash A$.*

Proof: Suppose $\alpha \cup \beta \vdash A$ and, for any $B \in \beta$, $((B \supset A) \supset A) \in \gamma$. There is a finite number of formulas $C_1, \ldots, C_n \in \alpha$ $(n > 0)$ and a finite number of formulas $D_1, \ldots, D_m \in \beta$ $(m > 0)$ such that $C_1, \ldots, C_n, D_1, \ldots, D_m \vdash A$. Hence, by Lemma 1, $C_1, \ldots, C_n, D_2, \ldots, D_m \vdash (D_1 \supset A)$. Consequently, by Lemma 2, $C_1, \ldots, C_n, D_2, \ldots, D_m, ((D_1 \supset A) \supset A) \vdash A$. Applying the same reasoning to the other D_i we obtain $C_1, \ldots, C_n, ((D_1 \supset A) \supset A), \ldots, ((D_m \supset A) \supset A) \vdash A$. Consequently, $\alpha \cup \gamma \vdash A$.

Lemma 4 *If $\alpha \cup \xi_A \vdash A$, then $\alpha \vdash A$.*

Proof: Suppose $\alpha \cup \xi_A \vdash A$ and let β be the set containing $((B \supset A) \supset A)$ for all $B \in \xi_A$. Hence $\alpha \cup \beta \vdash A$ by Lemma 3. As all members of β are theorems of the form $(((A \supset C) \supset A) \supset A)$, $Cn(\alpha \cup \beta) = Cn(\alpha)$. Hence $\alpha \vdash A$.

Corollary 2 *If $\alpha \nvdash A$, then $Cn(\alpha \cup \xi_A)$ is not trivial.*

Lemma 5 *If $Cn(\alpha)$ is not trivial, then Γ_α is not empty.*

Proof: Let the formulas be given in some determinate order $A_1, A_2, \ldots$. Let $\gamma_0 = \alpha$; let $\gamma_n = \gamma_{n-1} \cup \{A_n\}$ if $\gamma_{n-1} \vdash A_n$, and let $\gamma_n = \gamma_{n-1} \cup \xi_{A_n}$ if $\gamma_{n-1} \nvdash A_n$. Let γ be the set of all formulas which are in any set of the series $\gamma_0, \gamma_1, \ldots$. In view of Corollary 2 it is obvious that $\gamma \in \Gamma_\alpha$ if $Cn(\alpha)$ is not trivial.

Lemma 6 *Any $\gamma \in \Gamma$ has the following properties:*

1. *$A \in \gamma$ iff $\gamma \vdash A$*
2. *For some A, $A \notin \gamma$*
3. *$(A \supset B) \in \gamma$ iff $A \notin \gamma$ or $B \in \gamma$*
4. *$(A \,\&\, B) \in \gamma$ iff $A \in \gamma$ and $B \in \gamma$*
5. *$(A \vee B) \in \gamma$ iff $A \in \gamma$ or $B \in \gamma$*
6. *$\sim\sim A \in \gamma$ iff $A \in \gamma$*
7. *$\sim(A \supset B) \in \gamma$ iff $A \in \gamma$ and $\sim B \in \gamma$*
8. *$\sim(A \,\&\, B) \in \gamma$ iff $\sim A \in \gamma$ or $\sim B \in \gamma$*
9. *$\sim(A \vee B) \in \gamma$ iff $\sim A \in \gamma$ and $\sim B \in \gamma$*
10. *If $A \notin \gamma$, then $\sim A \in \gamma$.*

Proof: I only prove items 5 and 10. Proofs of the others are either obvious or analogous to the proof of 5 or 10. For 5, we clearly have $(A \vee B) \in \gamma$ if $A \in \gamma$ or $B \in \gamma$ (from 1 and Axioms III.1-2). To prove the converse, suppose that $(A \vee B) \in \gamma$, $A \notin \gamma$ and $B \notin \gamma$. As γ is implication-saturated, $(A \supset A) \in \gamma$ and $(B \supset A) \in \gamma$. But then $A \in \gamma$ by 1 and Axiom III.3, which contradicts the supposition. For 10, notice that $(A \vee \sim A) \in \gamma$ (from property 1 and Axiom VIII.1) and hence, by property 5, that $A \in \gamma$ or $\sim A \in \gamma$.

Lemma 7 *$\alpha \vdash A$ iff, for all $\gamma \in \Gamma_\alpha$, $A \in \gamma$.*

Proof: One direction is obvious. For the other, suppose $\alpha \nvdash A$. Hence $Cn(\alpha \cup \xi_A)$ is not trivial (by Corollary 2) and consequently $\Gamma_{\alpha \cup \xi_A} \neq \emptyset$ (by

Lemma 5). But for any $\gamma \in \Gamma_{\alpha \cup \xi_A}$ we have $\gamma \in \Gamma_\alpha$ (by the definition of Γ_α) and $A \notin \gamma$ (by properties 2 and 3 of Lemma 6). Hence, for some $\gamma \in \Gamma_\alpha$, $A \notin \gamma$.

Now we come to the second step which consists in linking semantic derivability with the members of Γ. To this end I define, for each valuation function, the set of formulas to which it assigns the value 1.

Definition δ_v is the set of all A such that $v(A) = 1$.
Definition Δ is the set of all nontrivial δ_v.
Definition Δ_α is the set of all $\gamma \in \Delta$ such that $\alpha \subseteq \gamma$.

These definitions enable us to express any statement about valuation functions as statements about members of Δ, as in Lemma 8.

Lemma 8 *$\alpha \models A$ iff, for all $\gamma \in \Delta_\alpha$, $A \in \gamma$.*

Proof: Valuation functions that assign the value 1 to all formulas, a fortiori assign the value 1 to A. Hence, the (standard) definition of $\alpha \models A$ is equivalent to '$v(A) = 1$ for any valuation function v such that δ_v is not trivial and $v(B) = 1$ for all $B \in \alpha$'. This in turn is equivalent to 'for all $\gamma \in \Delta_\alpha$, $A \in \gamma$'.

Lemma 9 *If γ has properties 2-10 from Lemma 6, then $\gamma \in \Delta$.*

The proof is obvious and left to the reader.

Corollary 3 $\Gamma_\alpha \subseteq \Delta_\alpha$.

Lemma 10 *If $\alpha \models A$, then, for all $\gamma \in \Gamma_\alpha$, $A \in \gamma$.*

Proof: Suppose $\alpha \models A$. Hence, for all $\gamma \in \Delta_\alpha$, $A \in \gamma$ (from Lemma 8). But then, for all $\gamma \in \Gamma_\alpha$, $A \in \gamma$ (by Corollary 3).

Theorem 2 *If $\alpha \models A$, then $\alpha \vdash A$.*

Proof: Immediate in view of Lemma 7 and Lemma 10.

Corollary 4 *If $\models A$, then $\vdash A$.*

In the remaining part of this paper I discuss the applicability of the proof method to other propositional logics, and its use for turning semantic systems into axiomatic systems and vice versa. In order to clarify the matter, I mention some results which are easily provable but were not needed for the completeness proof:

$\gamma \in \Gamma$ iff γ has properties 1-10 from Lemma 6.
$\gamma \in \Delta$ iff γ has properties 1-10 from Lemma 6.
$\Gamma_\alpha = \Delta_\alpha$.
$\alpha \vdash A$ iff $\alpha \models A$ iff, for all $\gamma \in \Gamma_\alpha$, $A \in \gamma$.
$\vdash A$ iff $\models A$ iff, for all $\gamma \in \Gamma$, $A \in \gamma$.

This means that we are able to characterize a logic completely in terms of properties of the nontrivial, deductively closed, implication-saturated sets of formulas. Hence, we may expect that there are a number of logics for which it should be easy to turn an axiomatic characterization into a characterization in terms of properties of the members of Γ, and to turn the latter into a semantic characterization, *and the other way around.* I shall prove two theorems in this connection.

Let us first consider the implicational fragment of *PC*. Its axiomatic characterization, which I shall call *IA*, consists of Axioms I.1-3 and of the two rules; its semantic characterization, *IS*, consists of the semantic clauses 0 and 1. In order to adapt the preceding proof to *IA* and *IS*, simply restrict the properties in Lemma 6 to 1-3, and drop from the proofs of Theorem 1 and Lemmas 6 and 9 all references to other axioms, semantic clauses, and properties of the $\gamma \in \Gamma$.

Let *IS*+ be the result of adding to *IS* a number of clauses of the following form:

(◦) If $v(A_1) = \ldots = v(A_n) = 1$ and $v(B_1) = \ldots = v(B_m) = 0$, then $v(C) = k$,

where k is either 0 or 1 and $0 \leqslant n, m$ (if $n = m = 0$, the clause reduces to $v(C) = k$). The following definition will further the readability of the proof of Theorem 3.

Definition $X =_{df} (((\ldots((B_1 \supset B_2) \supset B_2) \supset \ldots) \supset B_m) \supset B_m)$.

The first three dots denote left parentheses only; B_1 occurs only once in X, all other B_i twice.

Theorem 3 *For any IS+, there is an effective procedure to articulate an axiomatic system IA+ (an extension of IA) such that the preceding proof method applies to IA+ and IS+.*

Proof: We start from *IS*, *IA*, and properties 1-3 from Lemma 6. For any further semantic clause (of the form (◦)) contained in *IS*+, we proceed as follows, according as k is 0 or 1 and m is or is not equal to 0.

Case 1. $k = 1$ and $m > 0$. Add to the properties in Lemma 6:

If $A_1 \in \gamma, \ldots, A_n \in \gamma, B_1 \notin \gamma, \ldots, B_m \notin \gamma$, then $C \in \gamma$,

and add as an axiom to *IA*:

$A_1 \supset (A_2 \supset \ldots (A_n \supset ((X \supset C) \supset C)) \ldots)$.

The adaptation of the proofs of Theorem 1 and Lemma 9 is obvious. To the proof of Lemma 6 we add the following:

> Suppose that $A_1 \in \gamma, \ldots, A_n \in \gamma, B_1 \notin \gamma, \ldots, B_m \notin \gamma$. From $B_1 \notin \gamma$ follows $(B_1 \supset B_2) \in \gamma$ by property 3. From $(B_1 \supset B_2) \in \gamma$ and $B_2 \notin \gamma$ follows, again by property 3, $((B_1 \supset B_2) \supset B_2) \notin \gamma$ and hence $(((B_1 \supset B_2) \supset B_2) \supset B_3) \in \gamma$. Proceeding in the same way for $B_3, \ldots, B_m$ we finally arrive at $X \notin \gamma$ and hence $(X \supset C) \in \gamma$. But $A_1, \ldots, A_n, (X \supset C) \vdash C$ (from the axiom). Hence $C \in \gamma$.

Case 2. $k = 1$ and $m = 0$. Add to the properties in Lemma 6:

If $A_1 \in \gamma, \ldots, A_n \in \gamma$, then $C \in \gamma$,

and add as an axiom to *IA*:

$A_1 \supset (A_2 \supset \ldots (A_n \supset C) \ldots)$.

The adaptation of the proofs of Theorem 1 and Lemmas 6 and 9 is obvious.

Case 3. $k = 0$ and $m > 0$. Add to the properties in Lemma 6:

If $A_1 \in \gamma, \ldots, A_n \in \gamma, B_1 \notin \gamma, \ldots, B_m \notin \gamma$, then $C \notin \gamma$,

and add as an axiom to *IA*:

$$A_1 \supset (A_2 \supset \ldots (A_n \supset (C \supset X)) \ldots).$$

The adaptation of the proof of Theorem 1 and Lemma 9 is obvious. Add to the proof of Lemma 6:

Suppose $A_1 \in \gamma, \ldots, A_n \in \gamma, B_1 \notin \gamma, \ldots, B_m \notin \gamma$. It follows from the axiom that $A_1, \ldots, A_n \vdash (C \supset X)$. Hence $(C \supset X) \in \gamma$. But $X \notin \gamma$ (proof as in Case 1). Hence $C \notin \gamma$ (by property 3).

Case 4. $k = 0$ and $m = 0$. Add to the properties in Lemma 6:

If $A_1 \in \gamma, \ldots, A_n \in \gamma$, then $C \notin \gamma$,

and add as an axiom to *IA*:

$$A_1 \supset (A_2 \supset \ldots (A_n \supset (C \supset D)) \ldots)$$

where D is a variable that occurs neither in C nor in any A_i. Again, the adaptation of the proofs of Theorem 1 and Lemma 9 is obvious. Add to the proof of Lemma 6:

Suppose $A_1 \in \gamma, \ldots, A_n \in \gamma$ and consider any E such that $E \notin \gamma$ (there is such a formula by property 2). $A_1 \supset (A_2 \supset \ldots (A_n \supset (C \supset E)) \ldots)$ is a theorem of *IA*+ (from the axiom by Uniform Substitution), and hence $A_1, \ldots, A_n \vdash (C \supset E)$. Consequently, $(C \supset E) \in \gamma$. From this and $E \notin \gamma$ follows $C \notin \gamma$ (by property 3).

This completes the proof.

Let us now turn to the opposite case in which an axiomatic system is given. Let *IA*+ be any axiomatic system arrived at by adding axioms to *IA* (these axioms may contain any propositional connectives and any nonlogical constants). For the proof of Theorem 4 we need one further definition. Consider any one-to-one relation between variables and metavariables.

Definition $\uparrow A$ is the result of replacing each occurrence of each variable by an occurrence of the corresponding metavariable.

Theorem 4 *For any IA+, there is an effective procedure to articulate a semantics IS+ such that the completeness-proof method applies to IA+ and IS+.*

Proof: We start again from *IA*, *IS*, and properties 1-3 from Lemma 6. For any further Axiom A, add to the properties in Lemma 6:

$$\uparrow A \in \gamma$$

and add as a semantic clause to *IS*:

$$v(\uparrow A) = 1.$$

The adaptation of the proofs of Theorem 1, Lemma 6, and Lemma 9 is obvious.

By way of an example, consider Schütte's system Φ_v, which consists of the two rules and of the following Axioms: I.1-2, II.1-3, III.1-3, IV.1-2, V.1-3, VI.1-3, VII.1-3, and VIII.1, together with:

I.3′ $\curlywedge \supset p$
IV.3′ $\sim\curlywedge$.

The application of the present method leads immediately to the result that the semantics of this system consists of clauses 0-8 together with '$v(\curlywedge) = 0$'. (Given properties 1-10 from Lemma 6, '$\curlywedge \notin \gamma$' indeed turns out to be equivalent to the conjunction of '$\sim\curlywedge \in \gamma$' and '$(\curlywedge \supset A) \in \gamma$'. Schütte's Φ_r is exactly as Φ_v except for having VIII.2 instead of VIII.1 as an axiom:

VIII.2 $(p \,\&\, \sim p) \supset q$.

Applying the present completeness-proof method, we readily find that the semantics of this system consists of clauses 0-7, together with 'If $v(A) = 1$, then $v(\sim A) = 0$' and '$v(\sim\curlywedge) = 1$'. In the same way, the proof method applies to all systems presented in [1]-[6], except for C_ω.

The proof method applies to still other kinds of logics. I mention only one point in this connection. Any deduction rule of the form

If $\vdash A_1, \ldots, \vdash A_n$, then $\vdash B$

corresponds to a semantic clause:

If, for all v', $v'(A_1) = \ldots = v'(A_n) = 1$, then $v(B) = 1$,

and to the following property of the $\gamma \in \Gamma$:

If, for all $\delta \in \Gamma$, $A_1 \in \delta, \ldots, A_n \in \delta$, then $B \in \gamma$.

The adaptation of the proofs of Theorem 1 and Lemmas 6 and 9 is obvious.

As a final comment I mention that a semantics arrived at in the way described in the proof of Theorem 4 will not always be very "natural". On the other hand, the characterization of a logic by means of a set of properties of the nontrivial, deductively closed, implication-saturated sets (i.e., of the $\gamma \in \Gamma$) will make it quite easy to find a more natural two-valued semantics, if there is one. In this connection I refer to what I said about Φ_v. Consider also da Costa's and Alves's semantics for da Costa's calculi $C_n (0 \leqslant n < \omega)$, which were devised independently of the present completeness-proof method (see [5]). These semantic systems contain the clause

If $v(B^{(n)}) = v(A \supset B) = v(A \supset \sim B) = 1$, then $v(A) = 0$,

which seems quite unnatural (and is unnatural in the sense that, as will become clear immediately, the value assigned to A is wholly irrelevant to the value assigned to $B^{(n)}$). The application of the present proof method reveals immediately that the preceding clause may be replaced by the more natural

$v(A^{(n)}) = 1$ iff $v(A) = 0$ or $v(\sim A) = 0$ (i.e., iff $v(A) \neq v(\sim A)$).

It also reveals that the axiom scheme

$$B^{(n)} \supset ((A \supset B) \supset ((A \supset \sim B) \supset \sim A))$$

may be replaced by

$$(A \,\&\, \sim A) \supset (A^{(n)} \supset B).$$

This reformulation too is clearer.

NOTES

1. Other paraconsistent logics are based on some relevant implication (see [7]), on intuitionist implication, e.g., da Costa's C_ω (see [4]), or on some many-valued implication, e.g., Kleene's three-valued logic.
2. This paraconsistent logic is Schütte's Φ_v (see [8], p. 74) restricted to formulas that do not contain the constant $\curlywedge$ (which may be regarded within this system as the conjunction of all formulas).

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