Notre Dame Journal of Formal Logic
Volume XX, Number 4, October 1979
NDJFAM

ENTAILMENT AND PROOF

HUNG HIN-CHUNG

1 *The proof, the thesis, and the paradox* Apparently, "Today is Monday and today is not Monday" does not entail "Plato loves Socrates". Yet the following proof that it does, looks valid. Hence the paradox.

Proof: Let "$p \cdot -p$" stand for the first proposition, and "q" stand for the second. Then

(1) $(p \cdot -p)$ entails p.
(2) p entails $(p \vee q)$.
(3) $(p \cdot -p)$ entails $-p$.
(4) $-p$ and $(p \vee q)$ together entail q.
(5) Applying the principle of transitivity of entailment to (1) and (2), we get: $(p \cdot -p)$ entails $(p \vee q)$.
(6) Applying the principle of transitivity of entailment to (3), (5) and (4), we get: $(p \cdot -p)$ entails q.
(The principle of transitivity of entailment is: If $P_1, \ldots, P_n$ entail Q_1, and $P_1, \ldots, P_n$ entail $Q_2, \ldots \ldots$, and $P_1, \ldots, P_n$ entail Q_m, and if $Q_1, \ldots, Q_m$ entail R, then $P_1, \ldots, P_n$ entail R.)

It can be seen that the proof applies to all contradictions. Since it is adopted from Lewis [6], we shall call the thesis that any contradiction entails any proposition whatsoever, Lewis' thesis. We shall call the proof, Lewis' proof, and the paradox, Lewis' paradox.

The aim of the paper is to arrive at the *best* interpretation of entailment by analysing the role it plays in deductive proofs. We shall see that according to this interpretation, Lewis' proof is valid, the paradox is a misunderstanding, and that Lewis' thesis stands (quite contrary to our intuition).

2 *Reasons for the rejection of the thesis* Many philosophers reject Lewis' thesis. The major reasons are as follows:

a. The thesis is counter-intuitive. It simply does not sound right to

Received June 6, 1975

say that "Today is Monday and today is not Monday" entails "Plato loves Socrates" ([3], p. 212).

b. For a proposition A to entail another, B, A must be connected in meaning with B ([3], p. 214). But $(p \cdot -p)$ and q are not so connected.

c. For A to entail B, A must be relevant to B (see [1], [2], and [3], p. 222). But $(p \cdot -p)$ is not relevant to q.

d. For A to entail B, A must be a reason for B [11]. But $(p \cdot -p)$ is not a reason for q.

3 *Roots of the reasons* "Entailment" is a comparatively vague word, though commonly used in philosophy. Its vagueness allows quite a few interpretations and, according to a number of them, Lewis' thesis does not hold. Here are the interpretations:

1. A entails B if and only if A is a valid deductive argument[1] for B. (For simplicity, here as well as elsewhere in this section, we present entailment as if it were a relation between two propositions, even though strictly speaking, it is a relation between a set of propositions and a single proposition). $(p \cdot -p)$ is obviously not even an argument for q, let alone a valid deductive argument for q. So $(p \cdot -p)$ does not entail q. It is quite obvious that this interpretation lurks behind the mind of many when they give reasons b, c or d (Section **2**) for the rejection of Lewis' thesis. However, if we adopt this interpretation, steps (5) and (6) of Lewis' proof (Section **1**) do not hold because valid deductive argument is not transitive. We can see its nontransitivity from the following example: We can validly argue that today is Friday by pointing out that yesterday was Thursday. And we can validly argue that yesterday was Thursday by pointing out today is Friday. But I don't think anyone will accept the argument "today is Friday, therefore today is Friday". Not only do steps (5) and (6) not hold, I think most of us will reject (1) and (3) as well. So according to this interpretation, we not only have to reject Lewis' thesis, we have to reject the proof of the thesis as well. But then we have no paradox.

2. A entails B if and only if A is a deductive proof[1] of B. Obviously no one will accept $(p \cdot -p)$ as a proof of q. So again Lewis' thesis does not hold. This is one of the interpretations which give rise to reasons b, c, and d for the rejection of the thesis. But as in the last interpretation, steps (5) and (6) no longer hold because deductive proof is nontransitive. We can see it from the following example: Let K be the axioms of Euclid. Let P be Pythagoras' theorem. A student is asked to prove P. Would "K, therefore P" be acceptable?

3. It is not uncommon to have a few typical cases in mind when we interpret "entailment". These cases are to serve as paradigms. Here are some of them:

a. "Montreal is to the north of New York, and New York is to the north of Washington" entails "Montreal is north of Washington" [5]. "No P are M and some M are S" entails "Some S are not P". In each of the two cases, it is claimed that the antecedents are not just a simple aggregate.

They function together as a joint force ([9], p. 444). Neither antecedent yields the consequent. It is when they are combined that the consequent is brought forth. The consequent is a new discovery [5]. Let us say that in such cases the consequent is a logical resultant of the antecedents.

b. "This is red" entails "This is coloured". "This is green" entails "This is extended". "This is triangular" entails "This is trilateral". We shall say these are cases where the consequent is a logical part of the antecedent.

c. "This is a bachelor" entails "This is an unmarried man". "No S are P" entails "No P are S". "This is longer than that" entails "That is shorter than this". We shall say these are cases where the consequent is logically synonymous to the antecedent.

Suppose some philosopher has these three relations in mind when he interprets entailment. He may then take the sum[2] of these three relations as the *definiens* of entailment. Let us say that B is a logical offspring of A if and only if B is either a logical resultant, or a logical part of, or a logical synonym of A. Let us interpret: "A entails B" as "B is a logical offspring of A". With this interpretation, obviously we have to reject Lewis' thesis. But as with the other two interpretations before, we have to reject Lewis' proof as well, because "logical offspring" is not transitive. (Of course, we have to reject steps (1), (2), (3), and perhaps (4) as well.) It can be seen that "logical offspring" is nontransitive from the following example: Let L be "There are no featherless bipeds", M be "There are no feathered bipeds", and N be "There are no bipeds". Now N is a logical offspring of L and M (because N is a logical resultant of L and M). M, on the other hand, is a logical offspring of N (because M is a logical part of N). But M is not a logical offspring of L and M. Since logical offspring is not transitive, Lewis' proof does not hold. So again there is no paradox.

4. Suppose we reinforce the notion "logical offspring" with the following notions:

a. P is said to be a logical factor of $(P \cdot Q)$ where "." is the truth-functional conjunction).

b. Q is said to be the logical remainder of $((P \vee Q) \cdot -P)$ (where "v" is the truth-functional disjunction).

c. $(P \vee Q)$ is said to be a logical understatement of P. Let us define: B is a logical offspring of A if and only if B is either a logical resultant, or a logical part, or a logical synonym, or a logical factor, or a logical remainder, or a logical understatement of A. Let us interpret: "A entails B" as "B is a logical offspring of A (according to this new definition of logical offspring)". Now steps (1), (2), (3) and (4) of Lewis' proof hold. But (5) and (6) still do not hold because this new notion of entailment is still nontransitive. This can be seen from the following example: Let K be "Today is Sunday". Then (L and M) is a logical offspring of $((L \text{ and } M) \cdot K)$ (because the former is a logical factor of the latter). But N is not a logical offspring of $((L \text{ and } M) \cdot K)$. So again Lewis' proof does not hold, and the paradox is resolved.

We see how each of the four interpretations of entailment (1-4, above) not only justifies the rejection of Lewis' thesis, but provides a solution to the paradox as well. Such approaches to the problem are quite common (Nelson, Duncan-Jones, von Wright, Lewy, Watling, Geach, Anderson, Belnap, etc.). But are these interpretations acceptable?

We notice that each of the interpretations, in solving the paradox, pays a price. Entailment for each of the four cases is nontransitive. Perhaps this price is too great. A nontransitive notion of entailment, while paradox-free, may prove to be impotent and useless (see Sections **5**, **6** and **7**). In throwing out the paradoxic bath water, we may have thrown out the entailment baby as well. We must look at the use of entailment in science and reasonings. How does it function there? Would a nontransitive notion of entailment serve the purposes it is originally designed for? In what context do we find the deployment of entailment? Isn't it essential that the notion be transitive in these contexts?

The expression "entailment" was introduced by Moore to mean the converse of "follow logically from", "deducible from", etc. [8]. And "follow logically from", "deducible from", etc., are employed (mostly and essentially) in deductive proofs and deductive arguments. They are not the same notions as deductive proof nor as deductive argument. Rather they occur inside deductive proofs and deductive arguments. They contribute to (and play the key role in) the notions "proof" and "argument".

In the following sections, we shall proceed to analyse "proof" and the role entailment plays in proofs. (The notion "argument" is similar.) From the analysis, we shall give the best interpretation of entailment possible. We shall see that under this interpretation Lewis' proof is valid, his thesis correct, and there is no paradox.

4 *The direct acceptability proof* There are many kinds of proof. Obviously, in this paper, we are interested only in one particular kind, viz., the deductive proof. (Henceforth, by proof, I shall mean deductive proof.) The deductive proof, according to Szabo [10] can be traced as far back as the Eleatic school of dialectic. There, in proofs, one starts with one or more hypotheses and through arguments arrives at the proposition which one set out to prove. The Greek word for hypothesis is ὑπόθεσις. Szabo pointed out: "The Greek word ὑπόθεσις derives from the preposition ὑπό and the verb τίθεσθαι and signifies, in fact, that which two conversationalists, the partners in a debate, mutually agree to accept as the basis and starting point of their debate ([10], p. 3). In such a proof, there is an audience. The aim is to get the audience to accept a certain proposition. The technique is to start with some propositions which the audience accept. These are the hypotheses (ὑπόθεσις). Suppose one intends to prove Q. One may start with hypotheses $P_1, \ldots, P_n$. From these n propositions, step by step, one arrives at Q. Let us illustrate this kind of proof:

A: "Here in London, there are two people who have exactly the same number of hairs on their heads."
B: "It may be the case. But it is highly improbable."

A: "It is not improbable. On the contrary, I am quite certain that it is the case."
B: "Prove it".
A: "Do you accept that there are at most one million hairs on any individual's head?"
B: "I accept."
A: "Isn't it true that the population of London is around ten million?"
B: "Yes."
A: "If you accept that there are at most one million hairs on any individual's head, then you must accept that there are at most only one million and one distinctive types of heads as far as their number of hairs is concerned."
B: "Yes."
A: "If there are at most one million and one distinct types of heads (as far as their number of hairs is concerned), and there are ten million heads (in London) to go into these one million and one types, isn't it true that at least some one of the types has to accommodate more than one head? In other words, at least two of the people (in London) have exactly the same number of hairs on their heads."[3]

We are acquainted with proofs in mathematics which starts with axioms and definitions. Euclid's geometry is a prototype. On the other hand, our proof above *does not* start with axioms nor definitions. A starts with something that B accepts. The audience of the proof is B. The aim of the proof is to convince B to accept a certain proposition. The proof has the following form:

(0) $P_1, \ldots, P_n$ are acceptable (to you).
(1) $P_1, \ldots, P_n$ acc Q_1
(2) Q_1 acc Q_2
. . . .
(m) Q_{m-1} acc Q_m
(m + 1) Q_m is acceptable (to you)[4].

Here "acc" is a certain relation between two sets of propositions. Members of the first set are sometimes called premisses or antecedents. The second set is a unit set whose only member is sometimes called the conclusion or consequent. For the sake of convenience, we shall not write "$\{P_1, \ldots, P_n\}$ acc $\{Q_1\}$" but shall abbreviate it to "$P_1, \ldots, P_n$ acc Q_1", when there is little risk of confusion.

We don't know yet what "acc" is. All we know is that such proofs hold if and only if "acc" transmits acceptability, i.e., if K is acceptable and if K acc H, then H is acceptable. Let us read "X acc Y" as "Y is *acceptable relative* to X". We shall call the relation "*relative acceptability*". The reading of the relation "acc" just proposed is a mere matter of pure convention. Other ways of reading "acc" would serve as well, and we should not because of the reading proposed take "acc" as anything more than a relation which transmits acceptability.

Now the proof works in the following manner: The audience is asked to accept $P_1, \ldots, P_n$. Then via steps (1) to (m), he is forced to accept Q_m through the transmissivity of acceptability of acc. Let us call such proofs *direct acceptability proofs*.

As I remarked earlier, this type of proof already existed, as claimed by Szabo, in the Eleatic school of dialectic. Someone wants to convince another of a certain proposition. He starts with one or more hypotheses and, by argument, arrives at the proposition concerned. The Greek ὑπόθεσις does not have the same meaning as the word "hypothesis" today. It doesn't mean a tentative statement or a theory. It means a proposition acceptable to both parties, the speaker and the audience, as the starting point of a debate. With different audiences, the speaker may start with different hypotheses. The popularity of this sort of proof has not waned with the ages. People usually employ such proofs when they want to convince an audience, whether in parliament or in the market-place.

There are many uses of the term "entailment". It cannot be denied that one of the senses employed coincides with that of relative acceptability in direct acceptability proofs. Suppose the direct acceptability proof is the only context where we ever employ entailment. Then entailment can be identified with relative acceptability. In this case, entailment can be any notion that transmits acceptability. In Section **3**, we went over three logical notions, viz., logical resultant, logical part, and logical synonymy. Each of these three notions can be seen to transmit acceptability if we identify acceptability with truth. So if we confine the use of entailment to direct acceptability proofs, any of the three notions can be taken as the notion "entailment". There is no apparent reason why we should prefer one or the other. None of the three has a higher claim. But if we confine ourselves to the use of only one of the three notions, the power of direct acceptability proof will be severely limited.

We should therefore identify acc as the sum of all logical relations which transmit acceptability[2] so that each of the three notions now becomes a subnotion of this comprehensive notion. We shall call this interpretation of acc the *first optimum interpretation*. It is optimum because this is the most comprehensive relation that satisfies the condition of acceptability-transmissivity. This notion of acc shall be known as the *first notion of optimum relative acceptability* or acc_1. And we can see that if "entailment" is confined to its use in direct acceptability proof, it can be taken as identical with acc_1. This we shall call the *first optimum interpretation of entailment*. This first optimum interpretation being the most comprehensive interpretation is obviously the best interpretation in the confinement of direct acceptability proof.

The direct acceptability proof, according to Lucas ([7], p. 11), was later transformed into what can be called the universal acceptability proof by Plato. Plato was concerned not with acceptability by a certain particular audience, rather he was concerned with acceptability of all. So for him, the starting point of a proof is not some propositions that happen to suit a certain audience. They must be propositions universally accepted, not only universally accepted now but universally accepted now and forever. The universal proof was later taken up by Euclid in his geometry. It has since become the prototype of that proof.

5 *The direct relative acceptability proof* Universal proof held in the fields of metaphysics, mathematics, and science until the discovery of non-arithmetical algebra,[5] and the discovery of non-Euclidean geometry.[6] There a new notion of proof was conceived. In such a proof, one does not start with universally acceptable propositions. One does not even start with locally acceptable propositions, i.e., propositions acceptable to a certain audience. One simply starts with one or more propositions. These propositions may be false. They may even be inconsistent.[7] In such cases, it is obvious that the proof is not a proof of truth or acceptability. The proof is a proof of relative acceptability. It has the following general form:

(1) $P_1, \ldots, P_n \text{ acc } Q_1$
(2) $Q_1 \text{ acc } Q_2$
. . . .
(m) $Q_{m-1} \text{ acc } Q_m$
(m + 1) $P_1, \ldots, P_n \text{ acc } Q_m$.
(In contemporary mathematics, $P_1, \ldots, P_n$ are known as axioms, and Q_m is known as a theorem.)

We shall call this type of proof the *direct relative acceptability proof*. We see that this form of proof is derived from the acceptability proof by dropping line (0) and modifying line (m + 1) into a statement of relative acceptability. Since we start with hypotheses $P_1, \ldots, P_n$ which may or may not be acceptable by anyone, we have to drop line (0), and if line (0) is dropped, line (m + 1) cannot hold unless it is modified, as is done.

In our discussion of the acceptability proof, we said that it is sufficient and necessary that "acc" should transmit acceptability. But this condition is not sufficient here for we have to require that the relation be transitive as well. It may appear that since "acc" transmits acceptability, it must be transitive. This is not true. The property of transmissivity does not imply the property of transitivity. An example should make this clear. Let us take the sequence of natural numbers. Let us say that number b is a second successor of a if there is a number x such that b is a successor of x and x is a successor of a. Now "second successor of" transmits the property "evenness", i.e., if a is even, and b is a second successor of a, then b is even also. But "second successor of" is not transitive. So in order that the relative acceptability proof works, "acc" must be transitive as well as acceptability-transmissive.

Again, "acc" can have more than one interpretation to satisfy these two conditions. For example, we could have taken acc as the identity relation so that $P \text{ acc } Q$ if and only if $P = Q$. It is obvious that the identity relation satisfies both conditions. However, we want our "proof" as strong as possible, whilst the identity relation as acc would render the relative acceptability proof barren. So let us look for the most comprehensive relation which is both transitive and acceptability-transmissive.

Indeed, we need not look far for the first optimum interpretation of acc, introduced in the previous section, is such a relation. Even though acceptability-transmissivity does not imply transitivity, the sum of all acceptability-transmissive relations is transitive. The reasons are as follows: the composition[8] of two acceptability-transmissive relations is again acceptability-transmissive. In other words, if R_1 and R_2 are subnotions of optimum relative acceptability, then $R_1 \cdot R_2$ is also a subnotion. And certainly, if aR_1b and bR_2c, then $a(R_1 \cdot R_2)c$.

Relative acceptability proof can be seen to be more general than both the acceptability proof and the universal proof. Should one think that $P_1, \ldots, P_n$ are acceptable (universally acceptable), they need only to reintroduce this as line (0), and we have an acceptability proof (universal proof) right away.

As far as the search for knowledge is concerned, the universal proof is an improvement on the acceptability proof, because knowledge is objective—it does not vary from audience to audience. The relative acceptability proof later gained preference over the universal proof because scholars soon realized that the term "universally acceptable proposition" is too restrictive to be of much use. Nowadays, the only form of (direct) deductive proof employed in mathematics and science is this third type, the direct relative acceptability proof. Hilbert later in his *Grundlagen der Geometrie* generalised this type of proof into proof, not of propositions, but of propositional forms. Still later he formalized this notion in what is known as proof theory. But this need not concern us here.

6 *The indirect acceptability proof* So far we have only been studying the direct proofs. We must not overlook the all important indirect proofs. It is indirect proofs which were employed by Zeno of Elea to refute opinions of common sense. Corresponding to the direct acceptability proof is the *indirect acceptability proof* which has the following form:

(1) $P_1, \ldots, P_n, -R$ acc Q_1
(2) Q_1 acc Q_2
. . . .
(m) Q_{m-1} acc Q_m; therefore,
(m + 1) $P_1, \ldots, P_n, -R$ acc Q_m (from (1) to (m) by principle of transitivity of acc). But,
(m + 2) Q_m is unacceptable. Therefore,
(m + 3) $\{P_1, \ldots, P_n, -R\}$ is unacceptable (from (m + 2), by the principle of retransmission of unacceptability of acc). But,
(m + 4) $P_1, \ldots, P_n$ are acceptable. Therefore,
(m + 5) $-R$ is unacceptable (from (m + 4) and (m + 3), by the principle of mutual exclusion of acceptability-unacceptability, and the principle of unacceptability composition of unacceptable aggregate). Therefore,
(m + 6) R is acceptable (from (m + 5), by the principle of unacceptable negation).

The various principles employed are:

a. The principle of transitivity of acc.

b. The principle of retransmission of unacceptability of acc.

c. The principle of mutual exclusion of acceptability-unacceptability: If X is acceptable (unacceptable), then X is not unacceptable (acceptable).

d. The principle of unacceptability composition of unacceptable aggregate: If $\{X_1, \ldots, X_{m-1}, X_m\}$ is unacceptable, and $X_1, \ldots, X_{m-1}$ are acceptable, then X_m is unacceptable.

e. The principle of unacceptable negation: If $-X$ is unacceptable, then X is acceptable.

It can be seen that the indirect acceptable proof works if and only if these five principles are satisfied. Principles c, d, and e govern exclusively the notions "acceptability", "unacceptability" and "negation", while principles a and b govern the notion acc as well.

If the proof is conducted in the language of the two-valued classical propositional calculus, with acceptability identified with truth, and unacceptability identified with falsity, the principles c, d, and e can be seen to be satisfied. In this case, we can again identify acc as acc_1 (the first optimum relative acceptability), for acc_1 satisfies both a and b.

But if the proof is conducted in the three-valued propositional calculus of Łukasiewicz, the situation is different. It would seem reasonable to take 1 as the value of acceptability, and to take both $\frac{1}{2}$ and 0 as the values of unacceptability. But if we so interpret acceptability and unacceptability, we would see that, though principles c and d are satisfied, e is not satisfied. It is not the case that if $-X$ is $\frac{1}{2}$ or 0, then X is 1, because if $-X$ is $\frac{1}{2}$, X would be $\frac{1}{2}$ as well. Since e is not satisfied, the indirect acceptability proof does not hold. From step (m + 5), that $-R$ is unacceptable, we are not warranted to infer (m + 6), that R is acceptable.

However, if we take 1 as the value of acceptability, and 0 as the value of unacceptability, the principles c, d, and e are all satisfied. There is now a chance that the indirect proof works. We have only to think of a relation, which transmits acceptability and satisfies both principles a and b. Let us see if acc_1 will do. Acc_1 certainly transmits acceptability and satisfies a. Unfortunately, it does not satisfy b. For according to the definition of acc_1, $(X \cdot -X)$ acc_1 $(Y \cdot -Y)$.[9] But when $(Y \cdot -Y)$ is 0, $(X \cdot -X)$ is not necessarily 0. Therefore, unacceptability has not been retransmitted.

Let us define acc_2 as the sum of all logical relations which both transmit acceptability and retransmit unacceptability. Let us call acc_2 the *second optimum relative acceptability*. Now acc_2 transmits acceptability and satisfies both a and b. Moreover, this is the most comprehensive logical relation that does so. Hence the qualification "optimum".

It can be shown that in the three-valued logic of Łukasiewicz, acc_2 is the relation which holds between X and Y if and only if $[X] \leq [Y]$ for all truth-value assignments, i.e., X acc_2 Y if and only if the truth-value of X is not greater than that of Y for all truth-value assignments to the propositional variables in X and Y.[10]

Let us say that acceptability, unacceptability, and negation are given the *standard interpretation*, if they satisfy:

1. The principles of mutual exclusion of acceptability-unacceptability.
2. The principle of unacceptability composition of unacceptable aggregate.
3. The principle of unacceptable negation.

So in the language where acceptability, unacceptability, and negation are given this standard interpretation, the three direct proofs as well as the indirect acceptability proof hold, so long as acc is given the *second optimum interpretation* (i.e., interpreted as acc_2). And since we traditionally identify "entailment" with "acc" in these four proofs, it is only proper that entailment should be interpreted as acc_2. We shall call this interpretation of entailment the *second optimum interpretation.*

7 *The indirect relative acceptability proof* Corresponding to the direct relative acceptability proof are two indirect proofs, viz., the weak indirect relative acceptability proof and the strong indirect relative acceptability proof. We saw from the previous section that in indirect acceptability proofs, we establish the acceptability of a proposition R by deriving an unacceptable proposition Q_m from $-R$ together with a set of acceptable propositions $\{P_1, \ldots, P_n\}$. In indirect relative acceptability proofs, unacceptable propositions such as Q_m are not strong enough. We need contradictions.

The weak indirect relative acceptability proof has the following form:

(0) $\{P_1, \ldots, P_n\}$ is not self-contradictory.
(1) $P_1, \ldots, P_m, -R$ acc Q_1
(2) Q_1 acc Q_2
. . . .
(m) Q_{m-1} acc Q_m; therefore,
(m + 1) $P_1, \ldots, P_n, -R$ acc Q_m (by principle of transitivity of acc). But,
(m + 2) Q_m is a contradiction. Therefore,
(m + 3) $P_1, \ldots, P_n$ acc R (from (m + 2) and (m + 1), by the weak principle of self-contradictory aggregate).

The various principles employed are:

a. The principles of transitivity of acc
b. The weak principle of self-contradictory aggregate: If $\{X_1, \ldots, X_{m-1}\}$ is not self-contradictory, and if $\{X_1, \ldots, X_{m-1}, -X_m\}$ acc a contradiction, then $X_1, \ldots, X_{m-1}$ acc X_m.

The strong indirect relative acceptability proof has the same form as the weak variation except that line (0) is missing, and line (m + 3) is obtained through the strong principle of self-contradictory aggregate: If $\{X_1, \ldots, X_{m-1}, -X_m\}$ acc a contradiction, then $X_1, \ldots, X_{m-1}$ acc X_m.

What interpretation should we now give to acc so that in addition to the principles discussed previously, the present two principles of self-contradictory aggregate are satisfied? The natural candidate is acc_2 (the second optimum relative acceptability). But before we test whether acc_2 satisfies these two principles, perhaps we should first analyse the notion of contradiction, since both principles rest on this notion. A contradiction is the joint assertion of a proposition and its denial. In current logic, denial of a proposition P is taken as the negation of P, i.e., $-P$. In our standard interpretation of acceptability, unacceptability, and negation, the principle of unacceptable negation partially defines the logical relation of negation with the other two concepts. But this principle, by itself, is not sufficient to bring out all the properties of negation if negation is to be identified with denial. For we want not only that when a denial is unacceptable, the corresponding assertion is acceptable, but also that when an assertion is acceptable, the corresponding denial is unacceptable, and also that an assertion is unacceptable when and only when its corresponding denial is acceptable. So let us introduce the principle of acceptability-negation alternation: X is acceptable if and only if $-X$ is unacceptable, and X is unacceptable if and only if $-X$ is acceptable. And let us strengthen the *standard interpretation* of acceptability, unacceptability, and negation with this principle.

How shall we interpret "contradiction" in this strengthened standard interpretation? A contradiction, being the joint assertion and denial of a proposition, is obviously the couple set $\{X, -X\}$, where X is any proposition. And we must take contradictions as unacceptable under all circumstances.[11] With this interpretation of contradiction, we have now a *standard interpretation of acceptability, unacceptability, negation, and contradiction.*

Now let us return to the problem: Does acc_2 satisfy the two further conditions, i.e., the two principles of self-contradictory aggregate, so that acc_2 works in the two indirect relative acceptability proofs, as well as the four proofs previously discussed? The answer is affirmative. The reasons are as follows: first of all we need only to show that acc_2 satisfies the strong principle, because the weak principle is a logical consequence of the strong one. The strong principle says: If $X_1, \ldots, X_{m-1}, -X_m$ acc contradiction, then $X_1, \ldots, X_{m-1}$ acc X_m. So we need only to show that:

(1) whenever $X_1, \ldots, X_{m-1}$ are acceptable, then X_m is acceptable, and
(2) whenever X_m is unacceptable, $\{X_1, \ldots, X_{m-1}\}$ is unacceptable, provided that $X_1, \ldots, X_{m-1}, -X_m$ acc_2 contradiction.

Now contradictions by definition are unacceptable under all circumstances. By the principle of retransmission of unacceptability, $\{X_1, \ldots, X_{m-1}, X_m\}$ is unacceptable in all circumstances, since $X_1, \ldots, X_{m-1}, X_m$ acc_2 contradiction.

(1) Suppose $X_1, \ldots, X_{m-1}$ are acceptable. Then since $\{X_1, \ldots, X_{m-1}, X_m\}$ is unacceptable, then $-X_m$ must be unacceptable (by the principle of unacceptable aggregate). Since $-X_m$ is unacceptable, X_m is acceptable (by the principle of acceptability-negation alternation).
(2) Suppose X_m is unacceptable. Then $-X_m$ is acceptable (by the principle of acceptability-negation alternation). Since $\{X_1, \ldots, X_{m-1}, -X_m\}$ is unacceptable, and $-X_m$ is acceptable, $\{X_1, \ldots, X_{m-1}\}$ must be unacceptable (by the principle of unacceptable aggregate).

This completes the proof.

So for a language where acceptability, unacceptability, negation, and contradiction have the strengthened standard interpretation, acc_2 yields all the three direct proofs as well as all the three indirect proofs. Moreover this is the most comprehensive relation that works in these six proofs in the sense that all other logical relations that also work are subnotions of acc_2. Since entailment performs exactly the same functions as acc in the six proofs, it is obvious that entailment should be equated to acc_2. That is, entailment should be given the second optimum interpretation, i.e., to be identified with the sum of all logical relations which both transmit acceptability and retransmit unacceptability.

8 *Conclusion* The search for an adequate interpretation of entailment started with Lewis' paradox. The paradox's two opposing arms are: $(p \cdot -p)$ entails q, and $(p \cdot -p)$ does not entail q. The justification of the first is Lewis' proof, and the justification of the second is a number of intuitively appealing interpretations of entailment.

Instead of relying on shaky intuitions, we analysed the relation between entailment and proof in detail. We succeeded in isolating the structure of entailment in the context of the six deductive proofs.

We discovered that the second notion of optimum relative acceptability is the most comprehensive logical relation that satisfies this structure. This led us to identify this notion with entailment. This is our recommended interpretation. It is not an interpretation that depends on intuition or psychological appeal. It is an interpretation that results from meticulous analysis of the actual functioning of entailment in all the traditional deductive proofs. We can claim that our interpretation is the correct interpretation.

If our interpretation is correct, then the second arm of the paradox does not hold. $(p \cdot -p)$ in propositional calculus does entail q, provided we identify acceptability with truth, unacceptability with falsity, negation with the connective "not-", and contradiction with any set of propositions of the form $\{X, -X\}$. Lewis' proof is valid, for every step in the proof both transmits acceptability and retransmits unacceptability.

We propose this as the solution of the paradox, and the second optimum relative acceptability as the interpretation of entailment.

NOTES

1. This term is to be taken in its ordinary non-technical sense.
2. The sum of relations $R_1, \ldots, R_n$ is the relation S such that xSy if and only if xR_1y or . . . or xR_ny.
3. This example is adapted from Cohen and Nagel [4].
4. This is a simplified form. Some proofs of this type may not be "linear". For example, line (2) may be: P_2, P_5 acc Q_2 and line (3) may be: Q_1, Q_2 acc Q_3. But the simplification will not affect the subsequent arguments.

5. W. R. Hamilton (1843) and H. Grassmann (1844).

6. N. I. Lobachevsky (1829) and J. Bolyai (1832).

7. Saccheri and Lobachevsky did not know if their postulates were consistent.

8. The composition of two relations R and S is the relation C such that xCy if and only if there is z such that xRz and zSy. We shall write C as $R.S$.

9. Since $(X\ .\ -X)$ never attains the value 1, acceptability is trivially transmitted to $(Y\ .\ -Y)$.

10. If $X\ \mathrm{acc}_2\ Y$, then (1) if $X = 1$, then $Y = 1$; (2) if $X = \frac{1}{2}$, then $Y = \frac{1}{2}$ or 1; (3) if $X = 0$, then $Y = 0$ or $\frac{1}{2}$ or 1. In other words, $[X] \leqslant [Y]$. On the other hand, if $[X] \leqslant [Y]$, then both 1 is transmitted and 0 is retransmitted. In other words, $X\ \mathrm{acc}_2\ Y$.

11. So in the three-valued logic of Łukasiewicz, $(X\ .\ -X)$ is not a contradiction, unless we take both 0 and 1/2 as values of unacceptability.

REFERENCES

[1] Anderson, A. R., and N. D. Belnap, "Entailment," pp. 76-110 in *Logic and Philosophy*, ed., G. Iseminger, Appleton-Century-Crofts, New York, 1968.

[2] Anderson, A. R., and N. D. Belnap, "Tautological entailments," *Philosophical Studies*, vol. 13 (1962), pp. 9-24.

[3] Bennett, J., "Entailment," *The Philosophical Review*, vol. 68 (1969), pp. 197-236.

[4] Cohen, M. R., and E. Nagel, *An Introduction to Logic and Scientific Method,* Harcourt, Brace and Co., New York, 1934.

[5] Ewing, A. C., *The Fundamental Questions of Philosophy*, Routledge & Kegan Paul, London, 1951.

[6] Lewis, C. I., and C. H. Langford, *Symbolic Logic*, second edition, Dover Publications, New York, 1959.

[7] Lucas, J. R., "Plato and the axiomatic method," pp. 11-14 in *Problems in the Philosophy of Mathematics*, ed., I. Lakatos, North-Holland, Amsterdam, 1967.

[8] Moore, G. E., "External and internal relations," *Proceedings of the Aristotelian Society*, vol. 20 (1920), pp. 40-62.

[9] Nelson, E. J., "Intensional relations," *Mind*, vol. 39 (1930), pp. 440-453.

[10] Szabo, A., "Greek dialectic and Euclid's axiomatics," pp. 1-8 in *Problems in the Philosophy of Mathematics*, ed., I. Lakatos, North-Holland, Amsterdam, 1967.

[11] Watling, J., "Entailment," *Aristotelian Society*, supplementary volume 32 (1958), pp. 143-156.

Waikato University
Hamilton, New Zealand