Notre Dame Journal of Formal Logic
Volume XX, Number 4, October 1979
NDJFAM

THE Ω-SYSTEM AND THE Ł-SYSTEM OF MODAL LOGIC

JEAN PORTE

1 *Definition* The Ω-system is a logistic system, the alphabet of which consists of a denumerable set of propositional variables ($p_1, p_2, \ldots$), and of three connectives: $\Rightarrow$ (implication), $\neg$ (negation), and Ω. Ω is a 0-ary connective, i.e., a propositional constant. The well-formed formulas are defined as usual[1] throughout this paper,* the letters "x", "y", and "z" represent arbitrary wffs. We have three axiom schemas and a rule:

$$x \Rightarrow (y \Rightarrow x)$$
$$(x \Rightarrow (y \Rightarrow z)) \Rightarrow ((x \Rightarrow y) \Rightarrow (x \Rightarrow z))$$
$$(\neg x \Rightarrow \neg y) \Rightarrow (y \Rightarrow x)$$
$$\frac{x,\ x \Rightarrow y}{y}$$

The system is different from the well-known Frege-Łukasiewicz system for the classical propositional calculus, for the wffs are not the same: they may contain the symbol Ω, but Ω does not appear explicitly in the postulates.

The Ω-system may be considered as a modal system when possibility P and necessity N are defined as follows:

$$Px = \Omega \Rightarrow x$$
$$Nx = \neg P \neg x$$

whence $Nx = \Omega \wedge x$.

The chief result of this paper is that this modal system is (in a certain sense) identical with the Ł-system of modal logic. The Ł-system is defined in Łukasiewicz [13] (see Harrop [7] or Rose [23], if that paper is not available); see also Łukasiewicz [12], Anderson [1], Smiley [24], Church [4].

*This paper is the development of a communication given to the "Colloquium on non-classical logics" (Helsinki, 1962). Only a mimeographed abstract had been circulated.

Received May 15, 1978

2 Let us first consider an extension of the propositional calculus in which we have simply added a supplementary propositional variable, p_ω (with obvious extensions of the wff and of the postulates). Nothing is changed. The new system is isomorphic to the classical one (use the one-one correspondence $p_1 \to p_2, p_2 \to p_3, \ldots, p_\omega \to p_1$). Moreover, this isomorphism extends to one between the propositional calculus and its characteristic matrix. We will summarize this latter fact by saying that the propositional calculus and the new system are not only *syntactically isomorphic* but also *semantically isomorphic.*

3 In the system defined in section **2**, let us now replace p_ω by a *constant* Ω. A constant differs from a propositional variable only from the semantical point of view. When a propositional logistic system is coupled with a matrix, a propositional variable takes (for various assignments) every value of the matrix—while a constant takes only one value, always the same for various assignments. Then our new system, *the Ω-system, is syntactically isomorphic to the propositional calculus*—but it cannot be semantically isomorphic to the classical calculus (whatever matrix the Ω-system is coupled with).

4 Let us now consider the classical 2-value characteristic matrix, $\mathfrak{M}$, of the propositional calculus, the elements of which will be named here: 1 (truth) and 2 (falsity). It is known that the 4-value matrix $\mathfrak{M} \times \mathfrak{M}$ is also characteristic for the propositional calculus, *cf.* [9] and [10].

If we try to use the matrix $\mathfrak{M} \times \mathfrak{M}$ for the Ω-system the only difficulty lies in the interpretation of Ω. But Ω plays the same syntactical role as p_ω in the system of section **3**. We will find exactly all the theses of the system of section **3** by interpreting p_ω as 1 in an assignment, and by 2 in another. This is not possible for Ω, since it is a constant. But we will get the same result by interpreting Ω by 1 in one factor of the product $\mathfrak{M} \times \mathfrak{M}$ and by 2 in the other factor. This amounts to interpreting Ω by the constant value (1, 2) in $\mathfrak{M} \times \mathfrak{M}$.[2]

5 Eventually we get a characteristic matrix for the Ω-system:

for Ω: (1, 2)
designated value: (1, 1)

$x \Rightarrow y$

x \ y	(1, 1)	(1, 2)	(2, 1)	(2, 2)	$\neg x$
(1, 1)	(1, 1)	(1, 2)	(2, 1)	(2, 2)	(2, 2)
(1, 2)	(1, 1)	(1, 1)	(2, 1)	(2, 1)	(2, 1)
(2, 1)	(1, 1)	(1, 2)	(1, 1)	(1, 2)	(1, 2)
(2, 2)	(1, 1)	(1, 1)	(1, 1)	(1, 1)	(1, 1)

6 Renaming the elements of the matrix, we get:

for Ω: 2
designated value: 1

$x \Rightarrow y$

$x \backslash y$	1	2	3	4	$\neg x$
1	1	2	3	4	4
2	1	1	3	3	3
3	1	2	1	2	2
4	1	1	1	1	1

7 If now we interpret possibility and necessity as abbreviations,

$$Px = \Omega \Rightarrow x$$
$$Nx = \Omega \wedge x$$

the matrix of section **6** will be completed by

x	1	2	3	4
Px	1	1	3	3
Nx	2	2	4	4

But this, together with the preceding tables for $\Rightarrow$ and $\neg$, and with 1 as the sole designated value, is just the Łukasiewicz matrix that Smiley [24] has proved characteristic for the Ł-system. Thus every computation in the Ł-system is reduced to a computation in the classical propositional calculus. The Ł-system is reduced to near triviality. A decision procedure for the Ł-system follows at once. A decision procedure was of course available since we knew a 4-elements characteristic matrix, but translating an Ł-formula into a propositional formula may involve a shorter computation.

8 As easy consequence of this fact, we can get what follows (for every formula x of the Ł-system):

(i)
$$\vdash NPx \Leftrightarrow Nx$$
$$\vdash PNx \Leftrightarrow Px$$

(which contrasts with the results in Lewis' systems: in S5 we have $\vdash NPx \Leftrightarrow Px$ and $\vdash PNx \Leftrightarrow Nx$);

(ii) for no formula x we can get $\vdash Nx$.

These features are rather odd in modal logics. Such oddities are probably the reasons that the Ł-system is not cited in Feys-Dopp [6].

9 The Ł-system satisfies, however, the condition of Łukasiewicz for a "good" modal system, namely (see Prior [22], p. 3):

(a) for every formula x, if $\vdash Nx$ then $\vdash x$
(b) there is a formula x such as not-$\vdash Nx$ and $\vdash x$
(c) not for every formula x, $\vdash \neg Nx$
(a′) for every formula x, if $\vdash x$ then $\vdash Px$
(b′) there is a formula x such as not $\vdash x$ and $\vdash Px$
(c′) not for every formula x, $\vdash Px$

(d) $\vdash Px \Leftrightarrow \neg N \neg x$

(d′) $\vdash Nx \Leftrightarrow \neg P \neg x$.

Only (b′) is not obvious. We will verify it by putting

$$x = a \Rightarrow Na$$

(a being an arbitrary wff). Then

$$\text{not } \vdash x, \text{ for not } \vdash a \Rightarrow (\Omega \wedge a)$$
$$\vdash Pa, \text{ for } \vdash \Omega \Rightarrow (a \Rightarrow (\Omega \wedge a)).$$

10 In what precedes, N is "interpreted" by means of Ω. If we consider the Ω-system and the Ł-system as two different formal systems, we have used a translation, τ, of the second into the first that maps theses into theses and non-theses into non-theses:

$$\begin{aligned} \tau(p_i) &= p_i \\ \tau(x \Rightarrow y) &= \tau(x) \Rightarrow \tau(y) \\ \tau(\neg x) &= \neg \tau(x) \\ \tau(Nx) &= \neg(\Omega \Rightarrow \neg \tau(x)). \end{aligned}$$

Conversely, there is a translation, τ' of the Ω-system into the Ł-system which maps theses into theses and non-theses into non-theses, namely:

$$\begin{aligned} \tau'(p_i) &= p_i \\ \tau'(x \Rightarrow y) &= \tau'(x) \Rightarrow \tau'(y) \\ \tau'(\neg x) &= \neg \tau'(x) \\ \tau'(\Omega) &= Pa \Rightarrow Na, \end{aligned}$$

where a is a constant (but arbitrary) wff.[3]

In this way Ω is "interpreted" as an "abbreviation" of a particular formula of the Ł-system. The Ω-system and the Ł-system are then "equipollent", in the sense of Porte [18], chapter 12.

11 *Historical remarks* The notion of "necessity" may be motivated by the vague idea that a formula is "necessary" when it is a consequence of premisses at least as strong as those which make a formula simply "true" (or rather "acceptable").[4] This could perhaps be made precise, without involving too many "oddities", by using *two* formal systems (one larger and the other smaller). But the authors went another way: to represent necessity (and/or the dual notion of possibility) *within one* formal system. Such methods lead inevitably to consideration of formulas like NNx, NPx, $NPNPx$, etc., which have few or no intuitive meanings! But this way has been followed by, among others, Lewis and Langford [11], Boll [2], Curry [5], Łukasiewicz [12] and me in [16], [19], and this paper.

The method used here could probably be used in a similar work starting from the first-order predicate calculus (instead of the propositional calculus). Reinhardt and Boll tried to do that (see Boll [2], who cites unpublished works of Reinhardt in 1944; see also Boll and Reinhardt [3]). While Boll's text is not up to modern standards of clarity and precision, I was able to extract from it ideas which have led me to the present work.

It is possible to generalize the Ω-system by introducing (in the propositional calculus) any number of constants similar to Ω: $\Omega_1, \ldots, \Omega_n$. We will eventually get a characteristic matrix with 2^{n+1} elements, and we can define n unary connectives similar to N (oral remark of Daniel Lacombe).

NOTES

1. Ω standing alone is a wff.
2. This simple proof has been found by Daniel Lacombe during a lecture that I gave in Paris (Institut Henri Poincaré) a few months after the Colloquium; my original proof was more complex.
3. For instance, a propositional variable, such as p_1.
4. In mathematical logic, it is usual to use the word 'true' only with a semantical meaning.

REFERENCES

[1] Anderson, A. R., "On the interpretation of a modal system of Łukasiewicz," *The Journal of Computing Systems*, vol. 1 (1953), pp. 213-219.

[2] Boll, M., *Manuel de logique scientifique*, Dunod, Paris (1948).

[3] Boll, M. and J. Reinhardt, "Une interprétation des modalités aristotéliciennes," *Congrès International de Philosophie des Sciences, 1949*, Hermann, Paris (1951), pp. 15-17.

[4] Church, A., Review of Smiley [24], *The Journal of Symbolic Logic*, vol. 27 (1962), p. 113.

[5] Curry, H. B., *A Theory of Formal Deducibility*, Notre Dame University Press, Notre Dame, Indiana, 1950 (written in 1948).

[6] Feys, R. and J. Dopp, *Modal Logics*, Gauthier-Villars and Nauwelaerts, Paris and Leuven, 1965.

[7] Harrop, R., Review of Łukasiewicz [12], [13], [14], [15] (and other papers), *The Journal of Symbolic Logic*, vol. 25 (1959), pp. 293-296.

[8] Hughes, G. E. and M. J. Cresswell, *An Introduction to Modal Logic*, Methuen, London, 1968.

[9] Kalicki, J., "Note on truth-tables," *The Journal of Symbolic Logic*, vol. 15 (1950), pp. 174-181.

[10] Kalicki, J., "A test for the existence of tautologies," *The Journal of Symbolic Logic*, vol. 15 (1950), pp. 182-184.

[11] Lewis, C. I. and C. H. Langford, *Symbolic Logic*, Dover, New York, 1932.

[12] Łukasiewicz, J., "A system of modal logic," *Actes du 11e Congrès International de Philosophie*, vol. 14 (volume complémentaire et communications du Colloque de Logique), North-Holland Co., Amsterdam and Nauwelaerts, Leuven (1953), pp. 82-83.

[13] Łukasiewicz, J., "A system of modal logic," *The Journal of Computing Systems*, vol. 1 (1954), pp. 111-149.

[14] Łukasiewicz, J., "On a controversial problem of Aristotle modal logic," *Dominican Studies*, vol. 7 (1954), pp. 114-128.

[15] Łukasiewicz, J., "Arithmetic and modal logic," *The Journal of Computing Systems*, vol. 1 (1953), pp. 213-219.

[16] Porte, J., "Recherches sur les logiques modales," *Le raisonnement en mathématiques et en sciences expérimentales*, (70e Colloque International du C.N.R.S., Paris, septembre 1955), Paris (éditions du Centre National de la Recherche Scientifique) (1958), pp. 117-128.

[17] Porte, J., "The Ω-system and the Ł-system of modal logic," *Colloquium on non-classical logics, Helsinki, 1962*; Helsinki, mimeographed, 2 pp. (communication: on August 26, 1962; plus two lectures at the seminary of mathematical logic in Paris: October 31, 1962 and November 7, 1963).

[18] Porte, J., *Recherches sur la théorie générale des systèmes formels et sur les systèmes connectifs*, Gauthier-Villars, Paris and Nauwelaerts, Leuven (1965).

[19] Porte, J., *A research in modal logic*, to be published (a development of Porte [16]).

[20] Prior, A. N., "The interpretation of two systems of modal logic," *The Journal of Computing Systems*, vol. 1 (1953), pp. 201-208.

[21] Prior, A. N., *Formal Logic*, The Clarendon Press, Oxford (1955).

[22] Prior, A. N., *Time and Modality*, Cambridge University Press, (1956).

[23] Rose, A., Review of Łukasiewicz [13], *Mathematical Reviews*, vol. 15 (1954), p. 2.

[24] Smiley, T., "On Łukasiewicz Ł-modal system," *Notre Dame Journal of Formal Logic*, vol. II (1961), pp. 149-153.

[25] Thomas, I., "Note on a modal system of Łukasiewicz," *Dominican Studies*, vol. 6 (1953), pp. 167-170.

Ecole Normale Superieure
Algiers, Algeria