Notre Dame Journal of Formal Logic
Volume XX, Number 4, October 1979
NDJFAM

EQUATIONAL TWO AXIOM BASES FOR BOOLEAN ALGEBRAS AND SOME OTHER LATTICE THEORIES

BOLESŁAW SOBOCIŃSKI

In this paper it will be shown that the postulate systems for Boolean algebras and six other lattice theories can be reduced to sets containing only two equational axioms. As far as I know such axiomatizations of the theories under investigation are not mentioned in the literature.

1 For this end we have to prove the following nine Propositions[1]:

PI *An algebraic system $\langle A, \cup\rangle$ is a join semilattice, if it satisfies the following two postulates:*

$A1$ $[a]: a \epsilon A .\supset. a = a \cup a$
$A2$ $[abc]: a, b, c \epsilon A .\supset. a \cup (b \cup c) = c \cup (a \cup b)$

PII *An algebraic system $\langle A, \cap\rangle$ is a meet semilattice, if it satisfies the following two postulates:*

$B1$ $[a]: a \epsilon A .\supset. a = a \cap a$
$B2$ $[abc]: a, b, c \epsilon A .\supset. a \cap (b \cap c) = c \cap (a \cap b)$

PIII *An algebraic system $\langle A, \cup, -\rangle$ is a Boolean algebra, if it satisfies the following two postulates: A1 and*

$C1$ $[abcd]: a, b, c, d \epsilon A .\supset. a \cup (b \cup c) = (-(-c \cup d) \cup -(-c \cup -d)) \cup (a \cup b)$

PIV *An algebraic system $\langle A, \cup, \cap, O, I\rangle$ is a lattice with universal bounds, if it satisfies the following two postulates:*

$D1$ $[abc]: a, b, c \epsilon A .\supset. (b \cap (c \cap a)) \cup a = a$
$D2$ $[abcdef]: a, b, c, d, e, f \epsilon A .\supset. ((a \cap (b \cap c)) \cup d) \cup e$
$= ((((c \cap I) \cup O) \cap (a \cap b)) \cup e) \cup ((f \cup d) \cap d)$

PV *An algebraic system $\langle A, \cup, \cap, ^{\perp}\rangle$ is an ortholattice, if it satisfies the following two postulates: D1 and*

1. Throughout this paper, A indicates arbitrary but fixed carrier set. The so-called closure axioms are assumed tacitly.

Received December 8, 1978

E1 $[abcdefgh]: a, b, c, d, e, f, g, h \epsilon A .\supset. ((a \cap (b \cap (f \cup c))) \cup d) \cup e$
$= ((((g \cap g^{\perp}) \cup (c^{\perp} \cap f^{\perp})^{\perp}) \cap (a \cap b)) \cup e) \cup ((h \cup d) \cap d)$

PVI *An algebraic system* $\langle A, \cup, \cap, O, I\rangle$ *is a modular lattice with universal bounds, if it satisfies the following two postulates: D1 and*

F1 $[abcdefmn]: a, b, c, d, e, f, m, n \epsilon A .\supset. (((a \cap (b \cup c)) \cap (m \cap n)) \cup d) \cup e$
$= ((((n \cap I) \cup O) \cap ((a \cap ((b \cap (a \cup c)) \cup c)) \cap m)) \cup e) \cup ((f \cup d) \cap d)$

PVII *An algebraic system* $\langle A, \cup, \cap, {}^{\perp}\rangle$ *is a modular ortholattice, if it satisfies the following two postulates: D1 and*

G1 $[abcdefghmn]: a, b, c, d, e, f, g, h, m, n \epsilon A .\supset.$
$(((a \cap (b \cup c)) \cap (m \cap (g \cup n))) \cup d) \cup e = ((((h \cap h^{\perp}) \cup (n^{\perp} \cap g^{\perp})^{\perp})$
$\cap ((a \cap ((b \cap (a \cup c)) \cup c)) \cap m)) \cup e) \cup ((f \cup d) \cap d)$

PVIII *An algebraic system* $\langle A, \cup, \cap, {}^{\perp}\rangle$ *is an orthomodular lattice, if it satisfies the following two postulates: D1 and*

H1 $[abcdefghmn]: a, b, c, d, e, f, g, h, m, n \epsilon A .\supset.$
$((((a^{\perp} \cap (a \cup ((b \cup c) \cup d))) \cup a) \cap (m \cap n)) \cup g) \cup e$
$= ((n \cap (((h^{\perp} \cap h) \cup (((d^{\perp} \cap c^{\perp})^{\perp} \cup b) \cup a)) \cap m)) \cup e) \cup ((f \cup g) \cap g)$

PIX *An algebraic system* $\langle A, \cup, \cap, O, I\rangle$ *is a distributive lattice with universal bounds, if it satisfies the following two postulates: D1 and*

K1 $[abcdefmn]: a, b, c, d, e, f, m, n \epsilon A .\supset.$
$(((a \cap (b \cup c)) \cap (m \cap n)) \cup d) \cup e = ((((n \cap I) \cup O) \cap (((a \cap b)$
$\cup (a \cap c)) \cap m)) \cup e) \cup ((f \cup d) \cap d)$

Remarks:

1. Each postulate system presented above contains only two axioms and these axioms are mutually independent as can be proved easily. Of course, in axiom systems **PV**, **PVII**, and **PVIII** the primitive operations are not mutually independent. It is self-evident that the axioms belonging to **PIII**, **PIV**, **PV**, and **PIX** are theses of the respective theories. On the other hand, since formula

$$[abc]: a, b, c \epsilon A .\supset. a \cap (b \cup c) = a \cap ((b \cap (a \cup c)) \cup c)$$

is a modular lattice theorem, *cf.* [1], p. 39, and formula

$$[abcd]: a, b, c, d \epsilon A .\supset. (a^{\perp} \cap (a \cup ((b \cup c) \cup d))) \cup a = ((d^{\perp} \cap c^{\perp})^{\perp} \cup b) \cup a$$

is an orthomodular lattice theorem, *cf.* [7], pp. 317-319, we know that in the systems **PVI**, **PVII**, and **PVIII** their axioms are theses of the respective theories. For these reasons only the converses will be proved in this paper.

2. Pamanabhan [6] has established that a semilattice can be formalized using only two equational axioms which have the same lengthes as *A1* and *A2*. The new axiomatizations of semilattices presented in **PI** and **PII** are discussed in this paper, since these axiom systems are deeply involved in the construction of the axioms *C1, D2, E1, F1, G1, H1,* and *K1*.

3. Byrne [2] has shown that there is a base for Boolean algebras containing only two axioms. However, his axiomatization is not equational and, moreover, in some respect is longer than given in **PIII**.

4. It should be noted that forms of postulates given in **PIV-PIX** are suggested by Kalman's axiom system for lattices, *cf.* [5], but, obviously, the deductions presented below in sections **4-9** differ in several points from Kalman's proofs.

2 *Proofs of* PI *and* PII

2.1 Let us assume *A1* and *A2*. Then:

A3 $[ab]: a, b \epsilon A .\supset. a \cup b = b \cup a$

PR $[ab]$: Hp (1) $.\supset.$

$a \cup b = a \cup (b \cup b) = b \cup (a \cup b)$ [1; *A1*, *a/b*; *A2*, *c/b*]

$= b \cup (b \cup a) = b \cup (b \cup (a \cup a))$ [*A2*, *a/b*, *b/a*, *c/b*; *A1*]

$= b \cup (a \cup (b \cup a)) = (b \cup a) \cup (b \cup a) = b \cup a$

[*A2*, *a/b*, *b/a*, *c/a*; *A2*, *a/b*, *b/a*, *c/b* ∪ *a*; *A1*, *a/b* ∪ *a*]

A4 $[abc]: a, b, c \epsilon A .\supset. a \cup (b \cup c) = (a \cup b) \cup c$ [*A2*; *A3*]

Since we have *A1*, *A3*, and *A4*, the proof of Proposition **I** is complete.

2.2 Since **PI** holds, we know automatically that *B1* and *B2* imply

B3 $[ab]: a, b \epsilon A .\supset. a \cap b = b \cap a$ [*B1*; *B2*]

and

B4 $[abc]: a, b, c \epsilon A .\supset. a \cap (b \cap c) = (a \cap b) \cap c$ [*B2*; *B3*]

Whence, Proposition **II** is proven.

3 *Proof of* PIII

Let us assume *A1* and *C1*. Then:

C2 $[ab]: a, b \epsilon A .\supset. a = (-(-a \cup b) \cup -(-a \cup -b)) \cup a$

PR $[ab]$: Hp (1) $.\supset.$

$a = a \cup a = a \cup (a \cup a) = (-(-a \cup b) \cup -(-a \cup -b)) \cup (a \cup a)$

[1; *A1*; *A1*; *C1*, *b/a*, *c/a*, *d/b*]

$= (-(-a \cup b) \cup -(-a \cup -b)) \cup a$ [*A1*]

C3 $[ab]: a, b \epsilon A .\supset. a = -(-a \cup b) \cup -(-a \cup -b)$

PR $[ab]$: Hp (1) $.\supset.$

$a = (-(-a \cup b) \cup -(-a \cup -b)) \cup a$ [1; *C2*]

$= (-(-a \cup b) \cup -(-a \cup -b)) \cup ((-(-a \cup b) \cup -(-a \cup -b)) \cup a)$ [*C2*]

$= (-(-a \cup b) \cup -(-a \cup -b)) \cup ((-(-a \cup b) \cup$
$-(-a \cup -b)) \cup (-(-a \cup b) \cup -(-a \cup -b)))$

[*C1*, *a*/-(-*a* ∪ *b*) ∪ -(-*a* ∪ -*b*), *b*/-(-*a* ∪ *b*) ∪ -(-*a* ∪ -*b*), *c/a*, *d/b*]

$= (-(-a \cup b) \cup -(-a \cup -b)) \cup (-(-a \cup b) \cup -(-a \cup -b))$

[*A1*, *a*/-(-*a* ∪ *b*) ∪ -(-*a* ∪ -*b*)]

$= -(-a \cup b) \cup -(-a \cup -b)$ [*A1*, *a*/-(-*a* ∪ *b*) ∪ -(-*a* ∪ -*b*)]

A2 $[abc]: a, b, c \epsilon A .\supset. a \cup (b \cup c) = c \cup (a \cup b)$ [*C1*; *C3*, *a/c*, *b/d*]

A3 $[ab]: a, b \epsilon A .\supset. a \cup b = b \cup a$ [*A1*; *A2*; *cf.* section **2.1**]

A4 $[abc]: a, b, c \epsilon A .\supset. a \cup (b \cup c) = (a \cup b) \cup c$ [*A2*; *A3*]

Since *A1* and *C1* imply *A4*, *A3*, and *C3*, we have Huntington's well known postulate system for Boolean algebras, *cf.* [3], pp. 280-286, and [4]. Thus, the proof of Proposition **III** is complete.

4 *Proof of* **P IV** Let us assume *D1* and *D2*. Then:

D3 $[def]: d, e, f \epsilon A . \supset . d \cup e = e \cup ((f \cup d) \cap d)$

PR $[def]: \text{Hp}(1) . \supset .$

$d \cup e = ((d \cap (e \cap d)) \cup d) \cup e = ((((d \cap I) \cup O) \cap (d \cap e)) \cup e)$
$\cup ((f \cup d) \cap d)$ [1; *D1*, a/d, b/d, c/e; *D2*, a/d, b/e, c/d]
$= e \cup ((f \cup d) \cap d)$ [*D1*, a/e, $b/(d \cap I) \cup O$, c/d]

B1 $[a]: a \epsilon A . \supset . a = a \cap a$

PR $[a]: \text{Hp}(1) . \supset .$

$a = (a \cap (a \cap a)) \cup a = a \cup ((a \cup (a \cap (a \cap a))) \cap (a \cap (a \cap a)))$
[1; *D1*, b/a, c/a; *D3*, $d/a \cap (a \cap a)$, e/a, f/a]
$= ((a \cup (a \cap (a \cap a))) \cap (a \cap (a \cap a))) \cup (((a \cap (a \cap a)) \cup a) \cap a)$
[*D3*, d/a, $e/((a \cup (a \cap (a \cap a))) \cap (a \cap (a \cap a)))$, $f/a \cap (a \cap a)$]
$= ((a \cup (a \cap (a \cap a))) \cap (a \cap (a \cap a))) \cup (a \cap a)$ [*D1*, b/a, c/a]
$= a \cap a$ [*D1*, $a/a \cap a$, $b/a \cup (a \cap (a \cap a))$, c/a]

Remark 5: Concerning the proofs of *D3* and *B1*, *cf.* Kalman [5].

D4 $[ab]: a, b \epsilon A . \supset . (b \cap a) \cup a = a$ [*D1*, c/a; *B1*]

A1 $[a]: a \epsilon A . \supset . a = a \cup a$ [*D4*, b/a; *B1*]

A3 $[ab]: a, b \epsilon A . \supset . a \cup b = b \cup a$ [*D3*, d/a, e/b, f/a; *A1*, *B1*]

D5 $[abcde]: a, b, c, d, e \epsilon A . \supset . ((a \cap (b \cap c)) \cup d) \cup e$
$= ((((c \cap I) \cup O) \cap (a \cap b)) \cup e) \cup d$ [*D2*, f/d; *A1*, a/d; *B1*, a/d]

D6 $[abc]: a, b, c \epsilon A . \supset . a \cap (b \cap c) = (((c \cap I) \cup O) \cap (a \cap b))$

PR $[abc]: \text{Hp}(1) . \supset .$

$a \cap (b \cap c) = (a \cap (b \cap c)) \cup (a \cap (b \cap c))$ [1; *A1*, $a/a \cap (b \cap c)$]
$= ((a \cap (b \cap c)) \cup (a \cap (b \cap c))) \cup (a \cap (b \cap c))$ [1; *A1*, $a/a \cap (b \cap c)$]
$= ((((c \cap I) \cup O) \cap (a \cap b)) \cup (a \cap (b \cap c))) \cap (a \cap (b \cap c))$
[*D5*, $d/a \cap (b \cap c)$, $e/a \cap (b \cap c)$]
$= ((a \cap (b \cap c)) \cup (((c \cap I) \cup O) \cap (a \cap b))) \cup (a \cap (b \cap c))$
[*A3*, $a/a \cap (b \cap c)$, $b/((c \cap I) \cap O) \cap (a \cap b)$]
$= ((((c \cap I) \cup O) \cap (a \cap b)) \cup (a \cap (b \cap c))) \cup (((c \cap I) \cup O) \cap (a \cap b))$
[*D5*, $d/((c \cap I) \cup O) \cap (a \cap b)$, $e/a \cap (b \cap c)$]
$= ((a \cap (b \cap c)) \cup (((c \cap I) \cup O) \cap (a \cap b))) \cup (((c \cap I) \cup O) \cap (a \cap b))$
[*A3*, $a/((c \cap I) \cup O) \cap (a \cap b)$, $b/a \cap (b \cap c)$]
$= ((((c \cap I) \cup O) \cap (a \cap b)) \cup (((c \cap I) \cup O) \cap (a \cap b)))$
$\cup (((c \cap I) \cup O) \cap (a \cap b))$
[*D5*, $d/((c \cap I) \cup O) \cap (a \cap b)$, $e/((c \cap I) \cup O) \cap (a \cap b)$]
$= (((c \cap I) \cup O) \cap (a \cap b)) \cup (((c \cap I) \cup O) \cap (a \cap b))$
[*A1*, $a/((c \cap I) \cup O) \cap (a \cap b)$]
$= ((c \cap I) \cup O) \cap (a \cap b)$ [*A1*, $a/((c \cap I) \cup O) \cap (a \cap b)$]

Remark 6: The proof of *D6* is not difficult, but it is rather cumbersome. The deductions used here will be explained by the following schema:

Put $\alpha = a \cap (b \cap c)$ and $\beta = ((c \cap I) \cup O) \cap (a \cap b)$. Then *D5* will have the form

Z1 $[abcde]: a, b, c, d, e \in A .\supset. (\alpha \cup d) \cup e = (\beta \cup e) \cup d$

Hence:

Z2 $[abc]: a, b, c \in A .\supset. \alpha = \beta$

PR $[abc]$: Hp (1) $.\supset.$

$\alpha = \alpha \cup \alpha = (\alpha \cup \alpha) \cup \alpha = (\beta \cup \alpha) \cup \alpha = (\alpha \cup \beta) \cup \alpha$ [1; *A1*; *A1*; *Z1*; *A3*]

$= (\beta \cup \alpha) \cup \beta = (\alpha \cup \beta) \cup \beta = (\beta \cup \beta) \cup \beta = \beta \cup \beta = \beta$

[*Z1*; *A3*; *Z1*; *A1*; *A1*]

Below, the same schema of deductions will be used several times.

D7 $[a]: a \in A .\supset. a = ((a \cap I) \cup O) \cap a$

PR $[a]$: Hp (1) $.\supset.$

$a = a \cap a = a \cap (a \cap a) = ((a \cap I) \cup O) \cap (a \cap a)$ [1; *B1*; *B1*; *D6*, b/a, c/a]

$= ((a \cap I) \cup O) \cap a$ [*B1*]

D8 $[a]: a \in A .\supset. a = (a \cap I) \cup O$

PR $[a]$: Hp (1) $.\supset.$

$a = ((a \cap I) \cup O) \cap a = ((a \cap I) \cup O) \cap (((a \cap I) \cup O) \cap a)$ [1; *D7*; *D7*]

$= ((a \cap I) \cup O) \cap (((a \cap I) \cup O) \cap ((a \cap I) \cup O))$

[*D6*, $a/(a \cap I) \cup O$, $b/(a \cap I) \cup O$, c/a]

$= ((a \cap I) \cup O) \cap ((a \cap I) \cup O)$ [*B1*, $a/(a \cap I) \cup O$]

$= (a \cap I) \cup O$ [*B1*, $a/(a \cap I) \cup O$]

B2 $[abc]: a, b, c \in A .\supset. a \cap (b \cap c) = c \cap (a \cap b)$ [*D6*; *D8*, a/c]

B3 $[ab]: a, b \in A .\supset. a \cap b = b \cap a$ [*B1*; *B2*; *cf.* section **2.2**]

B4 $[abc]: a, b, c \in A .\supset. a \cap (b \cap c) = (a \cap b) \cap c$ [*B2*; *B3*]

D9 $[ab]: a, b \in A .\supset. a = a \cup (a \cap b)$ [*D4*; *A3*, $b/b \cup a$; *B3*]

D10 $[ab]: a, b \in A .\supset. a = a \cap (a \cup b)$

PR $[ab]$: Hp (1) $.\supset.$

$a = a \cup a = a \cup ((b \cup a) \cap a) = ((b \cup a) \cap a) \cup ((b \cup a) \cap a)$

[1; *A1*; *D3*, d/a, e/a, f/b; *D3*, d/a, $e/(b \cup a) \cap a$, f/b]

$= (b \cup a) \cap a = a \cap (b \cup a) = a \cap (a \cup b)$

[*A1*, $a/(b \cup a) \cap a$; *B3*, $a/b \cup a$; *A3*]

D11 $[abc]: a, b, c \in A .\supset. (a \cup b) \cup c = (a \cup c) \cup b$

PR $[abc]$: Hp (1) $.\supset.$

$(a \cup b) \cup c = ((a \cap a) \cup b) \cup c = ((a \cap (a \cap a)) \cup b) \cup c$ [1; *B1*; *B1*]

$= ((((a \cap I) \cup O) \cap (a \cap a)) \cup c) \cup b$ [*D5*, b/a, c/a, d/b, e/c]

$= ((a \cap (a \cap a)) \cup c) \cup b = ((a \cap a) \cup c) \cup b$ [*D8*; *B1*]

$= (a \cup c) \cup b$ [*B1*]

A4 $[abc]: a, b, c \in A .\supset. a \cup (b \cup c) = (a \cup b) \cup c$

PR $[abc]$: Hp (1) $.\supset.$

$a \cup (b \cup c) = (b \cup c) \cup a = (b \cup a) \cup c = (a \cup b) \cup c$

[1; *A3*, $b/b \cup c$; *D11*, a/b, b/c, e/a; *A3*]

D12 $[a]: a \in A .\supset. a = a \cup O$

PR $[a]$: Hp (1) $.\supset.$

$a = a \cup a = a \cup ((a \cap I) \cup O) = (a \cup (a \cap I)) \cup O = a \cup O$

[1; *A1*; *D8*; *A4*, $b/a \cap I$, c/O; *D9*, b/I]

D13 $[a]: a \in A .\supset. a = a \cap I$

PR $[a]$: Hp (1) $.\supset.$

$a = (a \cap I) \cup O = a \cap I$ [1; *D8*; *D12*, $a/a \cap I$]

It has been proved above that *D1* and *D2* imply *A3, A4, D9, B3, B4, D10, D12* and *D13*. Therefore, the proof of Proposition **IV** is complete.

5 *Proof of* **PV** In this and the next sections the proofs of many theses are either the same or entirely analogous to the deductions given in section **4**. In such cases their proofs will be presented here in abbreviate form. Now let us assume *D1* and *E1*. Then:

D3 $[def]: d, e, f \epsilon A .\supset. d \cup e = e \cup ((f \cup d) \cap d)$

PR $[def]:$ Hp (1) $.\supset.$

$d \cup e = ((d \cap (e \cap d)) \cup d) \cup e$ [1; *D1, a/d, b/d, c/e*]

$= ((d \cup (e \cup ((d \cap (d \cap d)) \cup d))) \cup d) \cup e$ [*D1, a/d, b/d, c/d*]

$= ((((d \cap d^{\perp}) \cup (d^{\perp} \cap (d \cap (d \cap d))^{\perp})^{\perp}) \cap (d \cap e)) \cup e) \cup ((f \cup d) \cap d)$

[*E1, a/d, b/e, c/d,* $f/d \cap (d \cap d)$, *g/d, h/f*]

$= e \cup ((f \cup d) \cap d)$ [*D1, a/e,* $b/(d \cap d^{\perp}) \cup (d^{\perp} \cap (d \cap (d \cap d))^{\perp})^{\perp}$, *c/d*]

B1 $[a]: a \epsilon A .\supset. a = a \cap a$ [*D1, D3, cf.* proof of *D3* in section **4**]

D4 $[ab]: a, b \epsilon A .\supset. (b \cap a) \cup a = a$ [*D1*; *B1*]

A1 $[ab]: a \epsilon A .\supset. a = a \cup a$ [*D4*; *B1*]

A3 $[ab]: a, b \epsilon A .\supset. a \cup b = b \cup a$ [*D3*; *A1*; *B1*]

E2 $[abcdefg]: a, b, c, d, e, f, g \epsilon A .\supset. ((a \cap (b \cap (f \cup c))) \cup d) \cup e$

$= ((((g \cap g^{\perp}) \cup (c^{\perp} \cap f^{\perp})^{\perp}) \cap (a \cap b)) \cup e) \cup d$ [*E1*; *A1*; *B1*]

E3 $[abcfg]: a, b, c, f, g \epsilon A .\supset. a \cap (b \cap (f \cup c))$

$= ((g \cap g^{\perp}) \cup (c^{\perp} \cap f^{\perp})^{\perp}) \cap (a \cap b)$

[*A1*; *E2*; *A3*; *cf.* proof of *D6* and Remark 6 in section **4**]

E4 $[abc]: a, b, c \epsilon A .\supset. a \cup b = ((c \cap c^{\perp}) \cup (a^{\perp} \cap b^{\perp})^{\perp}) \cap (b \cup a)$

PR $[abc]:$ Hp (1) $.\supset.$

$a \cup b = (a \cup b) \cap (a \cup b) = (a \cup b) \cap ((a \cup b) \cap (a \cup b))$

[1; *B1*; $a/a \cup b$; *B1*, $a/a \cup b$]

$= (a \cup b) \cap ((a \cup b) \cap (b \cup a))$ [*A3*]

$= ((c \cap c^{\perp}) \cup (a^{\perp} \cap b^{\perp})^{\perp}) \cap ((a \cup b) \cap (a \cup b))$

[*E3*, $a/a \cup b$, $b/a \cup b$, *f/b, g/c, c/a*]

$= ((c \cap c^{\perp}) \cup (a^{\perp} \cap b^{\perp})^{\perp}) \cap (a \cup b)$ [*B1*, $a/a \cup b$]

$= ((c \cap c^{\perp}) \cup (a^{\perp} \cap b^{\perp})^{\perp}) \cap (b \cup a)$ [*A3*]

E5 $[abc]: a, b, c \epsilon A .\supset. a \cup b = (c \cap c^{\perp}) \cup (a^{\perp} \cap b^{\perp})^{\perp}$

PR $[abc]:$ Hp (1) $.\supset.$

$a \cup b = ((c \cap c^{\perp}) \cup (a^{\perp} \cap b^{\perp})^{\perp}) \cap (b \cup a)$ [1, *E4*]

$= ((c \cap c^{\perp}) \cup (a^{\perp} \cap b^{\perp})^{\perp}) \cap (a \cup b)$ [*A3*]

$= ((c \cap c^{\perp}) \cup (a^{\perp} \cap b^{\perp})^{\perp}) \cap (((c \cap c^{\perp}) \cup (a^{\perp} \cap b^{\perp})^{\perp}) \cap (b \cup a))$ [*E4*]

$= ((c \cap c^{\perp}) \cup (a^{\perp} \cap b^{\perp})^{\perp}) \cap (((c \cap c^{\perp}) \cup (a^{\perp} \cap b^{\perp})^{\perp}) \cap ((c \cap c^{\perp})$
$\cup (a^{\perp} \cap b^{\perp})^{\perp}))$

[*E3*, $a/(c \cap c^{\perp}) \cup (a^{\perp} \cap b^{\perp})^{\perp}$, $b/(c \cap c^{\perp}) \cup (a^{\perp} \cap b^{\perp})^{\perp}$, *f/b, c/a, g/c*]

$= ((c \cap c^{\perp}) \cup (a^{\perp} \cap b^{\perp})^{\perp}) \cap ((c \cap c^{\perp}) \cup (a^{\perp} \cap b^{\perp})^{\perp})$

[*B1*, $a/(c \cap c^{\perp}) \cup (a^{\perp} \cap b^{\perp})^{\perp}$]

$= (c \cap c^{\perp}) \cup (a^{\perp} \cap b^{\perp})^{\perp}$ [*B1*, $a/(c \cap c^{\perp}) \cup (a^{\perp} \cap b^{\perp})^{\perp}$]

B2 $[abc]: a, b, c \epsilon A .\supset. a \cap (b \cap c) = c \cap (a \cap b)$

PR $[abc]:$ Hp (1) $.\supset.$

$a \cap (b \cap c) = a \cap (b \cap (c \cup c))$ [1; *A1, a/c*]

$= ((c \cap c^{\perp}) \cup (c^{\perp} \cap c^{\perp})^{\perp}) \cap (a \cap b)$ [*E3*, *f/c*, *g/c*]

$= (c \cup c) \cap (a \cap b) = c \cap (a \cap b)$ [*E5*, *a/c*, *b/c*; *A1*, *a/c*]

B3 $[ab]: a, b \in A . \supset . a \cap b = b \cap a$ [*B1*; *B2*; *cf.* section **2.2**]

B4 $[abc]: a, b, c \in A . \supset . a \cap (b \cap c) = (a \cap b) \cap c$ [*B2*; *B3*]

D9 $[ab]: a, b \in A . \supset . a = a \cup (a \cap b)$ [*D4*; *A3*; *B3*]

D10 $[ab]: a, b \in A . \supset . a = a \cap (a \cup b)$

[*A1*; *D3*; *B3*; *cf.* proof of *D10* in section **4**]

D11 $[abc]: a, b, c \in A . \supset . a \cup (b \cup c) = (a \cup c) \cup b$

PR $[abc]$: Hp (1) $. \supset .$

$(a \cup b) \cup c = ((a \cap a) \cup b) \cup c = ((a \cap (a \cap a)) \cup b) \cup c$ [1; *B1*; *B1*]

$= ((a \cap (a \cap (a \cup a))) \cup b) \cup c$ [*A1*]

$= ((((a \cap a^{\perp}) \cup (a^{\perp} \cap a^{\perp})^{\perp}) \cap (a \cap a)) \cup c) \cup b$

[*E2*, *b/a*, *c/a*, *f/a*, *d/b*, *e/c*, *g/a*]

$= (((a \cup a) \cap (a \cap a)) \cup c) \cup b$ [*E5*, *b/a*, *c/a*]

$= ((a \cap a) \cup c) \cup b = (a \cup c) \cup b$ [*A1*; *B1*; *B1*]

A4 $[abc]: a, b, c \in A . \supset . a \cup (b \cup c) = (a \cup b) \cup c$ [*A3*; *D11*; *A3*]

E6 $[a]: a \in A . \supset . a = a^{\perp\perp}$

PR $[a]$: Hp (1) $. \supset .$

$a = a \cup a = (a^{\perp} \cap a^{\perp\perp}) \cup (a^{\perp} \cap a^{\perp})^{\perp}$ [1; *A1*; *E5*, *b/a*, $c/a^{\perp}$]

$= (a^{\perp} \cap a^{\perp\perp}) \cup a^{\perp\perp} = a^{\perp\perp}$ [*B1*; $a/a^{\perp}$; *D4*, $a/a^{\perp\perp}$, $b/a^{\perp}$]

E7 $[ab]: a, b \in A . \supset . a = (b \cap b^{\perp}) \cup a$ [*A1*; *E5*, *b/a*, *c/b*; *B1*, $a/a^{\perp}$; *E6*]

E8 $[ab]: a, b \in A . \supset . a \cap a^{\perp} = b \cap b^{\perp}$

PR $[ab]$: Hp (1) $. \supset .$

$a \cap a^{\perp} = (b \cap b^{\perp}) \cup (a \cap a^{\perp}) = (a \cap a^{\perp}) \cup (b \cap b^{\perp}) = b \cap b^{\perp}$

[1; *E7*, $a/a \cap a^{\perp}$; *A3*, $a/b \cap b^{\perp}$, $b/a \cap a^{\perp}$; *E7*, $a/b \cap b^{\perp}$, *b/a*]

Df1 $[a]: a \in A . \supset . a \cap a^{\perp} = O$ [*E8*]

E9 $[a]: a \in A . \supset . a = a \cup O$ [*E7*, *b/a*; *A3*, $b/a \cap a^{\perp}$; *Df1*]

E10 $[ab]: a, b \in A . \supset . a \cup b = (a^{\perp} \cap b^{\perp})^{\perp}$ [*E5*, *c/b*, *E7*, $a/(a^{\perp} \cap b^{\perp})^{\perp}$]

E11 $[ab]: a, b \in A . \supset . (a \cup b)^{\perp} = a^{\perp} \cap b^{\perp}$ [*E10*; *E6*, $a/a^{\perp} \cap b^{\perp}$]

E12 $[ab]: a, b \in A . \supset . (a \cap b)^{\perp} = a^{\perp} \cup b^{\perp}$

[*E11*, $a/a^{\perp}$, $b/b^{\perp}$; *E6*; *E6*, *a/b*; *E6*, $a/a^{\perp} \cup b^{\perp}$]

Df2 $O^{\perp} = I$ [*Df1*]

E13 $[a]: a \in A . \supset . a \cup a^{\perp} = I$

PR $[a]$: Hp (1) $. \supset .$

$a \cup a^{\perp} = (a^{\perp} \cap a^{\perp\perp})^{\perp} = (a^{\perp} \cap a)^{\perp} = (a \cap a^{\perp}) = O^{\perp} = I$

[1; *E10*, $b/a^{\perp}$; *E6*; *B3*, $b/a^{\perp}$; *Df1*; *Df2*]

E14 $[a]: a \in A . \supset . a = a \cap I$ [*E10*, $b/a^{\perp}$; *E13*]

Thus, the theses *A3, A4, D9, B3. B4, D10, D12, D13, E12, E11, E6, Df1* and *E13* follow from *D1* and *E1*. Therefore, Proposition **PV** holds, *cf.* [1], p. 52.

6 *Proof of* **PVI** Let us assume *D1* and *F1*. Then:

D3 $[def]: d, e, f \in A . \supset . d \cup e = e \cup ((f \cup d) \cap d)$

PR $[def]$: Hp (1) $. \supset .$

$d \cup e = (((d \cap (d \cup d)) \cap (e \cap d)) \cup d) \cup e$

[1; *D1*, *a/d*, $b/d \cap (d \cup d)$, *c/e*]

$= ((((d \cap I) \cup O) \cap ((d \cap ((d \cap (d \cup d))) \cup d)) \cap e)) \cup e \cup ((f \cup d) \cap d)$
[$F1$, a/d, b/d, c/d, m/e, n/d]
$= e \cup ((f \cup d) \cap d)$ [$D1$, a/e, $b/(d \cap I) \cup O$, $c/d \cap ((d \cap (d \cup d)) \cup d)$]

Since *D1* and *F1* imply *D3* and since *B1*, *D4*, and *A1* follow from *D1* and *D3* alone, *cf.* section **4**, the following deductions hold:

D2 $[abcdef]: a, b, c, d, e, f \epsilon A .\supset. ((a \cap (b \cap c)) \cup d) \cup e$
$= ((((c \cap I) \cup O) \cap (a \cap b)) \cup e) \cup ((f \cup d) \cap d)$

PR $[abcdef]$: Hp (1) $.\supset.$
$((a \cap (b \cap c)) \cup d) \cup e = (((a \cap (a \cup a)) \cap (b \cap c)) \cup d) \cup e$ [1; *B1*; *A1*]
$= ((((c \cap I) \cup O) \cap ((a \cap ((a \cap (a \cup a)) \cup a)) \cup b)) \cup e \cup ((f \cup d) \cap d)$
[$F1$, b/a, c/a, m/b, n/c]
$= ((((c \cap I) \cup O) \cap (a \cap b)) \cup e) \cup ((f \cup d) \cap d)$ [*A1*; *B1*]

Since *D2* is a consequence of *D1* and *F1*, we know that the system $\{D1, F1\}$ is a lattice with universal bounds. Hence, we have at our disposal all theses proven in section **4**. Therefore,

F2 $[abcdemn]: a, b, c, d, e, m, n \epsilon A .\supset.$
$(((a \cap (b \cup c)) \cap (m \cap n)) \cup d) \cup e$
$= ((n \cap ((a \cap ((b \cap (a \cup c)) \cup c)) \cap m)) \cup e) \cup d$ [*F1*; *D8*; *A1*; *B1*]

F3 $[abcmn]: a, b, c, m, n \epsilon A .\supset. (a \cap (b \cup c)) \cap (m \cap n)$
$= n \cap ((a \cap ((b \cap (a \cup c)) \cup c)) \cap m)$
[*F2*; *A1*; *A3*; *cf.* proof of *D6* and Remark 6 in section **4**]

F4 $[abc]: a, b, c \epsilon A .\supset. a \cap (b \cup c) = a \cap ((b \cap (a \cup c)) \cup c)$

PR $[abc]$: Hp (1) $.\supset.$
$a \cap (b \cup c) = (a \cap (b \cup c)) \cap (a \cap (b \cup c))$ [1; $B1$, $a/a \cap (b \cup c)$]
$= (a \cap (b \cup c)) \cap ((a \cap (b \cup c)) \cap (a \cap (b \cup c)))$ [$B1$; $a/a \cap (b \cup c)$]
$= (a \cap (b \cup c)) \cap ((a \cap ((b \cap (a \cup c)) \cup c)) \cap (a \cap (b \cup c)))$
[$F3$, $m/a \cap (b \cup c)$, $n/a \cap (b \cup c)$]
$= (a \cap (b \cup c)) \cap ((a \cap ((b \cap (a \cup c)) \cup c)) \cap ((a \cap ((b \cap (a \cup c)) \cup c)))$
[$F3$, $m/a \cap ((b \cap (a \cup c)) \cup c$; $n/a \cap (b \cup c)$]
$= (a \cap ((b \cap (a \cup c)) \cup c)) \cap ((a \cap ((b \cap (a \cup c)) \cup c))$
$\cap (a \cap ((b \cap (a \cup c)) \cup c)))$
[$F3$, $m/a \cap ((b \cap (a \cup c)) \cup c)$, $n/a \cap ((b \cap (a \cup c)) \cup c)$]
$= (a \cap ((b \cap (a \cup c)) \cup c)) \cup (a \cap ((b \cap (a \cup c)) \cup c))$
[$B1$, $a/a \cap ((b \cap (a \cup c)) \cup c)$]
$= a \cap ((b \cap (a \cup c)) \cup c)$ [$B1$, $a/a \cap ((b \cap (a \cup c)) \cup c)$]

Remark 7: The deductions used here will be explained by the following schema:

Put $\alpha = a \cap (b \cup c)$ and $\beta = a \cap ((b \cap (a \cup c)) \cup c)$. Then *F3* will have the form

Z3 $[abcmn]: a, b, c, m, n \epsilon A .\supset. \alpha \cap (m \cap n) = n \cap (\beta \cap m)$

Hence:

Z4 $[abc]: a, b, c \epsilon A .\supset. \alpha = \beta$

PR $[abc]$: Hp (1) $.\supset.$

$\alpha = \alpha \cap \alpha = \alpha \cap (\alpha \cap \alpha) = \alpha \cap (\beta \cap \alpha)$ [1; *B1*; *B1*; *Z3*]
$= \alpha \cap (\beta \cap \beta) = \beta \cap (\beta \cap \beta) = \beta \cap \beta = \beta$ [*Z3*; *Z3*; *B1*; *B1*]

Below, the same schema of deductions will be used several times.

Since in the field of any lattice *F4* is inferentially equivalent to the standard axiom of modular lattices

M1 $[abc]: a, b, c \in A . a \leqslant c .\supset. a \cup (b \cap c) = (a \cup b) \cap c,$

cf. [1], p. 39, the proof of Proposition **VI** is complete.

7 *Proof of* **P VII** Let us assume *D1* and *G1*. Then:

D3 $[def]: d, e, f \in A .\supset. d \cup e = e \cup ((f \cup d) \cap d)$

PR $[def]$: Hp (1) $.\supset.$

$d \cup e = (((d \cap (d \cup d)) \cap (e \cap d)) \cup d) \cup e$ [1; *D1*, $b/d \cap (d \cup d)$, c/e, a/d]

$= (((d \cap (d \cup d)) \cap (e \cap ((d \cap (d \cap d)) \cup d))) \cup d) \cup e$ [*D1*, b/d, c/d, a/d]

$= ((((d \cap d^{\perp}) \cup (d^{\perp} \cap (d \cap (d \cap d))^{\perp})^{\perp}) \cap ((d \cap ((d \cap (d \cup d)) \cup d)) \cap e))) \cup e) \cup ((f \cup d) \cap d)$

[*G1*, a/d, b/d, c/d, m/e, $g/d \cap (d \cap d)$, n/d, h/d]

$= e \cup ((f \cup d) \cap d)$

[*D1*, $b/(d \cap d^{\perp}) \cup (d^{\perp} \cap (d \cap (d \cap d))^{\perp})^{\perp}$, $c/d \cap ((d \cap (d \cup d)) \cap d)$, a/e]

Thus, *D1* and *G1* imply *D3* and, therefore, also *B1*, *D4*, and *A1*, *cf.* section **4**. Whence:

E1 $[abcdefgh]: a, b, c, d, e, f, g, h \in A .\supset.$

$((a \cap (b \cap (f \cup c))) \cup d) \cup e = ((((g \cap g^{\perp}) \cup (c^{\perp} \cap f^{\perp})^{\perp}) \cap (a \cap b)) \cup e) \cup ((h \cup d) \cap d)$ [*G1*, b/a, c/a, m/b, n/c, g/f, f/h, h/g; *A1*; *B1*]

Since we have *D1* and *E1*, we know that the system under investigation is an ortholattice. Then:

G2 $[abcdemn]: a, b, c, d, e, m, n \in A .\supset.$

$(((a \cap (b \cup c)) \cap (m \cap n)) \cup d) \cup e$

$= ((n \cap ((a \cap ((b \cap (a \cup c)) \cup c)) \cap m)) \cup e) \cup d$

PR $[abcdemn]$: Hp (1) $.\supset.$

$(((a \cap (b \cup c)) \cap (m \cap n)) \cup d) \cup e$

$= (((a \cap (b \cup c)) \cap (m \cap (n \cup n))) \cup d) \cup e$ [1; *A1*, a/n]

$= ((((n \cap n^{\perp}) \cup (n^{\perp} \cap n^{\perp})^{\perp}) \cap ((a \cap ((b \cap (a \cup c)) \cap d)) \cap m)) \cup e) \cup ((d \cup d) \cap d)$ [*G1*, g/n, h/n, f/d]

$= ((n \cap ((a \cap ((b \cap (a \cup c)) \cup c)) \cap m)) \cup e) \cup (d \cap d)$

[*E5*, a/n, b/n, c/n; *A1*, a/p; *A1*, a/d]

$= ((n \cap ((a \cap ((b \cap (a \cup c)) \cup c)) \cap m)) \cup e) \cup d$ [*B1*, a/d]

G3 $[abcmn]: a, b, c, m, n \in A .\supset. (a \cap (b \cup c)) \cap (m \cap n)$

$= n \cap ((a \cap ((b \cap (a \cup c)) \cup c)) \cap m)$

[*G2*; *A1*; *A3*; *cf.* proof of *D6* and Remark 6 in section **4**]

F4 $[abc]: a, b, c \in A .\supset. a \cap (b \cup c) = a \cap ((b \cap (a \cup c)) \cup c)$

[*G3*; *B1*; *cf.* proof of *F4* and Remark 7 in section **6**]

Since the system $\{D1, G1\}$ is an ortholattice and implies *F4*, the proof of Proposition **VII** holds, *cf.* section **6**.

8 *Proof of* **PVIII** Let us assume *D1* and *H1*. Then:

D3 $[def]: d, e, f \epsilon A .\supset. d \cup e = e \cup ((f \cup d) \cap d)$

PR $[def]:$ Hp (1) $.\supset.$

$d \cup e = ((((d^{\perp} \cap (d \cup ((d \cup d) \cup d))) \cup d) \cap (e \cap d)) \cup d) \cup e$

$[1, D1, b/(d^{\perp} \cap (d \cup ((d \cup d) \cup d))) \cup d, c/e, a/d]$

$= ((d \cap (((d^{\perp} \cap d) \cup (((d^{\perp} \cap d^{\perp})^{\perp} \cup d) \cup d)) \cap e)) \cup e) \cup ((f \cup d) \cap d)$

$[H1, a/d, b/d, c/d, m/e, n/d, g/d, h/d]$

$= e \cup ((f \cup d) \cap d)$

$[D1, b/d, c/(d^{\perp} \cap d) \cup (((d^{\perp} \cap d^{\perp})^{\perp} \cup d) \cup d), a/e]$

Since *D1* and *D3* imply *B1, D4, A1,* and *A3,* we can proceed as follows:

H2 $[abcdeghmn]: a, b, c, d, e, g, h, m, n \epsilon A .\supset.$

$((((a^{\perp} \cap (a \cup ((b \cup c) \cup d))) \cup a) \cap (m \cap n)) \cup g) \cup e$

$= ((n \cap (((h^{\perp} \cap h) \cup (((d^{\perp} \cap c^{\perp})^{\perp} \cup b) \cup a)) \cap m)) \cup e) \cup g$

$[H1, f/g; A1, a/g; B1, a/g]$

H3 $[abcdhmn]: a, b, c, d, h, m, n \epsilon A .\supset.$

$((a^{\perp} \cap (a \cup ((b \cup c) \cup d))) \cup a) \cap (n \cap n)$

$= n \cap (((h^{\perp} \cap h) \cup (((d^{\perp} \cap c^{\perp})^{\perp} \cup b) \cup a)) \cap m)$

[*H2*; *A1*; *A3*; *cf.* proof of *D6* and Remark 6 in section **4**]

H4 $[ahmn]: a, h, m, n \epsilon A .\supset. a \cap (m \cap n)$

$= n \cap (((h^{\perp} \cap h) \cup ((a^{\perp\perp} \cup a) \cup a)) \cap m)$

PR $[ahmn]:$ Hp (1) $.\supset.$

$a \cap (m \cap n) = ((a^{\perp} \cap a) \cup a) \cap (m \cap n)$ $[1; D4, b/a^{\perp}]$

$= ((a^{\perp} \cap (a \cup ((a \cup a) \cup a))) \cup a) \cap (m \cap n)$ $[A1]$

$= n \cap (((h^{\perp} \cap h) \cup (((a^{\perp} \cap a^{\perp})^{\perp} \cup a) \cup a)) \cap m)$ $[H3, b/a, c/a, d/a]$

$= n \cap (((h^{\perp} \cap h) \cup ((a^{\perp\perp} \cup a) \cup a)) \cap m)$ $[B1, a/a^{\perp}]$

H5 $[ah]: a, h \epsilon A .\supset. a = (h^{\perp} \cap h) \cup ((a^{\perp\perp} \cup a) \cup a)$

[*H4*; *B1*; *cf.* proof of *F4* and Remark 7 in section **6**]

B2 $[abc]: a, b, c \epsilon A .\supset. a \cap (b \cap c) = c \cap (a \cap b)$ $[H4, m/b, n/c; H5]$

B3 $[ab]: a, b \epsilon A .\supset. a \cap b = b \cap a$ $[B1; B2]$

D10 $[ab]: a, b \epsilon A .\supset. a = a \cap (a \cup b)$ [*D4*; *B3*; *A3*; *cf.* section **4**]

H6 $[a]: a \epsilon A .\supset. a = (a^{\perp\perp} \cup a) \cup a$

PR $[a]:$ Hp (1) $.\supset.$

$a = (((a^{\perp\perp} \cup a) \cup a)^{\perp} \cap ((a^{\perp\perp} \cup a) \cup a)) \cup ((a^{\perp\perp} \cup a) \cup a)$

$[1; H5, h/(a^{\perp\perp} \cup a) \cup a]$

$= (a^{\perp\perp} \cup a) \cup a$ $[D4, b/((a^{\perp\perp} \cup a) \cup a)^{\perp}, a/(a^{\perp\perp} \cup a) \cup a]$

H7 $[ah]: a, h \epsilon A .\supset. a = (h^{\perp} \cap h) \cup a$ $[H5; H6]$

H8 $[ab]: a, b \epsilon A .\supset. a = a \cup (b \cap b^{\perp})$ $[H7, h/b; B3, a/b^{\perp}; A3, b/b \cap b^{\perp}]$

H9 $[abcdh]: a, b, c, d, h \epsilon A .\supset.$

$(a^{\perp} \cap (a \cup ((b \cup c) \cup d))) \cup a = (h^{\perp} \cap h) \cup (((d^{\perp} \cap c^{\perp})^{\perp} \cup b) \cup a)$

[*H3*; *B1*; *cf.* proof of *F4* and Remark 7 in section **6**]

H10 $[abcd]: a, b, c, d \epsilon A .\supset. a \cup ((a \cup ((b \cup c) \cup d)) \cap a^{\perp})$

$= ((d^{\perp} \cap c^{\perp})^{\perp} \cup b) \cup a$

PR $[abcd]:$ Hp (1) $.\supset.$

$a \cup ((a \cup ((b \cup c) \cup d)) \cap a^{\perp}) = ((a \cup ((b \cup c) \cup d)) \cap a^{\perp}) \cup a$

$[1; A3, b/(a \cup ((b \cup c) \cup d)) \cap a^{\perp}]$

$= (a^{\perp} \cap (a \cup ((b \cup c) \cup d))) \cup a$ [$B3$, $a/a^{\perp}$, $b/a \cup ((b \cup c) \cup d)$]
$= (b^{\perp} \cap b) \cup (((d^{\perp} \cap c^{\perp})^{\perp} \cup b) \cup a)$ [$H9$, h/b]
$= ((d^{\perp} \cap c^{\perp})^{\perp} \cup b) \cup a$ [$H7$, $a/((d^{\perp} \cap c^{\perp})^{\perp} \cup b) \cup a$, h/b]

Thus, *H10*, *H8*, and *D19* are the consequences of *D1* and *H1*. Since it has been proved in [7] that the set $\{H10, H8, D10\}$ is a postulate system for orthomodular lattices, the proof of Proposition **VIII** holds.

9 *Proof of* **PIX** Let us assume *D1* and *K1*. Then, in an entirely analogous way to section **6**, we obtain *D3*, *B1*, *D4*, *A1*, *A3*, and *D2* from *D1* and *K1*. Hence the system $\{D1, K1\}$ is a lattice with universal bounds. Therefore, we can prove in the same manner as in section **6** the following theses:

K2 $[abcdemn]: a, b, c, d, e, m, n \in A \;.\supset.\; (((a \cap (b \cup c)) \cap (m \cap n)) \cup d) \cup e$
$= ((n \cap (((a \cap b) \cup (a \cap c)) \cap m)) \cup e) \cup d$ [*K1*; *D8*; *A1*; *B1*]

K3 $[abcmn]: a, b, c, m, n \in A \;.\supset.\; ((a \cap (b \cup c)) \cap (m \cap n))$
$= n \cap (((a \cap b) \cup (a \cup c)) \cap m)$
[*K1*; *A1*; *A3*; *cf.* proof of *D6* and Remark 6 in section **4**]

K4 $[abc]: a, b, c \in A \;.\supset.\; a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$
[*K3*; *B1*; *cf.* proof of *F4* and Remark 7 in section **6**]

Since the system $\{D1, K1\}$ is a lattice with universal bounds and implies *K4*, the proof of Proposition **IX** is complete.

10 In another paper I shall discuss some problems related to the kind of axiomatizations investigated here.

REFERENCES

[1] Birkhoff, G., *Lattice Theory*, third (new) edition. American Mathematical Society Colloquium Publications, Providence, Rhode Island, vol. XXV (1967).

[2] Byrne, L., "Two brief formulations of Boolean algebra," *Bulletin of the American Mathematical Society*, vol 52 (1966), pp. 269-272.

[3] Huntington, E. V. "New sets of independent postulates for the algebra of logic, with special reference to Whitehead and Russell's Principia Mathematica," *Transactions of the American Mathematical Society*, vol 35 (1933), pp. 274-304.

[4] Huntington, E. V., "Boolean algebra. A correction," *Transactions of the American Mathematical Society*, vol. 35 (1933), pp. 557-558.

[5] Kalman, J. A., "Two axiom definition for lattices," *Revue Roumaine de Mathematiques Pures et Appliquées*, vol. 13 (1968), pp. 669-679.

[6] Padmanabhan, R., "On axioms for semi-lattices," *Canadian Mathematical Bulletin*, vol. 9 (1966), pp. 357-358.

[7] Sobociński, B., "A short equational axiomatization of orthomodular lattices," *Notre Dame Journal of Formal Logic*, vol. XVII (1976), pp. 317-320.

University of Notre Dame
Notre Dame, Indiana