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GENERALIZATION IN FIRST-ORDER LOGIC

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Dealing initially with **QC**, the standard quantificational calculus of order one, I shall comment on a shortcoming, reported in 1956 by Montague and Henkin [18], in Church's 1944 account [2] of *a proof from hypotheses*, and sketch three ways of righting things.* The third, which exploits a trick of Fitch's and for this reason will be called *Fitch's account*, is the simplest of the three. I shall investigate it some, supplying fresh proof of **UGT**, the *Universal Generalization Theorem*. The proof holds good, it will turn out, as one passes from **QC** to **QC***, the presupposition-free variant of **QC**. Turning next to $\mathbf{QC}_=$, the standard quantificational calculus of order one with identity, and to the presupposition-free variant $\mathbf{QC}^*_=$ of $\mathbf{QC}_=$, I shall establish the lemmas needed there to obtain **UGT**. That given Fitch's account of a proof from hypotheses **UGT** holds for $\mathbf{QC}^*_=$ was argued in my recent *Truth-Value Semantics* [14], but the argument is circular, as Robert J. Cosgrove found out to my dismay.

The results submitted here are elementary, to be sure; but the difficulty that Montague and Henkin reported was quite a serious one, and ways of meeting it accordingly deserve attention. The results, by the way, are readily adapted to suit most (if not all) logics with quantifiers.

1.1 In most treatments of the calculus, **QC** has as its *primitive signs*:

(a) for each d from 0 on, aleph-zero predicate variables of degree d (to be referred to by means of 'F^d')[1]
(b) aleph-zero individual variables, say, 'x', 'y', 'z', 'x'', 'y'', 'z'', etc. (to be referred to by means of X and Y)
(c) the three logical operators: '$\sim$', '$\supset$', and '$\forall$'
(d) '(', ')', and ','.

1.2 It has as its *formulas* all finite sequences of primitive signs of **QC**

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(said sequences to be referred to by means of 'A', 'B', and 'C'). And it has as its *well-formed formulas* (wffs) all formulas of **QC** of any of the following four sorts:

(i) $F^d(X_1, X_2, \ldots, X_d)$, where $d \geqslant 0$
(ii) $\sim A$, in case A is a wff of **QC**
(iii) $(A \supset B)$, in case A and B are wffs of **QC**[2]
(iv) $(\forall X)A$, in case A is a wff of **QC**.

1.3 Further, any wff of **QC** counts as a *well-formed part* of itself; A counts as a well-formed part of the wffs $\sim A$ and $(\forall X)A$; A and B count as well-formed parts of the wff $A \supset B$; and any well-formed part of a well-formed part of a wff A counts as a well-formed part of A. An *occurrence* 0 of an individual variable X of **QC** in a wff A of **QC** is said to be *bound* if 0 is in a well-formed part of A of the sort $(\forall X)B$; 0 is said to be *free* in A if 0 is not bound in A; and the variable X itself is said to *occur free* in A if at least one occurrence of X in A is free. And I shall refer by means of '$A(Y/X)$' to A itself when at least one free occurrence of X in A is in a well-formed part of A of the sort $(\forall Y)B$, otherwise to the result of replacing every free occurrence of X in A by an occurrence of Y.

So much (at this point) for the primitive vocabulary and the grammar of **QC**.

1.4 There are numerous axiomatizations of **QC** in the literature. An especially serviceable one reckons as the *axioms of* **QC** all wffs of **QC** of any of the following six sorts:

A1. $A \supset (B \supset A)$
A2. $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
A3. $(\sim A \supset \sim B) \supset (B \supset A)$
A4. $(\forall X)(A \supset B) \supset ((\forall X)A \supset (\forall X)B)$
A5. $A \supset (\forall X)A$
A6. $(\forall X)A \supset A(Y/X)$,

where in the fifth case X does not occur free in A.

1.5 Given some such axiomatization, the pre-1956 literature would generally[3] own as a *proof in* **QC** *from a finite set S of wffs of* **QC** any column

$$\begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_p \end{array}$$

of wffs of **QC** such that, for each i from 1 through p: (i) A_i belongs to S, or (ii) A_i is an axiom of **QC**, or (iii) A_i is preceded by a wff $A_g \supset A_i$ for some g smaller than i, in which case A_i is said to *follow from A_g and $A_g \supset A_i$ by* MP (= *modus ponens*), or (iv) A_i is of the sort $(\forall X)A_h$ for some h smaller than i and some individual variable X of **QC** that does not occur free in any member of S, in which case A_i is said to *follow from A_h by* **UG**

(= Universal Generalization) and X is said to be (*universally*) *generalized upon*.

A wff A of **QC** would then be declared (a) *provable in* **QC** *from a finite set S of wffs of* **QC** if there is a proof $\mathfrak{P}$ in **QC** from S such that the last entry in $\mathfrak{P}$ is A, (b) *provable in* **QC** *from an infinite set S of wffs of* **QC** if A is provable in **QC** from at least one finite subset of S, and (c) *provable in* **QC** if A is provable in **QC** from $\emptyset$.

1.6 The account stems from Church [2], and so will be called *Church's account of a proof from hypotheses*.[4] As mentioned in the introductory paragraphs, it has a serious shortcoming. Indeed, Montague and Henkin have shown in [18] that—given Church's account—the wff '$(\forall y)(g(y) \supset g(y))$', though a semantic consequence of the set $\{g(y)\}$, is *not* provable in **QC** from $\{g(y)\}$.[5] So the following result, known of course as the *Strong Completeness Theorem for* **QC** and in which '$S \vdash A$' is to be understood as short for 'A is provable in **QC** from S':

(A) *If a wff A of* **QC** *is a semantic consequence of a set S of wffs of* **QC**, *then $S \vdash A$*,

cannot be had in [2].

1.7 It is easy to spot, incidentally, where (current versions of) Henkin's 1949 proof of (A) would break down in [2].[6] Having constructed an infinite array S_0, S_1, S_2, etc., of sets of wffs of **QC**, one goes on to show that $\sum_{i=0} S_i$ is sure to be syntactically consistent if each one of S_0, S_1, S_2, etc., is; and one does so by arguing that if $\sum_{i=0} S_i$ were syntactically inconsistent, then so would be some finite subset of $\sum_{i=0} S_i$, hence so would be some finite subset of one of S_0, S_1, S_2, etc., and hence so would be one of S_0, S_1, S_2, etc. Now this last step appeals to a familiar result:

(B) *If $S \vdash A$, then $S' \vdash A$ for any superset S' of S*,

which cannot be had in [2]. For proof, consider again the Montague-Henkin wff '$(\forall y)(g(y) \supset g(y))$'. Given Church's account of a proof from hypotheses, '$(\forall y)(g(y) \supset g(y))$' is provable in **QC** (= provable in **QC** from $\emptyset$). Yet, as Montague and Henkin showed, the wff is not provable in **QC** from the superset $\{g(y)\}$ of $\emptyset$.

1.8 Now for three solutions to this difficulty.[7]

There is a new (and welcome) trend in logic writings: (1) using for each type of variables one array of letters as *bound* variables and another array as *free* variables, and (2) reserving the label 'variables' for the letters that serve as bound variables and calling the other letters *parameters*.[8]

Under this convention, *which I heed throughout the balance of the paper*, several changes must be brought to the preceding material:

first, clause (a), Section **1.1**, must be edited to read

(a) for each d from 0 on, aleph-zero predicate parameters of degree d (to be referred to by means of 'F^d');

second, clause (b) must give way to the double clause

(b1) aleph-zero individual variables, say, 'x', 'y', 'z', 'x'', 'y'', 'z'', etc. (to be referred to by means of X and Y),

(b2) aleph-zero individual parameters, say, 'a', 'b', 'c', 'a'', 'b'', 'c'', etc. (to be referred to by means of 'P');

third, clause (i), Section **1.2**, must be edited to read

(i) $F^d(P_1, P_2, \ldots, P_d)$, where $d \geqslant 0$;

fourth, clause (iv) must be edited to read

(iv) $(\forall X)A$, in case the result $A(P/X)$ of putting an individual parameter P of **QC** everywhere for X in A is a wff of **QC**;[9]

fifth, Section **1.3**, which deals with the well-formed parts of a wff and with bound and free (occurrences of) individual variables, is needless;

sixth, the restriction on the individual variable X of axiom schema *A5* (in Section **1.4**) is similarly needless; and

seventh, axiom schema *A6* must be edited to read

A6. $(\forall X)A \supset A(P/X)$.

1.9 Using two arrays of individual signs disposes of the Montague-Henkin difficulty. First, draft the letter I to refer to the individual signs (i.e., the individual variables and the individual parameters) of **QC**; and then, generalizing the '$A(P/X)$' twice used in the preceding paragraph, draft '$[(A)(I'/I)]$' (when no ambiguity threatens: '$[A(I'/I)]$', '$(A)(I'/I)$', or plain '$A(I'/I)$') to refer to the result of putting (individual sign) I' everywhere for (individual sign) I in (formula) A. This done, edit clause (iv), Section **1.5**, as the distinction between variables and parameters requires:

(iv) A_i is of the sort $(\forall X)[A_h(X/P)]$ for some h smaller than i and some individual parameter P of **QC** that is foreign to each member of S,[10] in which case A_i is said to *follow from* A_h *by* **UG** and P to be *quasi-generalized upon*; and the Montague-Henkin problem is solved.

Proof that (B) holds true given this account of things can indeed be retrieved from [13], a text published some ten years after [18] and in which I used only one array of individual signs but met the Montague-Henkin difficulty in essentially the same manner as above.[11]

1.10 I have grown dissatisfied with the account, though. Deviating in this from Church, I would rather characterize provability from sets of hypotheses *at a stroke* rather than distinguish as [3] and [13] do between finite sets and infinite ones. Unfortunately, if you delete the qualifier 'finite' from the second line of **1.5**, edit clause (iv) of **1.5** to read as in the previous paragraph, and take a wff A of **QC** to be *provable in* **QC** *from a set* S (*be* S *finite or infinite*) *of wffs of* **QC** if there is a proof $\mathfrak{P}$ in **QC** from S such that the last entry in $\mathfrak{P}$ is A, then (B) no longer holds true. Indeed, a slight adaptation of the Montague-Henkin argument in [18] will show that '$(\forall y)(g(y) \supset g(y))$', though provable in **QC** from $\emptyset$, is no longer provable from the infinite superset $\{g(a), g(b), g(c), g(a'), g(b'), g(c'), \ldots\}$ of $\emptyset$.[12]

1.11 Another account, one which handles provability from hypotheses at a stroke, can be retrieved from pp. 94-98 of [10], which—Kleene tells me—were written up by 1942. Kleene's account is complicated, however, and Montague and Henkin accordingly devised a substitute one, to which I restrict myself.[13] To quote almost verbatim from [18], "where $\mathfrak{C}$ and $\mathfrak{C}'$ are columns of wffs of **QC**, call $\mathfrak{C}$ a *subcolumn of* $\mathfrak{C}'$ if and only if the wffs of $\mathfrak{C}$ appear among those of $\mathfrak{C}'$ in the same order which they have in $\mathfrak{C}$. (It is not required that two consecutive wffs in $\mathfrak{C}$ appear consecutively in $\mathfrak{C}'$.)" Next, tailoring matters to suit the distinction drawn here between variables and parameters, you understand

1. by a *proof in* **QC** any finite column $\mathfrak{C}$ of wffs of **QC** such that, for any wff A in $\mathfrak{C}$, A is an axiom of **QC**, or follows by MP from two earlier entries in $\mathfrak{C}$, or follows by **UG** (reading as in **1.5**) from an earlier entry in $\mathfrak{C}$, and
2. by a *proof in* **QC** *from a set S of wffs of* **QC** any finite column $\mathfrak{C}$ of wffs of **QC** such that, for any wff A in $\mathfrak{C}$, A belongs to S, or is an axiom of **QC**, or follows by MP from two earlier wffs in $\mathfrak{C}$, or is of the sort $(\forall X)[B(X/P)]$, where B is the last entry in a subcolumn of $\mathfrak{C}$ that qualifies as a proof in **QC**.

A wff A of **QC** is then declared *provable in* **QC** *from a set S of wffs of* **QC** if there is a proof $\mathfrak{P}$ in **QC** from S such that A is the last entry in $\mathfrak{P}$.

That (B) holds true given this second account of things is immediately evident.

1.12 As for Fitch's account, it stems from [5], a paper published in 1948 and hence antedating [18] by eight years. (Fitch, by the way, was unaware in 1948 of the difficulty eventually reported by Montague and Henkin,[14] and the latter were unaware in 1956 of Fitch's paper.) The account, tailored here to suit our distinction between variables and parameters, is of the simplest. You identify the *axioms of* **QC** recursively, first declaring any wff of **QC** of any of the sorts *A1-A6*, for example, an axiom of **QC** and next declaring any wff of **QC** of the sort $(\forall X)[A(X/P)]$ an axiom of **QC** if A itself is one.[15]

You then acknowledge as a *proof in* **QC** *from a set S of wffs of* **QC** any finite column $\mathfrak{C}$ of wffs of **QC** such that, for any wff A in $\mathfrak{C}$, A belongs to S, or is an axiom of **QC** (in the sense just defined), or follows by MP from two earlier entries in $\mathfrak{C}$; and you declare a wff A of **QC** *provable in* **QC** *from a set S of wffs of* **QC** if there is a proof $\mathfrak{P}$ in **QC** from S such that A is the last entry in $\mathfrak{P}$.[16]

That (B) holds true given this third account of things is immediately evident.

1.13 Proof of (A), the Strong Completeness Theorem for **QC**, calls for a number of lemmas besides (B). One of them is the Universal Generalization Theorem mentioned in the introductory paragraph. It runs:

(C) *If* $S \vdash A$, *then* $S \vdash (\forall X)[A(X/P)]$, *so long as* X *is foreign to* A *and* P *is foreign to* S, and helps to show that if any one of the sets S_0, S_1, S_2, etc. (Section **1.7**), say set S_n, were syntactically inconsistent, then so would set S_{n-1} be.

That (C) holds true given Leblanc's account of a proof from hypotheses is immediately evident. That it holds true given Montague and Henkin's account is shown in [18], pp. 133-134. And that it holds true given Fitch's account readily follows from (3.7.12) on pp. 336-337 of [17].

As perusal of [13], [18], and [17] will show, further lemmas needed to prove (A) and the various lemmas needed to prove the converse of (A):

(A′) *If $S \vdash A$, where S is a set of wffs and A is a wff of* **QC**, *then A is a semantic consequence of S,* (= *the Strong Soundness Theorem for* **QC**)

all hold true given Leblanc's account, given Montague and Henkin's, and given Fitch's. Each of our three accounts thus puts things to rights.

These preliminaries over with, I limit myself henceforth to Fitch's account, and with an eye to further results prove (C) anew.

2 I first establish (C) for the case where X is foreign to S (= Theorem 1), and obtain as a corollary that if $A(P/X)$ is provable in **QC**, then $(\forall X)A$ is sure to be (= Theorem 2). With Theorem 1 on hand, I then establish (C) for the general case (= Theorem 3). The resulting demonstration of (C) is admittedly longer than that of (3.7.12) in [17]. But *the portion of it that yields Theorem 1* (*and hence Theorem 2*) *does without axiom schema A6.* In the lemmas and theorems that follow '$S \vdash A$' is to be understood as 'A is provable in **QC** from S (in Fitch's sense)', and '$\vdash A$' as 'A is provable in **QC** (in Fitch's sense)'.

Lemma 1 (a) *If A is an axiom of* **QC**, *then so is $A(Y/X)$, so long as Y is foreign to A.*
(b) *If A belongs to S or is an axiom of* **QC**, *then $S \vdash A$.*
(c) *If $S \vdash A$ and $S \vdash A \supset B$, then $S \vdash B$.*
(d) *If $S \vdash A$, then there is a finite subset S' of S such that $S' \vdash A$.*
(e) *If $S \vdash A$, then $S' \vdash A$ for any superset S' of S.*
(f) *If $S \vdash (\forall X)(A \supset B)$, then $S \vdash (\forall X)A \supset (\forall X)B$.*
(g) *If $S \vdash (\forall X)(A \supset B)$ and $S \vdash (\forall X)A$, then $S \vdash (\forall X)B$.*
(h) *If $S \vdash A$, then $S \vdash (\forall X)A$, so long as X is foreign to A.*

Proof: (a) Proof of A is by cases. It uses three easily verified facts: (i) $(\sim A)(Y/X)$ is the same as $\sim[A(Y/X)]$; (ii) $(A \supset B)(Y/X)$ is the same as $A(Y/X) \supset B(Y/X)$; (iii) $((\forall X)A)(Y/X)$ is the same as $(\forall Y)[A(Y/X)]$; and (iv) if A is a wff of **QC**, then so is $A(Y/X)$, so long as Y is foreign to A. (b)-(h) Proofs of these, when not immediate, are routine.

Lemma 2 *Let X be foreign to S and to A. If there is a proof in* **QC** *from S whose last entry is A, then there is one to which X is foreign.*

Proof: Let the column made up of B_1, B_2, . . ., and B_p constitute a proof in **QC** from S whose last entry is A, and let Y be an individual variable of **QC** foreign to all of B_1, B_2, . . ., and B_p. (i) A routine induction shows that the column made up of $B_1(Y/X)$, $B_2(Y/X)$, . . ., and $B_p(Y/X)$ constitutes a proof in **QC** from S whose last entry is $B_p(Y/X)$. For suppose B_i $(1 \leqslant i \leqslant p)$ belongs to S. With X foreign to S, $B_i(Y/X)$ is sure to be the same as B_i and hence to belong to S. Or suppose B_i is an axiom of **QC**. With Y foreign to B_i, $B_i(Y/X)$ is sure by Lemma 1(a) to be an axiom of **QC** as well. Or suppose B_i is the ponential of, say, B_g and B_h, where B_h is $B_g \supset B_i$. Since $B_h(Y/X)$ and $B_g(Y/X) \supset B_i(Y/X)$ are the same, $B_i(Y/X)$ is sure to be the ponential of $B_g(Y/X)$ and $B_h(Y/X)$. (ii) Since X is foreign to $B_1(Y/X)$, $B_2(Y/X)$, . . ., and $B_p(Y/X)$, there is sure in view of (i) to be a proof in **QC** from S whose last entry is $B_p(Y/X)$ and to which X is foreign. But, with X foreign to A $(= B_p)$, $B_p(Y/X)$ is sure to be the same as B_p and hence as A. Hence there is sure to be a proof in **QC** from S whose last entry is A and to which X is foreign. Hence Lemma 2.

Theorem 1 *Let X be foreign to S and A, and P be foreign to S. If $S \vdash A$, then $S \vdash (\forall X)[A(X/P)]$.* (**UGT** *for* **QC**, *Special Case*)

Proof: Suppose $S \vdash A$. Since X is foreign to S and A, there is sure by Lemma 2 to be a proof in **QC** from S whose last entry is A and to which X is foreign. Let the column made up of B_1, B_2, . . ., and B_p constitute then such a proof. A routine induction shows that $S \vdash (\forall X)[B_i(X/P)]$ $(1 \leqslant i \leqslant p)$, and hence that $S \vdash (\forall X)[B_p(X/P)]$ $(= (\forall X)[A(X/P)])$. For suppose that B_i belongs to S and hence by Lemma 1(b) that $S \vdash B_i$. Since X is foreign to B_i, $S \vdash (\forall X)B_i$ by Lemma 1(h). But, with P foreign to S, B_i and $B_i(X/P)$ are sure to be the same. Hence $S \vdash (\forall X)[B_i(X/P)]$. Or suppose B_i is an axiom of **QC**. Since X is foreign to B_i, $(\forall X)[B_i(X/P)]$ is sure to be well-formed, and hence by the inductive clause in Fitch's account of an axiom of **QC** to qualify as an axiom of **QC**. Hence $S \vdash (\forall X)[B_i(X/P)]$ by Lemma 1(b). Or suppose B_i is the ponential of, say, B_g and B_h, where B_h is $B_g \supset B_i$. By the hypothesis of the induction $S \vdash (\forall X)[B_g(X/P)]$ and $S \vdash (\forall X)[(B_g \supset B_i)(X/P)]$. But $(\forall X)[(B_g \supset B_i)(X/P)]$ and $(\forall X)(B_g(X/P) \supset B_i(X/P))$ are the same. Hence $S \vdash (\forall X)(B_g(X/P) \supset B_i(X/P))$, and hence $S \vdash (\forall X)[B_i(X/P)]$, by Lemma 1(g).

Theorem 2 *Let X be foreign to A. If $\vdash A$, then $\vdash (\forall X)[A(X/P)]$.*

Proof by Theorem 1, with $\emptyset$ serving as S.

The reader will have noticed that, as promised, the foregoing proof of Theorem 1 (and hence that of Theorem 2) does without *A6*.

Lemma 3 $\vdash (\forall Y)((\forall X)A \supset A(Y/X))$.

Proof: Let P be an individual parameter of **QC** foreign to A. $(\forall X)A \supset A(P/X)$ counts as an axiom of **QC**. But, with $(\forall Y)((\forall X)A \supset A(Y/X))$ presumed to be well-formed, $(\forall Y)[((\forall X)A \supset A(P/X))(Y/P)]$ is likewise sure to be well-formed. Hence, by the inductive clause in Fitch's account of an axiom of **QC**, $(\forall Y)[((\forall X)A \supset A(P/X))(Y/P)]$ counts as an axiom of **QC**. But, with P foreign to A, $((\forall X)A \supset A(P/X))(Y/P)$ and $(\forall X)A \supset A(Y/X)$ are sure to be the same. Hence $(\forall Y)((\forall X)A \supset A(Y/X))$ counts as an axiom of **QC**. Hence Lemma 3 by Lemma 1(b).

Lemma 4 *Let* Y *be foreign to* $(\forall X)A$. *If* $S \vdash (\forall X)A$, *then* $S \vdash (\forall Y)[A(Y/X)]$.

Proof: Suppose $S \vdash (\forall X)A$. Then $(\forall X)A$ is sure to be well-formed. But, with $(\forall X)A$ well-formed and Y foreign to $(\forall X)A$, $(\forall Y)((\forall X)A \supset A(Y/X))$ is likewise sure to be well-formed. Hence $\vdash (\forall Y)((\forall X)A \supset A(Y/X))$ by Lemma 3, hence $\vdash (\forall Y)(\forall X)A \supset (\forall Y)[A(Y/X)]$ by Lemma 1(f), and hence $S \vdash (\forall Y)(\forall X)A \supset (\forall Y)[A(Y/X)]$ by Lemma 1(e). But, since $S \vdash (\forall X)A$ and Y is foreign to $(\forall X)A$, $S \vdash (\forall Y)(\forall X)A$ by Lemma 1(h). Hence $S \vdash (\forall Y)[A(Y/X)]$ by Lemma 1(c). Hence Lemma 4.

Theorem 3 *Let* X *be foreign to* A, *and* P *be foreign to* S. *If* $S \vdash A$, *then* $S \vdash (\forall X)[A(X/P)]$. (**UGT** *for* **QC**, *General Case*)

Proof: Suppose that $S \vdash A$, and hence by Lemma 1(d) that $S' \vdash A$ for some finite subset S' of S; and let Y be an individual variable of **QC** distinct from X and foreign to S' and A. Then $S' \vdash (\forall Y)[A(Y/P)]$ by Theorem 1, and hence $S \vdash (\forall Y)[A(Y/P)]$ by Lemma 1(e). But, with X foreign to A and Y distinct from X, X is sure to be foreign to $(\forall Y)[A(Y/P)]$. Hence $S \vdash (\forall X)[(A(Y/P))(X/Y)]$ by Lemma 4. But, with Y foreign to A, $(A(Y/P))(X/Y)$ and $A(X/P)$ are sure to be the same. Hence $S \vdash (\forall X)[A(X/P)]$. Hence Theorem 3.

The reader will have noticed that, though the proof of Lemma 3 resorts to *A6*, that of Lemma 4 and hence that of Theorem 3 do not. They merely presuppose that $(\forall X)((\forall X)A \supset A(Y/X))$, when well-formed, is provable in **QC**. The point will prove crucial further on.[17]

A version of Theorem 2 and one of Theorem 3 can be had which are closer to the interlim rule $\forall$I of Natural Deduction. Proof of Lemma 5 is immediate.

Lemma 5 *Let* P *be foreign to* A. *Then* A *and* $(A(P/X))(X/P)$ are the same.

Theorem 4 (a) *Let* P *be foreign to* $(\forall X)A$. *If* $\vdash A(P/X)$, *then* $\vdash (\forall X)A$. (b) *Let* P *be foreign to* S *and to* $(\forall X)A$. *If* $S \vdash A(P/X)$, *then* $S \vdash (\forall X)A$. ($\forall$I *for* **QC**)

Proof: (a) Suppose $\vdash A(P/X)$. Since X is foreign to $A(P/X)$, $\vdash (\forall X)[(A(P/X))(X/P)]$ by Theorem 2. But, being presumed to be foreign to $(\forall X)A$, P is sure to be foreign to A. Hence $\vdash (\forall X)A$ by Lemma 5. Hence (a). (b) Proof like that of (a), but using Theorem 3 in lieu of Theorem 2.

The foregoing proof of Theorem 4(a), the reader will have noticed, does without *A6*; and that of Theorem 4(b) merely presupposes that $(\forall Y)((\forall X)A \supset A(Y/X))$, when well-formed, is provable in **QC**.

Now for **QC***, the subcalculus of **QC** that grew out of [15] and [8], and is commonly known as *free logic* (*without identity*). Space prevents me from supplying a full-fledged semantics for **QC***. From a model-theoretic stance, suffice it to note that (i) $\emptyset$ counts in **QC*** as a domain and (ii) when the domain is non-empty, the individual parameters '*a*', '*b*', '*c*', etc., may go in **QC*** without values.[18] A Strong Soundness Theorem for **QC*** and (as established in [14], pp. 136-145) a Strong Completeness one can be had if *A6* in Section **1.8** is weakened to read

$A6^*$. $(\forall Y)((\forall X)A \supset A(Y/X))$,[19]

and the *Commutation Law for Universal Quantifiers*, to wit:

$A7^*$. $(\forall X)(\forall Y)A \supset (\forall Y)(\forall X)A$,

is adopted as an extra axiom schema.[20]

Wffs of **QC*** of the sort $(\forall Y)((\forall X)A \supset A(Y/X))$, once made into axioms of **QC***, are sure of course to be provable in **QC***. So, as announced in the introductory paragraph, the foregoing demonstration of Theorem 3 (and hence that of Theorem 4 as well) holds good for **QC***; and, starring '$\vdash$' to signal that the calculus at issue is **QC***, I conclude:

Theorem 5 (a) *Let X be foreign to A, and P be foreign to S. If* $S \vdash^* A$, *then* $S \vdash^* (\forall X)[A(X/P)]$. (**UGT** *for* **QC***)
(b) *Let P be foreign to S and to* $(\forall X)A$. *If* $S \vdash^* A(P/X)$, *then* $S \vdash^* (\forall X)A$. ($\forall$I *for* **QC***)

Fitch's account of a proof from hypotheses thus yields **UGT** (and $\forall$I) for both **QC** and **QC***.[21]

3 $\mathbf{QC}_=$, the (standard) quantificational calculus of order one with identity, has the same *primitive signs* as **QC**, *plus* of course the identity predicate '='. Its *formulas* are all finite sequences of primitive signs of $\mathbf{QC}_=$. Under our convention regarding variables and parameters, its *well-formed formulas* (wffs) are all formulas of the sort (i) in Section **1.8**, *plus* all those of the sort $(P = P')$,[22] *plus* all those of either of the sorts (ii)-(iii) of Section **1.2** (with '$\mathbf{QC}_=$' substituting there for '**QC**'), *plus* all those of the sort (iv) in **1.8** (with '$\mathbf{QC}_=$' substituting there for '**QC**'). And its *atomic wffs* are all those of the sort (i) in **1.8**, *plus* all those of the sort $P = P'$ above. (For brevity's sake I shall write '$(A \equiv B)$' in lieu of '$\sim((A \supset B) \supset \sim(B \supset A))$', and '$(\exists X)A$' in lieu of '$\sim(\forall X)\sim A$'.)

Among the numerous axiomatizations of $\mathbf{QC}_=$, three—retouched to suit the distinction between variables and parameters—rate mention at this point. The oldest and best known of them would own as the *axioms of* $\mathbf{QC}_=$ all wffs of any of the sorts *A1*-*A5* (in **1.4**), *plus* all those of the sort *A6* in **1.8**, *plus* all those of the sort

$$P = P \tag{1}$$

or the sort

$$P = P' \supset (A \supset A(P'/\!/P)), \tag{2}$$

where $A(P'/\!/P)$ is like A except for exhibiting P' *at zero or more places* where A exhibits P. Another, supplied by Tarski in [21], dispenses with *A6*, uses in lieu of (1) the axiom schema

$$(\exists X)(X = P),$$

and uses in lieu of (2) the axiom schema

$$P = P' \supset (A \supset A(P'/_1P)), \tag{3}$$

where (i) A is atomic and (ii) $A(P'/_1P)$ is like A except for exhibiting P' *at exactly one place* where A exhibits P.[23] And yet another, exploiting a suggestion of van Fraassen's,[24] is like the second, but uses in lieu of (3) the following two axiom schemata:

$$P = P' \supset (A \supset A(P'/P))$$

and

$$P = P' \supset (A(P'/P) \supset A),$$

where in either case A is presumed to be atomic.[25]

Given any of these axiomatizations, the pre-1956 literature would generally understand proofs from hypotheses in the manner of Church,[26] which of course blocks the *Strong Completeness Theorem* for $\mathbf{QC}_=$:

(D) *If a wff A of* $\mathbf{QC}_=$ *is a semantic consequence of a set S of wffs of* $\mathbf{QC}_=$, *then* $S \vdash A$.

Here as in **QC** things can be mended in at least three different ways: Leblanc's way, Montague and Henkin's, and Fitch's. Opting again for Fitch's, I shall acknowledge as the *axioms of* $\mathbf{QC}_=$ all wffs of any of the five sorts *A1*-*A5* (Section **1.4**), all those of any of the following three sorts:

B1. $(\exists X)(X = P)$,
B2. $P = P' \supset (A \supset A(P'/P))$, where A is atomic,
B3. $P = P' \supset (A(P'/P) \supset P)$, where A is atomic,

and all those of the sort $(\forall X)[A(X/P)]$, where A is an axiom of $\mathbf{QC}_=$; and I shall understand proofs from hypotheses as in **1.12**, paragraph 2 (with '$\mathbf{QC}_=$' substituting there for '**QC**').

Here as in **QC** my concern is with **UGT**, one of the main lemmas needed to obtain (D). Following Tarski's precedent in [22], I shall first establish

$$P = P' \supset (A \supset A(P'/P))$$

for any wff A of $\mathbf{QC}_=$, atomic or not. Borrowing from [15] and [22],[27] I shall then establish

$$(\exists X)(X = P) \supset ((\forall X)A \supset A(P/X)),$$

and get therefrom the counterpart for $\mathbf{QC}_=$ of Lemma 3 in II:

$$(\forall Y)((\forall X)A \supset A(Y/X)).$$

Since the proof of Theorem 3 in II uses only Lemma 3 and axioms whose counterparts for $\mathbf{QC}_=$ are available here, the way will be clear for **UGT**. Proofs of the other lemmas needed to obtain (D) will be found in [15].

In what follows I write '$\vdash_=$' to signal that the calculus at issue is $\mathbf{QC}_=$.

Lemma 6 (a) $\vdash_= A \supset ((A \supset B) \supset B)$.
(b) *If* $\vdash_= A \supset B$ *and* $\vdash_= B \supset C$, *then* $\vdash_= A \supset C$.
(c) *If* $\vdash_= A \supset (B \supset C)$, *then* $\vdash_= B \supset (A \supset C)$.
(d) *If* $\vdash_= A \supset (B \supset C)$ *and* $\vdash_= C \supset C'$, *then* $\vdash_= A \supset (B \supset C')$.
(e) *If* $\vdash_= A \supset (B \equiv C)$, *then* $\vdash_= A \supset (B \supset C)$ *and* $\vdash_= A \supset (C \supset B)$.
(f) *If* $\vdash_= A \supset (B \supset C)$ *and* $\vdash_= A \supset (C \supset B)$, *then* $\vdash_= A \supset (B \equiv C)$.
(g) *If* $\vdash_= A \supset (B \equiv B')$ *and* $\vdash_= A \supset (C \equiv C')$, *then* $\vdash_= A \supset ((B \supset C) \equiv (B' \supset C'))$.
(h) $\vdash_= (A \supset B) \supset ({\sim}B \supset {\sim}A)$.
(i) *If* $\vdash_= {\sim}A \supset B$, *then* $\vdash_= {\sim}B \supset A$.
(j) *If* $\vdash_= A \supset (B \equiv C)$, *then* $\vdash_= A \supset ({\sim}B \equiv {\sim}C)$.

Proof: left to the reader.

Lemma 7 (a) *If* $\vdash_= A(P/X) \supset B(P/X)$, *then* $\vdash_= (\forall X)A \supset (\forall X)B$, *so long as* P *is foreign to* $(\forall X)A$ *and to* $(\forall X)B$.
(b) *If* $\vdash_= A \supset (\forall X)(B \supset C)$, *then* $\vdash_= A \supset ((\forall X)B \supset (\forall X)C)$.
(c) *If* $\vdash_= A \supset (B(P/X) \supset C(P/X))$, *then* $\vdash_= A \supset ((\forall X)B \supset (\forall X)C)$, *so long as* P *is foreign to* A, $(\forall X)B$, *and* $(\forall X)C$, *and* X *is foreign to* A.
(d) *If* $\vdash_= A \supset (B(P/X) \equiv C(P/X))$, *then* $\vdash_= A \supset ((\forall X)B \equiv (\forall X)C)$, *so long as* P *and* X *are as in* (c).
(e) $\vdash_= (\forall X)(A \supset B) \supset ((\exists X)A \supset (\exists X)B)$.
(f) *If* $\vdash_= A(P/X) \supset B(P/X)$, *then* $\vdash_= (\exists X)A \supset (\exists X)B$, *so long as* P *is foreign to* $(\exists X)A$ *and to* $(\exists X)B$.
(g) $\vdash_= (\exists X)A \supset A$.
(h) $\vdash_= (\exists X)(A \supset B) \supset ((\forall X)A \supset B)$, *so long as* X *is foreign to* B.

Proof: (a) Suppose $\vdash_= A(P/X) \supset B(P/X)$, and hence $\vdash_= (A \supset B)(P/X)$. Suppose further that P is foreign to $(\forall X)A$ and to $(\forall X)B$, and hence to $(\forall X)(A \supset B)$ as well. Then $\vdash_= (\forall X)(A \supset B)$ by Theorem 4(a), and hence $\vdash_= (\forall X)A \supset (\forall X)B$ by Lemma 1(f). Hence (a).
(b) Suppose $\vdash_= A \supset (\forall X)(B \supset C)$. With $(\forall X)(B \supset C)$ presumed here to be well-formed, $(\forall X)(B \supset C) \supset ((\forall X)B \supset (\forall X)C)$ is sure to be well-formed and hence to count as an axiom of $\mathbf{QC}_=$. Hence $\vdash_= (\forall X)(B \supset C) \supset ((\forall X)B \supset (\forall X)C)$ by Lemma 1(b). Hence $\vdash_= A \supset ((\forall X)B \supset (\forall X)C)$ by Lemma 6(b). Hence (b).
(c) Suppose $\vdash_= A \supset (B(P/X) \supset C(P/X))$, and hence $\vdash_= A \supset (B \supset C)(P/X)$. Suppose further that X is foreign to A. Then A and $A(P/X)$ are the same, and hence $\vdash_= A(P/X) \supset (B \supset C)(P/X)$. Suppose finally that P is foreign to A, $(\forall X)B$, and $(\forall X)C$. Then P is sure to be foreign to $(\forall X)A$ and to $(\forall X)(B \supset C)$. Hence $\vdash_= (\forall X)A \supset (\forall X)(B \supset C)$ by (a). But, with A presumed here to be well-formed and with X foreign to A, $A \supset (\forall X)A$ is sure to be well-formed and hence to count as an axiom of $\mathbf{QC}_=$. Hence $\vdash_= A \supset (\forall X)A$ by Lemma 1(b), hence $\vdash_= A \supset (\forall X)(B \supset C)$ by Lemma 6(b), and hence $\vdash_= A \supset ((\forall X)B \supset (\forall X)C)$ by (b).
(d) Suppose $\vdash_= A \supset (B(P/X) \equiv C(P/X))$. Then $\vdash_= A \supset (B(P/X) \supset C(P/X))$ by Lemma 6(e). Suppose further that P is foreign to A, $(\forall X)B$, and $(\forall X)C$, and X is foreign to A. Then $\vdash_= A \supset ((\forall X)B \supset (\forall X)C)$. But by the same reasoning and under the same assumptions $\vdash_= A \supset ((\forall X)C \supset (\forall X)B)$. Hence $\vdash_= A \supset ((\forall X)B \equiv (\forall X)C)$ by Lemma 6(f).

(e) Let P be an individual parameter of $\mathbf{QC}_=$ foreign to $(\forall X)(A \supset B)$ and to $(\forall X)(\sim B \supset \sim A)$. $\vdash_= (A(P/X) \supset B(P/X)) \supset (\sim[B(P/X)] \supset \sim[A(P/X)])$ by Lemma 6(h). But $(A(P/X) \supset B(P/X)) \supset (\sim[B(P/X)] \supset \sim[A(P/X)])$ and $(A \supset B)(P/X) \supset (\sim B \supset \sim A)(P/X)$ are the same. Hence $\vdash_= (A \supset B)(P/X) \supset (\sim B \supset \sim A)(P/X)$, hence $\vdash_= (\forall X)(A \supset B) \supset (\forall X)(\sim B \supset \sim A)$ by (a), and hence $\vdash_= (\forall X)(A \supset B) \supset ((\forall X)\sim B \supset (\forall X)\sim A)$ by (b). But $\vdash((\forall X)\sim B \supset (\forall X)\sim A) \supset ((\exists X)A \supset (\exists X)B)$ by Lemma 6(h). Hence (e) by Lemma 6(b).
(f) Suppose $\vdash_= A(P/X) \supset B(P/X)$, and hence $\vdash_= (A \supset B)(P/X)$; suppose further that P is foreign to $(\exists X)A$ and to $(\exists X)B$, and hence to $(\forall X)(A \supset B)$. Then $\vdash_= (\forall X)(A \supset B)$ by Theorem 4(a). Hence $\vdash_= (\exists X)A \supset (\exists X)B$ by (e) and Lemma 1(c). Hence (f).
(g) With $(\exists X)A \supset A$ presumed here to be well-formed, $\sim A \supset (\forall X)\sim A$ is sure to be well-formed and hence to count as an axiom of $\mathbf{QC}_=$. Hence $\vdash_= \sim A \supset (\forall X)\sim A$ by Lemma 1(b), and hence $\vdash_= (\exists X)A \supset A$ by Lemma 6(i).
(h) Let P be an individual parameter of $\mathbf{QC}_=$ foreign to $(\forall X)A$ and to $(\forall X)((A \supset B) \supset B)$. $\vdash_= A(P/X) \supset ((A(P/X) \supset B(P/X)) \supset B(P/X))$ by Lemma 6(a). But $(A(P/X) \supset B(P/X)) \supset B(P/X)$ and $((A \supset B) \supset B)(P/X)$ are the same. Hence $\vdash_= A(P/X) \supset ((A \supset B) \supset B)(P/X)$, and hence $\vdash_= (\forall X)A \supset (\forall X)((A \supset B) \supset B)$ by (a). But $\vdash_= (\forall X)((A \supset B) \supset B) \supset ((\exists X)(A \supset B) \supset (\exists X)B)$ by (e). Hence $\vdash_= (\forall X)A \supset ((\exists X)(A \supset B) \supset (\exists X)B)$ by Lemma 6(b), and hence $\vdash_= (\exists X)(A \supset B) \supset ((\forall X)A \supset (\exists X)B)$ by Lemma 6(c). Suppose further that X is foreign to B. Since B is presumed here to be well-formed, $(\exists X)B \supset B$ is sure to be well-formed, and hence $\vdash_= (\exists X)B \supset B$ by (g). Hence (h) by Lemma 6(d).

Lemma 8 (a) $\vdash_= P = P' \supset (A \equiv A(P'/P))$.
(b) $\vdash_= P = P' \supset (A \supset A(P'/P))$.
(c) $\vdash_= P = P' \supset (A(P/X) \supset A(P'/X))$, *so long as P is foreign to A.*

Proof: (a) Proof of (a) is by mathematical induction on the number n of logical operators in A.
Basis: $n = 0$, in which case A is atomic. Then $P = P' \supset (A \supset A(P'/P))$ and $P = P' \supset (A(P'/P) \supset A)$, being presumed here to be well-formed, count as axioms of $\mathbf{QC}_=$. Hence (a) by Lemma 1(b) and Lemma 6(f).
Inductive Step: $n > 0$.
Case 1: A is a negation $\sim B$, and hence $A(P'/P)$ is $(\sim B)(P'/P)$.
$\vdash_= P = P' \supset (B \equiv B(P'/P))$ by the hypothesis of the induction, and hence $\vdash_= P = P' \supset (\sim B \equiv \sim[B(P'/P)])$ by Lemma 6(j). But $\sim[B(P'/P)]$ and $(\sim B)(P'/P)$ are the same. Hence $\vdash_= P = P' \supset (\sim B \equiv (\sim B)(P'/P))$.
Case 2: A is a conditional $B \supset C$, and hence $A(P'/P)$ is $(B \supset C)(P'/P)$.
$\vdash_= P = P' \supset (B \equiv B(P'/P))$ and $\vdash_= P = P' \supset (C \equiv C(P'/P))$ by the hypothesis of the induction, and hence $\vdash_= P = P' \supset ((B \supset C) \equiv (B(P'/P) \supset C(P'/P)))$ by Lemma 6(g). But $B(P'/P) \supset C(P'/P)$ and $(B \supset C)(P'/P)$ are the same. Hence $\vdash_= P = P' \supset ((B \supset C) \equiv (B \supset C)(P'/P))$.
Case 3: A is a quantification $(\forall X)B$, and hence $A(P'/P)$ is $((\forall X)B)(P'/P)$. Let P'' be an individual parameter of $\mathbf{QC}_=$ distinct from each of P and P', and foreign to $(\forall X)B$. $\vdash_= P = P' \supset (B(P''/X) \equiv (B(P''/X))(P'/P))$ by the hypothesis of the induction. But, with P'' distinct from each of P and P',

$(B(P''/X))(P'/P)$ and $(B(P'/P))(P''/X)$ are the same. Hence $\vdash_= P = P' \supset (B(P''/X) \equiv (B(P'/P))(P''/X))$. But P'' is sure to be foreign to $P = P'$, $(\forall X)B$, and $(\forall X)[B(P'/P)]$, and X is of course foreign to $P = P'$. Hence $\vdash_= P = P' \supset ((\forall X)B \equiv (\forall X)[B(P'/P)])$ by Lemma 7(d). But $(\forall X)[B(P'/P)]$ and $((\forall X)B)(P'/P)$ are the same. Hence $\vdash_= P = P' \supset ((\forall X)B \equiv ((\forall X)B)(P'/P))$.
(b) By (a) and Lemma 6(e).
(c) $\vdash_= P = P' \supset (A(P/X) \supset (A(P/X))(P'/P))$ by (b). Now suppose P to be foreign to A. Then $(A(P/X))(P'/P)$ and $A(P'/X)$ are sure to be the same. Hence (c).

Lemma 9 (a) $\vdash_= (\exists X)(X = P) \supset ((\forall X)A \supset A(P/X))$.
(b) $\vdash_= (\forall Y)(\exists X)(X = Y) \supset (\forall Y)((\forall X)A \supset A(Y/X))$.

Proof: (a) Let P' be an individual parameter of $\mathbf{QC}_=$ distinct from P and foreign to $(\forall X)A$. $\vdash_= P = P' \supset (A(P'/X) \supset A(P/X))$ by Lemma 8(c). But, with P' distinct from P and foreign to $(\forall X)A$ (and hence to A), $P' = P$ and $(X = P)(P'/X)$ are sure to be the same, and so are $A(P'/X) \supset A(P/X)$ and $(A \supset A(P/X))(P'/X)$. Hence $\vdash_= (X = P)(P'/X) \supset (A \supset A(P/X))(P'/X)$, and hence $\vdash_= (\exists X)(X = P) \supset (\exists X)(A \supset A(P/X))$ by Lemma 7(f). But, as X is foreign to $A(P/X)$, $\vdash_= (\exists X)(A \supset A(P/X)) \supset ((\forall X)A \supset A(P/X))$ by Lemma 7(h). Hence (a) by Lemma 6(b).
(b) Let P be an individual parameter of $\mathbf{QC}_=$ foreign to A. $\vdash_= (\exists X)(X = P) \supset ((\forall X)A \supset A(P/X))$ by (a). But, with $(\forall Y)(\exists X)(X = Y)$ presumed here to be well-formed, Y is sure to be distinct from X, and hence $(\exists X)(X = P)$ and $((\exists X)(X = Y))(P/Y)$ to be the same. Hence $\vdash_= ((\exists X)(X = Y))(P/Y) \supset ((\forall X)A \supset A(P/X))$. But, with $(\forall Y)((\forall X)A \supset A(Y/X))$ presumed here to be well-formed, Y is sure to be foreign to A, and hence $(\forall X)A \supset A(P/X)$ and $((\forall X)A \supset A(Y/X))(P/Y)$ to be the same. Hence $\vdash_= ((\exists X)(X = Y))(P/Y) \supset ((\forall X)A \supset A(Y/X))(P/Y)$. But, being foreign to A, P is sure to be foreign to $(\forall Y)(\exists X)(X = Y)$ and to $(\forall Y)((\forall X)A \supset A(Y/X))$. Hence (b) by Lemma 7(a).

Lemma 10 (a) $\vdash_= (\forall Y)(\exists X)(X = Y)$.
(b) $\vdash_= (\forall Y)((\forall X)A \supset A(Y/X))$.

Proof: (a) Let P be an individual parameter of $\mathbf{QC}_=$. $(\exists X)(X = P)$ counts as an axiom of $\mathbf{QC}_=$. Hence $\vdash_= (\exists X)(X = P)$ by Lemma 1(b). But, with $(\forall Y)(\exists X)(X = Y)$ presumed here to be well-formed, Y is sure to be distinct from X, and hence $(\exists X)(X = P)$ and $((\exists X)(X = Y))(P/Y)$ to be the same. Hence $\vdash_= ((\exists X)(X = Y))(P/Y)$, and hence (a) by Theorem 4(a). (b) By (a), Lemma 9(b), and Lemma 1(c).

Hence:

Theorem 6 (a) *Let X be foreign to A, and P be foreign to S. If $S \vdash_= A$, then $S \vdash_= (\forall X)[A(X/P)]$.* (**UGT** *for* $\mathbf{QC}_=$).
(b) *Let P be foreign to S and to $(\forall X)A$. If $S \vdash_= A(P/X)$, then $S \vdash_= (\forall X)A$.* ($\forall$I *for* $\mathbf{QC}_=$).

Proof: (a) Proof like that of Theorem 3, but using Lemma 10(b) in lieu of Lemma 3. (b) Proof like that of Theorem 4(b), but using (a) in lieu of Theorem 3.

The reader will have noticed that, though the proof of Lemma 10(a) resorts to *B1*, that of Lemma 10(b) and hence that of Theorem 6 do not. They merely presuppose that $(\forall Y)(\exists X)(X = Y)$, when well-formed, is provable in $\mathbf{QC}_=$.[28]

Now for $\mathbf{QC}^*_=$, the presupposition-free quantificational calculus of order one with identity, and proof there of **UGT**. A Strong Soundness Theorem for $\mathbf{QC}^*_=$ and (as established in [14], pp. 146-149) a Strong Completeness one can be had if *B1* (p. 844) is weakened to read

*B1**. $(\forall Y)(\exists X)(X = Y)$,

and the Law of Reflexivity for '=', to wit:

*B4**. $P = P$

is adopted as an extra axiom schema.[29]

Wffs of $\mathbf{QC}^*_=$ of the sort $(\forall Y)(\exists X)(X = Y)$, once made into axioms of $\mathbf{QC}^*_=$, are sure to be provable in $\mathbf{QC}^*_=$. So the foregoing demonstration of Theorem 6 holds good for $\mathbf{QC}^*_=$; and, writing '$\vdash^*_=$' to signal that the calculus at issue here is $\mathbf{QC}^*_=$, I conclude:

Theorem 7 (a) *Let X be foreign to A, and P be foreign to S. If $S \vdash^*_= A$, then $S \vdash^*_= (\forall X)[A(X/P)]$.* (**UGT** *for* $\mathbf{QC}^*_=$).
(b) *Let P be foreign to S and to $(\forall X)A$. If $S \vdash^*_= A(P/X)$, then $S \vdash^*_= (\forall X)A$.* ($\forall$I *for* $\mathbf{QC}^*_=$).

Fitch's account of a proof from hypotheses thus yields **UGT** (and $\forall$I) for $\mathbf{QC}^*_=$ as well as for $\mathbf{QC}_=$,[30] and the error detected by Cosgrove in [14] stands corrected.[31]

4 The results in Section **2** hold *mutatis mutandis* for most logics with quantifiers. Consider, for example, $\mathbf{QC}_2$, the quantificational calculus of order two. $\mathbf{QC}_2$ has as its primitive signs the signs of (b1)-(b2) in Section **1.8**, *plus* those of (c)-(d) in **1.1**, *plus* for each d from 0 on aleph-zero predicate variables and aleph-zero predicate parameters of degree d. Predicate variables of degree d are referred to by means of 'F^d', individual variables by means of 'X', variables in general (i.e., predicate variables and individual ones) by means of 'V', and parameters in general (i.e., predicate parameters and individual ones) by means of 'P'. $[(A)(P/V)]$ is to be the result of putting P everywhere for V in A, where (i) in case V is a predicate variable, P is a predicate parameter of the same degree as V, and (ii) in case V is an individual variable, P is an individual parameter; and $[(A)(V/P)]$ is to be understood in a like manner, but with 'P' and 'V' interchanged throughout.

The formulas of $\mathbf{QC}_2$ are all finite sequences of primitive signs of $\mathbf{QC}_2$. Its wffs are all formulas of $\mathbf{QC}_2$ of the sort (i) in **1.8**, *plus* all those of either of the sorts (ii)-(iii) in **1.2** (with '$\mathbf{QC}_2$' there for '$\mathbf{QC}$'), *plus* all those of the sort $(\forall V)A$, where for some parameter P of $\mathbf{QC}_2$ $A(P/V)$ is a wff of

QC$_2$. And its axioms are all those of the sorts *A1*-*A3* in **1.4**, *plus* all those of the sorts

$B1_2$. $(\forall V)(A \supset B) \supset ((\forall V)A \supset (\forall V)B)$
$B2_2$. $A \supset (\forall V)A$
$B3_2$. $(\forall V)A \supset A(P/V)$,

plus all those of the sort

$B4_2$. $(\exists F^d)(\forall X_1)(\forall X_2) \ldots (\forall X_d)(F^d(X_1, X_2, \ldots, X_d) \equiv A)$,

where the predicate variable F^d is foreign to A,[32] *plus* all those of the sort $(\forall V)[A(V/P)]$, where A is an axiom of **QC**$_2$.

Given Fitch's account of a proof from hypotheses, the counterparts for **QC**$_2$ of Lemmas 1-2 in **2** clearly hold true. So we may conclude as in **2**:

Theorem 8 *Let V be foreign to S and A, and P be foreign to S. If $S \vdash A$, then $S \vdash (\forall V)[A(V/P)]$.*

But the counterparts for **QC**$_2$ of Lemmas 3-4 also hold true, the counterpart of Lemma 3 reading

$$\vdash (\forall V')((\forall V)A \supset A(V'/V)),$$

where V and V' are either two predicate variables of the same degree or two individual variables, and $A(V'/V)$ is the result of putting V' everywhere for V in A; and the counterpart of Lemma 4 reading

If $S \vdash (\forall V)A$, then $S \vdash (\forall V')[A(V'/V)]$, so long as V' is foreign to $(\forall V)A$, where V, V', and $A(V'/V)$ are as for Lemma 3.

So we may conclude as in **2**:

Theorem 9 *Let V be foreign to A, and P be foreign to S. If $S \vdash A$, then $S \vdash (\forall V)[A(V/P)]$.* (**UGT** *for* **QC**$_2$).

Since the proof of Theorem 9 merely presupposes that $(\forall V')((\forall V)A \supset A(V'/V))$, when well-formed, is provable in **QC**$_2$, the theorem is also sure to hold true for the presupposition-free variant **QC**$_2^*$ of **QC**$_2$, a calculus gotten from **QC**$_2$ by dropping axiom schema $B3_2$ in favor of

$B3_2^*$. $(\forall V')((\forall V)A \supset A(V'/V))$,

dropping axiom schema $B4_2$, and adopting the Commutation Law for Universal Quantifiers

$B5_2^*$. $(\forall V)(\forall V')A \supset (\forall V')(\forall V)A$

as an extra axiom schema. So,

Theorem 10 *Let V be foreign to A, and P be foreign to S. If $S \vdash A$, then $S \vdash^* (\forall V)[A(V/P)]$.* (**UGT** *for* **QC**$_2^*$).

5 Appendix

5.1 To accommodate the many who have no access to [13], I supply proof of (B) for **QC** given Leblanc's account of a proof from hypotheses. I understand $[(A)(I_1'/I_1)]$ as in **1.9**, and—generalizing matters—understand $[(A)(I_1', I_2', \ldots, I_n'/I_1, I_2, \ldots, I_n)]$ as $[([(A)(I_1', I_2', \ldots, I_{n-1}'/I_1, I_2, \ldots, I_{n-1})])(I_n'/I_n)]$. For the occasion '$S \vdash A$' is of course short for 'A is provable in **QC** from S (in Leblanc's sense)'.

Lemma 11 *Let S be finite. If $S \vdash A$, then $S \cup \{B\} \vdash A$.*

Proof: Suppose $S \vdash A$; let the column made up of $A_1, A_2, \ldots,$ and A_p constitute a proof in **QC** from S whose last entry is A; let $P_1, P_2, \ldots,$ and P_k $(k \geq 0)$ be all the individual parameters of **QC** that are quasi-generalized upon in the column in question and occur in B; let $P_1', P_2', \ldots,$ and P_k' be k individual parameters of **QC** that are distinct from $P_1, P_2, \ldots,$ and P_k and are foreign to $A_1, A_2, \ldots, A_p$, and $S \cup \{B\}$; and let $X_1, X_2, \ldots,$ and X_k be k individual variables of **QC** foreign to $A_1, A_2, \ldots,$ and A_p. Then the column

1	$A_1(P_1', P_2', \ldots, P_k'/P_1, P_2, \ldots, P_k)$	
2	$A_2(P_1', P_2', \ldots, P_k'/P_1, P_2, \ldots, P_k)$	
⋮	⋮	
p	$A_p(P_1', P_2', \ldots, P_k'/P_1, P_2, \ldots, P_k)$	
$p+1$	$(\forall X_k)[A_p(P_1', P_2', \ldots, P_{k-1}', X_k/P_1, P_2, \ldots, P_k)]$	(**UG**, p)
$p+2$	$(\forall X_{k-1})(\forall X_k)[A_p(P_1', P_2', \ldots, P_{k-2}', X_{k-1}, X_k/P_1, P_2, \ldots, P_k)]$	(**UG**, $p+1$)
⋮	⋮	
$p+k$	$(\forall X_1)(\forall X_2) \ldots (\forall X_k)[A_p(X_1, X_2, \ldots, X_k/P_1, P_2, \ldots, P_k)]$	(**UG**, $p+k-1$)
$p+k+1$	$p+k \supset (\forall X_2)(\forall X_3) \ldots (\forall X_k)[A_p(X_2, X_3, \ldots, X_k/P_2, P_3, \ldots, P_k)]$	(*A6*)
$p+k+2$	$(\forall X_2)(\forall X_3) \ldots (\forall X_k)[A_p(X_2, X_3, \ldots, X_k/P_2, P_3, \ldots, P_k)]$	(MP, $p+k$, $p+k+1$)
$p+k+3$	$p+k+2 \supset (\forall X_3)(\forall X_4) \ldots (\forall X_k)[A_p(X_3, X_4, \ldots, X_k/P_3, P_4, \ldots, P_k)]$	(*A6*)
$p+k+4$	$(\forall X_3)(\forall X_4) \ldots (\forall X_k)[A_p(X_3, X_4, \ldots, X_p/P_3, P_4, \ldots, P_k)]$	(MP, $p+k+2$, $p+k+3$)
⋮	⋮	
$p+3k-2$	$(\forall X_k)[A_p(X_k/P_k)]$	(MP, $p+3k-4$, $p+3k-3$)
$p+3k-1$	$p+3k-2 \supset A_p$	(*A6*)
$p+3k$	A_p	(MP, $p+3k-2$, $p+3k-1$)

is sure to constitute a proof in **QC** from $S \cup \{B\}$ with A_p $(= A)$ as its last entry. For suppose A_i $(1 \leq i \leq p)$ belongs to S. Since $P_1, P_2, \ldots,$ and P_k are quasi-generalized upon in the original proof, they are sure to be foreign to S and hence to A_i. So $A_i(P_1', P_2', \ldots, P_k'/P_1, P_2, \ldots, P_k)$ is sure to be the same as A_i, and hence to belong to S. Or suppose A_i is an axiom of **QC**. Then by the same argument as on pp. 33-35 in [14] $A_i(P_1', P_2', \ldots, P_k'/P_1, P_2, \ldots, P_k)$ is sure to be an axiom of **QC**. Or suppose A_i follows from A_g and A_h by MP, and hence A_h, say, is of the sort $A_g \supset A_i$. Since $A_h(P_1', P_2', \ldots, P_k'/P_1, P_2, \ldots, P_k)$ and $A_g(P_1', P_2', \ldots, P_k'/P_1, P_2, \ldots, P_k) \supset A_i(P_1', P_2', \ldots, P_k'/P_1, P_2, \ldots, P_k)$ are the same, $A_i(P_1', P_2', \ldots, P_k'/P_1, P_2, \ldots, P_k)$ is sure to follow from $A_g(P_1', P_2', \ldots, P_k'/P_1, P_2, \ldots, P_k)$ and $A_h(P_1', P_2', \ldots, P_k'/P_1, P_2, \ldots, P_k)$ by MP. Or suppose A_i follows from A_h by **UG**, and hence is of the sort $(\forall X)[A_h(X/P)]$ for some individual parameter P of **QC** foreign to S; and suppose first that P is foreign to B. Since $((\forall X)[A_h(X/P)])(P_1',P_2', \ldots, P_k'/P_1,P_2, \ldots, P_k)$ and $(\forall X)[(A_h(P_1', P_2', \ldots, P_k'/P_1, P_2, \ldots, P_k))(X/P)]$ are the same and since P is sure to be foreign to $S \cup \{B\}$, $A_i(P_1',P_2', \ldots, P_k'/P_1,P_2, \ldots, P_k)$ is sure to follow from $A_h(P_1',P_2', \ldots, P_k'/P_1,P_2, \ldots, P_k)$ by **UG**. Suppose next that P occurs in B, and hence is one of P_1, P_2, . . ., and P_k, say P_j. Since $((\forall X)[A_h(X/P_j)])(P_1',P_2', \ldots, P_k'/P_1, P_2, \ldots, P_k)$ and $(\forall X)[(A_h(P_1', P_2', \ldots, P_k'/P_1, P_2, \ldots, P_k))(X/P_j')]$ are the same, and since—P_j' being sure to be foreign to $S \cup \{B\}$—$(\forall X)[(A_h(P_1',P_2', \ldots, P_k'/P_1,P_2, \ldots, P_k))(X/P_j')]$ follows from $A_h(P_1',P_2', \ldots, P_k'/P_1,P_2, \ldots, P_k)$ by **UG**, $((\forall X)[A_h(X/P_j)])(P_1',P_2', \ldots, P_k'/P_1,P_2, \ldots, P_k)$ is sure to follow from $A_h(P_1',P_2', \ldots, P_k'/P_1,P_2, \ldots, P_k)$ by **UG**. Hence $S \cup \{B\} \vdash A$.

Theorem 11 *If $S \vdash A$, then $S' \vdash A$ for any superset S' of S.*

Proof: Let S' be an arbitrary superset of S. *Case 1*: S' is finite, and hence is of the sort $S \cup \{B_1,B_2, \ldots, B_n\}$ for some n larger than 0. Then S is sure to be finite as well, and hence Theorem 11 by n applications of Lemma 11. *Case 2.1*: S' is infinite, but S is finite. Then Theorem 11 by definition. *Case 2.2*: Both S' and S are infinite. Suppose $S \vdash A$. Then by definition $S'' \vdash A$ for some finite subset S'' of S. But S'' is bound to be a finite subset of S' as well. Hence $S' \vdash A$ by definition.

5.2 I next establish that wffs of $\mathbf{QC}^*_=$ of the sort $(\forall X)(\forall Y)A \supset (\forall Y)(\forall X)A$ are provable in $\mathbf{QC}^*_=$ (given the axiomatization of $\mathbf{QC}^*_=$ in **3** and Fitch's account of a proof from hypotheses). The result follows of course from the Completeness Theorem in [14], but because of the difficulties attending $(\forall X)(\forall Y)A \supset (\forall Y)(\forall X)A$ in **QC** and **QC***, the demonstration that follows may be welcome. For brevity's sake I write '$(A \,\&\, B)$' in lieu of '$\sim(A \supset \sim B)$'.

Lemma 12 (a) *If* $\vdash^*_= A \supset (B \supset C)$ *and* $\vdash^*_= A' \supset (C \supset C')$, *then* $\vdash^*_= A \supset (A' \supset (B \supset C'))$.
(b) *If* $\vdash^*_= A \supset (B \supset (C \supset C'))$, *then* $\vdash^*_= (B \,\&\, C) \supset (A \supset C')$.
(c) *If* $\vdash^*_= (A \,\&\, B) \supset (C \supset C')$, *then* $\vdash^*_= C \supset (A \supset (B \supset C'))$.
(d) *If* $\vdash^*_= A(P/X) \supset (B \supset C(P/X))$, *then* $\vdash^*_= (\forall X)A \supset (B \supset (\forall X)C)$, *so long as* (i) *$P$ is foreign to $(\forall X)A$, B, and $(\forall X)C$, and* (ii) *X is foreign to B.*
(e) *If* $\vdash^*_= A(P/X) \supset (B \supset (C \supset C'(P/X)))$, *then* $\vdash^*_= (\forall X)A \supset (B \supset (C \supset (\forall X)C'))$, *so long as* (i) *$P$ is foreign to $(\forall X)A$, B, C, and $(\forall X)C$, and* (ii) *X is foreign to B and C.*
(f) $\vdash^*_= (\forall Y)[A(Y/X)] \supset (\forall X)A$.
(g) *If* $\vdash^*_= A \supset (\forall Y)[B(Y/X)]$, *then* $\vdash^*_= A \supset (\forall X)B$.
(h) *If* $\vdash^*_= A \supset (B \supset (\forall Y)[C(Y/X)])$, *then* $\vdash^*_= A \supset (B \supset (\forall X)C)$.

Proof: (a)-(c) Proofs left to the reader.
(d) Suppose $\vdash^*_= A(P/X) \supset (B \supset C(P/X))$, suppose P is as in (i), and suppose X is as in (ii). Then $\vdash^*_= B \supset (A(P/X) \supset C(P/X))$ by Lemma 6(c), hence $\vdash^*_= B \supset ((\forall X)A \supset (\forall X)C)$ by Lemma 7(c), and hence $\vdash^*_= (\forall X)A \supset (B \supset (\forall X)C)$ by Lemma 6(c).
(e) Suppose $\vdash^*_= A(P/X) \supset (B \supset (C \supset C'(P/X)))$, suppose P is as in (i), and

suppose X is as in (ii). Then $\vdash^*_= (B \& C) \supset (A(P/X) \supset C'(P/X))$ by (b), hence $\vdash^*_= (B \& C) \supset ((\forall X)A \supset (\forall X)C')$ by Lemma 7(c), and hence $\vdash^*_= (\forall X)A \supset (B \supset (C \supset (\forall X)C'))$ by (c).

(f) In Case X and Y are the same, proof of (f) is routine. So suppose X and Y are distinct from each other. Then $(\forall X)((\forall Y)[A(Y/X)] \supset (A(Y/X))(X/Y))$ is sure to be well-formed. Hence $\vdash^*_= (\forall X)((\forall Y)[A(Y/X)] \supset (A(Y/X))(X/Y))$ by Lemma 10(b), hence $\vdash^*_= (\forall X)((\forall Y)[A(Y/X)] \supset A)$, and hence $\vdash^*_= (\forall Y)[A(Y/X)] \supset (\forall X)A$ by Lemma 1(f).

(g) Proof by Lemma 12(g) and Lemma 6(b).

(h) Proof by Lemma 12(g) and Lemma 6(d).

Theorem 12 $\vdash^*_= (\forall X)(\forall Y)A \supset (\forall Y)(\forall X)A$.

Proof: Let P and P' be individual parameters of $\mathbf{QC}^*_=$ distinct from each other and foreign to $(\forall X)(\forall Y)A$, and let Z be an individual variable of $\mathbf{QC}^*_=$ foreign to $(\forall X)(\forall Y)A$. $\vdash^*_= (\exists X)(X = P) \supset ((\forall X)(\forall Y)A \supset ((\forall Y)A)(P/X))$ by Lemma 9(a), and hence $\vdash^*_= (\exists X)(X = P) \supset ((\forall X)(\forall Y)A \supset (\forall Y)[A(P/X)])$. Similarly, $\vdash^*_= (\exists Y)(Y = P') \supset ((\forall Y)[A(P/X)] \supset (A(P/X))(P'/Y))$ by Lemma 9(a), and hence $\vdash^*_= (\exists Y)(Y = P') \supset ((\forall Y)[A(P/X)] \supset (A(P'/Y))(P/X))$. So, $\vdash^*_= (\exists X)(X = P) \supset ((\exists Y)(Y = P') \supset ((\forall X)(\forall Y)A \supset (A(P'/Y))(P/X)))$ by Lemma 12(a), hence $\vdash^*_= ((\exists X)(X = Z))(P/Z) \supset ((\exists Y)(Y = P') \supset ((\forall X)(\forall Y)A \supset ((A(P'/Y))(Z/X))(P/Z)))$, and hence $\vdash^*_= (\forall Z)(\exists X)(X = Z) \supset ((\exists Y)(Y = P') \supset ((\forall X)(\forall Y)A \supset (\forall Z)[(A(P'/Y))(Z/X)]))$ by Lemma 12(e). But, since $(\forall Z)(\exists X)(X = Z)$ counts as an axiom of $\mathbf{QC}^*_=$, $\vdash^*_= (\forall Z)(\exists X)(X = Z)$ by Lemma 1(b). Hence $\vdash^*_= (\exists Y)(Y = P') \supset ((\forall X)(\forall Y)A \supset (\forall Z)[(A(P'/Y))(Z/X)])$ by Lemma 1(c), hence $\vdash^*_= (\exists Y)(Y = P') \supset ((\forall X)(\forall Y)A \supset (\forall X)[A(P'/Y)])$ by Lemma 12(h), hence $\vdash^*_= ((\exists Y)(Y = Z))(P'/Z) \supset ((\forall X)(\forall Y)A \supset ((\forall X)[A(Z/Y)])(P'/Z))$, and hence $\vdash^*_= (\forall Z)(\exists Y)(Y = Z) \supset ((\forall X)(\forall Y)A \supset (\forall Z)(\forall X)[A(Z/Y)])$ by Lemma 12(d). But, since $(\forall Z)(\exists Y)(Y = Z)$ counts as an axiom of $\mathbf{QC}^*_=$, $\vdash^*_= (\forall Z)(\exists Y)(Y = Z)$ by Lemma 1(b). Hence $\vdash^*_= (\forall X)(\forall Y)A \supset (\forall Z)(\forall X)[A(Z/Y)]$ by Lemma 1(c), hence $\vdash^*_= (\forall X)(\forall Y)A \supset (\forall Z)[((\forall X)A)(Z/Y)]$, and hence $\vdash^*_= (\forall X)(\forall Y)A \supset (\forall Y)(\forall X)A$ by Lemma 12(g).

NOTES

1. Predicate variables of degree 0 are of course statement variables.

2. When no ambiguity threatens, I shall write '$A \supset B$' in lieu of '$(A \supset B)$'.

3. Generally, *but not without fail*: the Kleene account [10] appeared in 1952.

4. See p. 45 of [2]. Church limits himself there to the case where S is finite. However, in [3], p. 310, he takes a wff A of $\mathbf{QC}_2$ (the quantificational calculus of order two) to be provable in $\mathbf{QC}_2$ from an infinite set S of wffs of $\mathbf{QC}_2$ if A is provable in $\mathbf{QC}_2$ from a finite subset of S. So the account in the text is close enough to Church's intentions to be attributed to Church.

5. Their proof, retouched to suit our axiomatization of $\mathbf{QC}$, is of utmost simplicity. To each wff A of $\mathbf{QC}$ assign a value $v(A)$ as follows: (a) in case A is of the sort (i) in **1.2**, $v(A) = 1$; (b) in case A is of the sort $\sim B$, $v(A) = 1 - v(B)$; (c) in case A is of the sort $B \supset C$, $v(A) = 1$ if $v(B) = 0$ or $v(C) = 1$, otherwise $v(A) = 0$; (d) in case A is of the sort $(\forall X)B$ and X is distinct

from 'y' or foreign to B, $v(A) = v(B)$, otherwise $v(A) = 0$. A routine induction shows that if the column

$$\begin{array}{c} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_p \end{array}$$

constitutes a proof in **QC** from a finite set S of wffs of the sort (i) in **1.2** and 'y' occurs free in at least one member of S, then $v(A_i) = 1$ for each i from 1 through p. (Note in particular that, with 'y' occurring free in at least one member of S, no entry in the column that follows from an earlier entry by **UG** can be of the sort $(\forall y)B$.) But $v(`(\forall y)(g(y) \supset g(y))') = 0$. So there can be no proof in **QC** from $\{g(y)\}$ with '$(\forall y)(g(y) \supset g(y))$' as its last entry.)

6. For a recent version of Henkin's proof, see [17], pp. 285-293. The original version is of course in [7].

7. Church's account in [3] of a proof in **QC** from a finite set of wffs of **QC** differs from that in [2]. Using three extra rules of inference (Alphabetic Change of Bound Variables, Substitution for Individual Variables, and Substitution for Predicate Variables), it yields (B) for finite S and S' (= *362 on pp. 199-200). It is, however, too intricate for review here.

   In some presentations of **QC** only *closed* wffs (i.e., only wffs in which there occur no free individual variables) can count as axioms, only proofs from sets of closed wffs can count as proofs, and hence only closed wffs are provable (be it from ϕ or from a non-empty set of wffs). In such presentations the Montague-Henkin difficulty does not arise, and an account of a proof from hypotheses for which (B) (and, as a corollary of Henkin's proof, (A)) holds true is easily had: adopt the axioms in either edition of [19] as your axioms and drop (iv) of **1.5**. However, open wffs matter as much–I believe–as closed ones, and the presentations of **QC** considered here strike me as unduly restricted.

8. Meyer and I used this terminology in [16] in connection with individual variables, and I have since used it regularly in connection with all types of variables.

9. Under the present wording of clause (iv), identical quantifiers can no longer overlap in a wff of **QC**. So, for example, when a conditional of the sort $A \supset (\forall X)A$ is well-formed, the individual variable X is sure to be foreign to the antecedent A, a point to bear in mind when coming to change number six.

10. Henceforth I shall abridge 'foreign to each member of S' as 'foreign to S'.

11. The proof, tailored to suit the present context, is reproduced in the Appendix (**5**).

12. Let $v(A)$ be defined as in Note 5 (but with clause (i) understood as in **1.8** rather than **1.2**). A routine induction will show that if the column

$$\begin{array}{c} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_p \end{array}$$

constitutes a proof in **QC** from $\{g(a), g(b), g(c), g(a'), g(b'), g(c'), \ldots\}$, then $v(A_i) = 1$ for each i from 1 through p. (Note in particular that, with each individual parameter of **QC** occurring in $\{g(a), g(b), g(c), g(a'), g(b'), g(c'), \ldots\}$, no entry in the column can follow from an earlier entry by **UG**.) But $v(`(\forall y)(g(y) \supset g(y))') = 0$, as before. Hence the conclusion in the text.

13. In [11], a 1967 publication of Kleene's, simplifications are brought to the account in [10], but the Montague-Henkin account still remains the easier one.

14. And remained unaware of it until the writer brought it to his attention in the early sixties.

15. Or, as [17] has it on p. 328, declaring any wff of **QC** of the sort $(\forall X)A$ an axiom of **QC** if, for some individual parameter P of **QC** foreign to $(\forall X)A$, $A(P/X)$ is an axiom of **QC**. The two characterizations amount to the same. For suppose, on one hand, that $A(P/X)$ is an axiom of **QC**, and hence by the characterization in the text that so is $(\forall X)[(A(P/X))(X/P)]$. If P is foreign to $(\forall X)A$ and hence to A, then $(A(P/X))(X/P)$ and A are sure to be the same, and hence $(\forall X)A$ is sure to be an axiom of **QC** (as the characterization in [17] would have it). Suppose, on the other hand, that A is an axiom of **QC**. With A and $(\forall X)[A(X/P)]$ both presumed here to be well-formed, X is sure to be foreign to A, and hence A and $(A(X/P))(P/X)$ have to be the same. So $(A(X/P))(P/X)$ is sure to be an axiom of **QC**, and hence by the characterization in [17] $(\forall X)[A(X/P)]$ is sure to be one as well (as the characterization in the text would have it).

16. In Montague and Henkin's account and in Fitch's I implicitly take a wff A of **QC** to be *provable in* **QC** if A is provable in **QC** from ϕ. Fitch in [5] merely deals with provability and the calculus he is concerned with is a modal extension of **QC**. But the account owes enough to Fitch to be credited to him.

17. The reader will also have noticed that *the above proof of* **UGT** *makes no use of axiom schemata A1-A3, and hence holds no matter one's axiom schemata for* '$\sim$' *and* '$\supset$'. It thus holds for a variety of first-order quantificational calculi.

18. A semantics for **QC*** of the truth-value sort will be found in [14], pp. 135-136, and one of the model-theoretic sort can be gotten from [24]. The model-theoretic semantics in [16] is slightly defective, as Shipley established in [20], and the correction offered in [14], p. 161, footnote 62, will not do the trick.

19. So far as I know, $A6^*$ made its first appearance in [12]. Though axiomatizations of $\mathbf{QC}^*_=$ (see the end of **3**) go back to 1959, the first axiomatization of **QC*** is probably to be found in [12].

20. In [16] Meyer and I assumed without further ado that $(\forall X)(\forall Y)A \supset (\forall Y)(\forall X)A$ was provable in **QC***. However, Trew questioned this in [23], and as all attempts to prove the Commutation Law in question have so far failed, we now incline to think with Trew that $A7^*$ is independent. The reader will recall the difficulties that Quine experienced with $(\forall X)(\forall Y)A \supset (\forall Y)(\forall X)A$ in [19]. With the closure $(\ )[A]$ of a wff A defined as in the 1940 edition of [19], he could not prove $(\ )[(\forall X)(\forall Y)A \supset (\forall Y)(\forall X)A]$ and hence adopted it as an extra axiom schema. Fitch showed in 1941 that proof of $(\ )[(\forall X)(\forall Y)A \supset (\forall Y)(\forall X)A]$ can be had if the definition of $(\ )[A]$ is amended; and, using yet another definition of $(\ )[A]$, so did Berry the very same year (see [4] and [1]). However, whether given Quine's original notion of a closure $(\ )[(\forall X)(\forall Y)A \supset (\forall Y)(\forall X)A]$ is independent in [19] and, in particular, whether $(\forall X)(\forall Y)A \supset (\forall Y)(\forall X)A$ itself is remain open questions.

21. The point made in Note 17 regarding *A1-A3* holds here as in **QC**.

22. When no ambiguity threatens, I shall write '$P = P'$' in lieu of '$(P = P')$'.

23. Tarski had an additional axiom schema: $(\forall X)A \supset A$, but Kalish and Montague showed it redundant in [9]. Note that, as Tarski uses but one run of individual variables, X may occur free in the consequent of his axiom schema.

24. The suggestion was in a letter than van Fraassen wrote to the author in early 1966.

25. That $P = P$ is provable in $\mathbf{QC}_=$ given the last two axiomatizations will be shown in Note 28. The proof there is essentially Tarski's in [22].

26. As would much of the post-1956 literature: surprisingly enough, the Montague-Henkin paper received little notice. There are exceptions, of course: [10] among pre-1956 publications and [13] among post-1956 ones.

27. That $(\exists X)(X = Y) \supset ((\forall X)A \supset A(Y/X))$ follows from $X = Y \supset (A \supset A(Y/_1X))$ (and $(\forall X)A \supset A$) was announced in [21], a 1951 abstract, but proof of the fact was supplied only in [22], a 1965 paper. I was unaware of [21] when I offered proof of $(\exists X)(X = P) \supset ((\forall X)A \supset A(P/X))$ from $P = P' \supset (A \supset A(P'/P))$ in [15], a 1959 paper, and Tarski was of course unaware of [15] when he elaborated [21] into [22].

28. Unlike the proof of Theorem 3, that of Theorem 6 makes recourse to *A1-A3*. This could of course be avoided by drafting, say, $(\forall Y)((\forall X)A \supset A(Y/X))$ as an extra axiom schema, but the resulting axiomatization of **QC** would be of little interest.

    That $P = P$ is provable in $\mathbf{QC}_=$ (given our axiomatization of $\mathbf{QC}_=$ earlier in **3** or its Tarski forebear) can be shown as follows. Let P' be an individual parameter of $\mathbf{QC}_=$ distinct from P. $\vdash_= P' = P \supset (P' = P \supset P = P)$ by Lemma 1(b); hence $\vdash_= P' = P \supset P = P$ by routine steps; hence $\vdash_= (X = P)(P'/X) \supset (P = P)(P'/X)$; hence $\vdash_= (\exists X)(X = P) \supset (\exists X)(P = P)$ by Lemma 7(f); and hence $\vdash_= (\exists X)(X = P) \supset P = P$ by Lemma 7(g) and Lemma 6(b). (That $(\exists X)(X = P) \supset P = P$ is provable in $\mathbf{QC}_=$, and provable in $\mathbf{QC}_=$ without recourse to *B1*, will be recalled in Note 30.) But $\vdash_= (\exists X)(X = P)$ by Lemma 1(b). Hence $\vdash_= P = P$ by Lemma 1(c).

29. It is easily seen that *B4**, shown in Note 28 to be provable in $\mathbf{QC}_=$, is independent of *A1-A5*, *B1**, and *B2-B3*. Let $v(P = P) = 0$ for any individual parameter P of $\mathbf{QC}^*_=$; let $v(A) = 1$ for any other atomic wff A of $\mathbf{QC}^*_=$; let $v({\sim}A) = 1 - v(A)$; let $v(A \supset B) = 1$ unless $v(A) = 1$ and $v(B) = 0$, in which case $v(A \supset B) = 0$; and let $v((\forall X)A) = 1$. As the reader may wish to verify, wffs of $\mathbf{QC}^*_=$ of any of the sorts *A1-A5*, *B1**, and *B2-B3* all evaluate to 1; wffs of the sort *B4**, on the other hand, evaluate to 0. (For further comments on *B4**, see Note 30). And, adapting an argument of Ermanno A. Bencivenga, it is easily seen that *B1** is independent of *A1-A5*, *B2-B3*, and *B4**. First, by a *Bencivenga sequence* for $\mathbf{QC}^*_=$ understand any (infinite) sequence of the sort $\langle \Sigma_1, \Sigma_2, \Sigma_3, \ldots \rangle$, where Σ_1, Σ_2, Σ_3, etc., are (possibly empty) sets of individual parameters of $\mathbf{QC}^*_=$. Then, *Seq* being a Bencivenga sequence for $\mathbf{QC}^*_=$, take $P = P$ to be true on *Seq* for any individual parameter P of $\mathbf{QC}^*_=$; take any other atomic wff of $\mathbf{QC}^*_=$ to be false on *Seq*; take ${\sim}A$ to be true on *Seq* if and only if A is false on *Seq*; take $A \supset B$ to be true on *Seq* if and only if A is false on *Seq* or B is true on *Seq*; and take $(\forall X)A$ to be true on *Seq* if and only if (i) $A(P/X)$ is true on *Seq* for every member P of Σ_1 in case X is 'x', (ii) $A(P/X)$ is true on *Seq* for every member P of Σ_2 in case X is 'y', (iii) $A(P/X)$ is true on *Seq* for every member P of Σ_3 in case X is 'z', etc. As the reader may wish to verify, wffs of $\mathbf{QC}^*_=$ of any of the sorts *A1-A5*, *B2-B3*, and *B4** are all true on any Bencivenga sequence in which Σ_1 is non-empty but Σ_2 is; wffs of the sort *B1**, on the other hand, are false on any such sequence.

30. The reader will notice that the proof of Theorem 7 makes no use of *B4**, which was drafted as an axiom schema of $\mathbf{QC}^*_=$ only for completeness' sake. Under an alternative treatment of identity $P = P$ could be weakened to read $P = P \supset (\exists X)(X = P)$. Since $(\exists X)(X = P) \supset P = P$ is already provable in $\mathbf{QC}^*_=$ (see Note 28), one would obtain $(\exists X)(X = P) \equiv P = P$, the counterpart in $\mathbf{QC}^*_=$ of a familiar theorem of *Principia Mathematica*.

31. The error occurs in the proof of *T5.3.15*, which presupposes $(\forall Y)((\forall X)A \supset A(Y/X))$ and yet is used to prove $(\forall Y)((\forall X)A \supset A(Y/X))$ in *T5.3.16*.

32. $B4_2$ is of course the Axiom of Comprehension. The axiomatization of $\mathbf{QC}_2$ used here stems from [7].

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