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AN EXAMINATION OF THE INFLUENCE OF BOOLE'S ALGEBRA ON PEIRCE'S DEVELOPMENTS IN LOGIC

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Peirce was led to the development of innovative improvements in logic through critical investigation of the writings of other logicians, e.g., the traditional system of syllogistic, Boole's algebra, Hamilton's system, Mill's logic.[1] In the following I consider his early reaction to Boole's algebra of logic. In his 1865 lectures on the philosophy of science presented at Harvard, Peirce indicated great respect for Boole's accomplishments but also pointed out many failings in Boole's system [1], [2]. By 1865, twelve years before the first publication of Schroeder, Peirce had read and had begun to work on revisions of the algebra of logic developed by Boole.

One of Boole's most significant achievements, Peirce said in Lecture 6 of his 1865 lecture series [2], was his contribution toward the development of an effective symbolic notation. Indeed, Boole's was not the first attempt at a symbolic notation for logic, but Peirce maintained it was more adequate than previous attempts in fulfilling the aims desired from using such a notation. Ordinary language, with its ambiguities and richness, is inadequate for the investigation of logical form, Peirce explained. The symbols most effective for the science of logic should have the powers of diagramming significant linguistic forms and of aiding in the analysis of the laws of the necessary relations between such forms. Peirce felt that Boole's symbolization was the first significant *approach* toward fulfilling these objectives,[2] an approach of great value and well worth studying, even though deficient in some respects.

In his 1865 lecture series [2] Peirce said that the application of Boole's calculus to ordinary reasoning involves two fundamental theorems, the first being:

1.1 $fx = xfx_1 + (1 - x) fx_0$.

This rule, Peirce said, enables us to interpret complex expressions. It is also presented in 1867 (3.9) in the following form:

1.2 $\phi x = \phi(1), x + \phi(0), (1 - x)$.

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This formula for the interpretation of complex expressions is derivable from the basic form for interpretation

1.3 $\phi x = i, x + j, (1 - x)$

where $\phi(1) = i$ and $\phi(0) = j$.
That is, substituting in 1.3 for $\phi(1)$, we get:

$\phi(1) = i, 1 + j, (1 - 1)$.

Thus,

$\phi(1) = i$

and for $\phi(0)$:

$\phi(0) = i, 0 + j, (1 - 0)$.

Thus,

$\phi(0) = j$.

Peirce presented an example of the application of this interpretation formula in [2]. Developing for a function of one variable, we get the following result:

If $f(x) = \frac{x}{2x+1}$, $f(1) = \frac{1}{3}$, and $f(0) = \frac{0}{1}$,
then, by 1.1: $f(x) = 1/3\ x + 0/1\ (1 - x)$.

Developing for two variables, Peirce said "we have to complicate the formula a little" [2]. Thus:

$$f(a,b) = abf(1,1) + a(1-b)f(1,0) + (1-ab)f(0,1) + (1-a)(1-b)f(0,0).$$

Boole left some logical expressions uninterpreted. Peirce saw this as a failing of Boole's system and presented rules for interpretation of the algebraic symbols. He said in [1] that any expression can be reduced to expressions in which we have relations of the following:

$0/1m$, $0/0m$, $1/0m$, $1/1m$, $1/1(1-m)$, $0/1(1-m)$, $1/0(1-m)$, $0/0(1-m)$.

By defining the meanings of these expressions we then can interpret any expression. These definitions given by Peirce are as follows (where '*m*' stands for 'man'):

$1/1m$	means all men
$1/1(1-m)$	means all things not men
$0/1m$	means no men
$0/1(1-m)$	means no things not men
$0/0m$	means some, all or no men
$0/0(1-m)$	means some, all or no things not men
$1/0m$	means there are no men
$1/0(1-m)$	means there are no things not men.

From the start of his career as a logician, Peirce held a semantic view of logic and maintained that the function of logic is essentially to serve as an analysis of natural language. Mathematics can accommodate uninterpreted expressions but logic cannot. Peirce distinguished logic from mathematics in terms of the aim of each, and explained:

> The mathematician asks what value this algebra has as a calculus. Can it be applied to unravelling a complicated question? Will it, at one stroke, produce a remote consequence? (4.239)

He defined a calculus as a system of symbols which enables a person with the aid of a few rules of transformation to pass from a premise to a conclusion in a swift and direct manner. This is not the concern of logic, for:

> The logician does not wish the algebra to have that character. On the contrary, the greater the number of distinct logical steps into which the algebra breaks up an inference will for him constitute a superiority of it over another which moves more swiftly to its conclusions. He demands that the algebra shall analyze a reasoning into its last elementary steps. (4.239).

As such, mathematics can as well investigate relations between empty signs, relating them by transformation rules. However, logic, according to Peirce, is never concerned with empty signs, with uninterpretable expressions.

It is from this viewpoint that Peirce, in his earliest public discussion of the algebra of logic, attempted to provide linguistic interpretations for the symbols of logical algebra. It is also from this viewpoint that we can understand his preoccupation late in his life with developing his system of logical graphs. Though perhaps not as effective a calculus as his earlier system of predicate logic, he judged this system to be far more effective for *logical analysis*, as it provides a clear, easily interpreted analysis of any deductive inference.

The second fundamental theorem presented in Lecture 6 of 1865 [2] is the following:

2.1 If $f(x) = 0$, then $f(1), f(0) = 0$.

Peirce presented the proof of this theorem in 1867 (3.11). This theorem was also presented by Boole. Peirce also independently presented the following (3.12):

$\phi(1) + \phi(0) = 1$ when $\phi(x) = 1$.

In the 1865 lecture series [2], Peirce presented an example of the application of the above rules to a syllogistic argument. This process involves the following steps:

1. Symbolize each premise so that it is equal to 0.
2. Combine the two expressions by addition and eliminate superfluous expressions (i.e., the middle term is eliminated).

He considered the argument:

No animals are vegetables.
All men are animals.
So, no men are vegetables.

Letting 'a' stand for 'animals', 'v' for 'vegetables' and 'm' for 'men', the premises can be symbolized as follows:

$av = 0$
$m(1 - a) = 0$.

That is, animals that are vegetables are equal to 0; men that are not animals are equal to 0. It is then necessary to eliminate the middle term 'a'. Now, since a is equal to 1 or 0, $f(1) = 0$ or $f(0) = 0$; $f(1)f(0) = 0$.

Thus, combining the above expressions by addition, we get:

$av + m(1 - a) = 0$, i.e., $f(a) = 0$.

This expression is in the normal form 1.3. We proceed as follows, eliminating the middle term 'a'

$f(1) = 1v + m(1 - 1) = v$. In that $f(a) = 0$, $f(1) = v = 0$.
$f(0) = 0v + m(1 - 0) = m$. In that $f(a) = 0$, $f(0) = m = 0$.

Now 'a' must be equal to either 1 or 0, thus we have in these considerations exhausted all possible interpretations of 'a'. Applying 2.1:

$$f(1)f(0) = vm = 0.$$

By eliminating 'a', we arrive at the conclusion: $vm = 0$, i.e., men that are vegetables are equal to 0, or, no men are vegetables.

This method is clearly too complicated to be of great interest, either as a method of performing inferences or of analyzing inferences. Peirce's revisions of Boole's algebra (and his eventual development of propositional and predicate calculi) were in part motivated by the complexity of Boole's system and the difficulty of interpretation of expressions noted above. In Lecture 6 of 1865 Peirce mentioned several other defects in Boole's algebra, defects which he attempted in subsequent years to correct. He pointed out that Boole's notation was not adequate to express all types of propositions and he was particularly dissatisfied with:

1. Boole's symbolization of particular propositions. Boole used the symbol 'v' to indicate the indefinite class; for example, he symbolized the particular proposition 'Some X's are Y's' as '$v \cdot x = v \cdot y$'. This method of symbolizing particular propositions, Peirce argued, is not adequate to indicate the existential presupposition of particular propositions.

2. Boole's symbolization of universal and conditional propositions using '=' as the connective. Boole symbolized 'All men are animals' as '$m = ma$', where 'm' denotes the class of men and 'a' denotes the class of animals. Conditional propositions were symbolized in the following way: Let 'A' stand for 'There is an east wind'; let 'B' stand for 'The barometer would rise'. Boole further introduced 'a', which stands for 'that portion of

time for which the proposition "*A*" is true' and '*b*', which stands for 'that portion of time for which the proposition "*B*" is true'. He symbolized the conditional proposition 'If there is an east wind, then the barometer would rise' as '$a = ab$' (i.e., '$a = a \cap b$'). According to Peirce, Boole's symbolization was not adequate to express the logical relation between the subject and predicate of a universal proposition or that between the antecedent and consequent of a conditional proposition.

3. Boole's mathematical symbols. Peirce recognized the need to clearly distinguish between mathematical and logical symbols.

In Peirce's 1865 lecture series, then, we find the motivating concerns that led to many of his later important developments. As a result of his dissatisfaction with Boole's algebra, in March of 1867 Peirce published a revision of the Boolean algebra (3.1ff). In this paper he distinguished logical from mathematical operations, presenting distinct logical symbols. He distinguished inclusive disjunction, as a logical notion, from exclusive disjunction used by Boole; this allowed Peirce to introduce the law of duality. In a paper of September of 1867 (3.20ff), Peirce dropped logical subtraction and logical division from his system of logic, for their use leads to uninterpretable expressions; he also introduced the notion of material identity, distinguishing it from mathematical equality.

Because of his dissatisfaction with the use of the identity theory of the copula in the work of Boole and also in the writings of Hamilton, De Morgan, and others, in 1870 Peirce introduced the symbol '$-\!\!<$' to indicate class inclusion and implication (3.47). Thus by 1870 Peirce presented a complete set of operations for what is commonly called Boolean algebra, clearly distinguishing, in his system, between the logical notions of class inclusion and identity. By using this new symbol, Peirce felt he was able to present a more adequate symbolization of universal and conditional propositions than Boole could in his system.

Peirce found the Boolean system inadequate to treat mathematical propositions without introducing relations.[3] In 1870, he presented his first published study on the logic of relations (3.45ff) and, in the same paper, attempted to symbolize the existential quantifier in such a way as to indicate existence ("case of the existence of--") to revise Boole's method of symbolizing particular propositions. Peirce's dissatisfaction with Boole's manner of symbolizing particular propositions led eventually to his independent discovery of quantifiers and indices.[4]

Schroeder, in letters and published writings, expresses indebtedness to Peirce for many of his own developments in logic. In the light of Peirce's direct contribution to Schroeder's work and Peirce's improvements of Boole's algebra of logic, it can be seen that Peirce played a significant role in contributing to the development of what is currently referred to as Boolean algebra or the Boole-Schroeder algebra of logic.

NOTES

1. *cf.* Peirce's critical discussion of other logicians in early manuscripts such as his lecture series of 1865, mss. 340-350; his lecture series of 1866, mss. 351-359; his lecture series of 1869, mss. 584-586.

2. In Ms. 344 Peirce discusses other notations (considering both geometrical and algebraic notations of other logicians) and attempts to show why these are very much inferior to Boole's notation.

3. For further discussion of Peirce's early work on relations, see my paper, "Peirce's early study of the logic of relations, 1865-1867," *Transactions of the Peirce Society*, vol. 10 (1974), pp. 63-75.

4. *cf.* 3.328ff. For the discovery of quantifiers, Peirce acknowledges a debt to his student, O. H. Mitchell; Peirce himself introduced indices.

REFERENCES

[1] Peirce, C. S., Ms. 342, Lecture 3. (All manuscript numbers refer to "The Charles S. Peirce Papers," at the Houghton Library of Harvard University, as catalogued by R. Robin in his *Annotated Catalogue of the Papers of Charles S. Peirce*, University of Massachusetts Press, 1967.)

[2] Peirce, C. S., Ms. 344, Lecture 6.

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