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A NOTE ON $p = mv$

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Professor Quine has long inveighed against the existence of truths invulnerable to empirical disconfirmation—in a word, necessary truths. As Quine sees it, *any* sentence whatsoever might come to be rejected on the basis of "recalcitrant experience." There are of course sentences that could not be rejected on a one-sentence—one-recalcitrant-experience basis, but for any sentence S, it might turn out that our entire account of the world would cohere better with the entirety of experience if S were rejected. Quine is entirely uncompromising, and counts as vulnerable such likely-looking exceptions as stipulative definitions in science. In "Necessary Truth"[1] he considers the ostensibly definitional truth that momentum (p) is equal to the product of mass (m) and velocity (v), and cites the revision it has undergone in the Special Theory of Relativity.

> A truth that might be cited [as logically necessary] is that momentum is proportional to velocity. This might be said to be logically or mathematically necessary on the ground that the word "momentum" is itself defined simply as short for "mass times velocity." But now imagine a physicist with some unexpected experimental findings to provide for. They conflict with his physical theory. There is no specific point in his theory that they conflict with. . . . [S]uppose the physicist hits upon a particularly neat repair, which involves revising slightly the law that momentum is proportional to velocity; he makes momentum proportional instead to, say, velocity divided by one minus the reciprocal of the square of the speed of light. . . . His modification of the proportionality of momentum to velocity will strike [his colleagues] in no other way than a modification in any other time-honored proposition of physics would strike them. . . . I am inclined to dismiss the idea that a special category of necessity, the logical or mathematical, is represented by the law that momentum is proportional to velocity.

Quine's diffident reference to "$p = mv$" as "a truth that might be cited" as logically necessary ought not mislead. Quine takes "$p = mv$" as paradigmatic of what proponents of logical necessity regard as definitional truth. He could, and presumably would, generalize his claim that "$p = mv$" is sensitive to factual findings to any proposed definitional truth whatever.[2]

G. Harman has amplified this account of how erstwhile definitional truths become indistinguishable from the empirical elements of their home

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theories.[3] Harman likens stipulative definitions to postulates, in particular postulated equivalences. At first, of course, it is pointless to challenge postulates, for we want to see where they lead. However, when changes in theory become required, "we will just as readily change a postulate as anything else."

Whatever the ultimate correctness of Quine's position, it is worth noting that his historical example is susceptible of a different interpretation. This alternative interpretation explains how the stipulation that "p" denote mv gave way to $p = m_0v/\sqrt{1 - v^2/c^2}$ (c the velocity of light in vacuo), and consequently $p \neq mv$, without having to concede that "$p = mv$" was a falsified definition.

Some early physicist introduced "p" into science by saying "Let mv be denoted 'p': therefore $p = mv$." What was the point of this introduction? It seems clear that the definiens contain a tacit factual assumption: that there is a velocity-independent inertial property of bodies (mass). Another way of stating this assumption is: mass is a velocity-independent property of bodies, so that there is such a phenomenon as *the* mass of a body. The point of "$p = mv$" was to give a short name for a dynamical property of a body *of a given mass* travelling at different speeds that property of a body which is a function (in particular the product) of *the* mass of the body and its velocity. Let "m_0" denote the rest mass of the body M and "m_v" the mass of M when M is travelling at velocity v. The factual assumptions underlying the use of "momentum" as an expression for a function of *the* mass of a body are $m_0 = m_v$ and, derivatively, $m_0v = m_vv$. Indeed, the *point* of "$p = mv$" is to assert that there is a property of a body which is the product of its velocity and its velocity-independent mass.

It turned out, however, that $m_v \neq m_0$. In point of fact $m_v = m_0/\sqrt{1 - v^2/c^2}$, and consequently $m_vv = m_0v/\sqrt{1 - v^2/c^2}$. When it was discovered that the use of "mv" as definiens for "p" rested on a factual error, the "definition" of which "mv" was the definiens was naturally rejected. Thus, the original introduction of "p" for "mv" can better be characterized, as the introduction of an abbreviation for a definite description than as a definition. If "Charlie" is introduced as short for "Sam's youngest brother," it is no surprise that the sentence "Charlie is Sam's youngest brother" is false if Sam has no brothers. And it would be a mistake to call

"Charlie" stands for "Sam's oldest brother"

a definition, just as it would be a mistake to construe "Charlie is not Sam's oldest brother" as the retraction of a definition.

I will now try by stages to explicate the claim that stipulatively introducing an abbreviation for a definite description (henceforth an ι-*definition*) rests on a factual assumption, and hence is not a genuine definition. It is tempting to say that

(1) Let "a" denote $(\iota x)Fx$

entails

(2) $(\exists x)Fx$,

and that this sins against the requirement that a definition must be non-creative,[4] in particular against the special case of the non-creativity requirement that a definition cannot entail an existence theorem not involving the definiendum. Similarly, it might be said,

(3) Let "p" denote the product of v and (ιx) (x is a velocity-independent inertial property of bodies)

entails

(4) $(\exists x)$ (x is a velocity-independent inertial property of bodies),

and hence sins against non-creativity. However, (1) violates non-creativity only if (2) is not already a theorem of **T**, the theory to which (1) is added. But since (4) is an assumption of **N** = classical mechanics, **N**′ = **N** + (3) is, in the standard terminology of footnote 4, a conservative extension of **N** and hence (3) is non-creative. The intuition that ι-definitions are creative will turn out to be well-founded; but it is clear that ι-definitions are not a subclass of creative ones.

One response to this is to say that, for most empirical theories **T**, the notion of "theorem of **T**" is too ill-defined for it ever to be clear whether (2) is a theorem of **T** independent of the addition of (1) to **T**. It would be better, however, to look for a more general characterization of "resting on a factual assumption" which explains the occasional indeterminacy of creativity as a special case. I will now introduce the notion of "pseudo-creativity" as such a general characterization, and suggest that "$p = mv$" is a pseudocreative definition. Adopting Shoenfield's notation, if "$f(x)$ is a function symbol to be introduced by definition into **T**, the *defining axiom* for "$f(x)$" is

(5) $f(x) = y \equiv D(x, y)$

where "$D(x, y)$" is a wff of **T**. **T**′ = **T** + (3) is a (conservative) extension by definition of a function symbol (henceforth an edf) if the existence-uniqueness (eu-) condition

(6) $(x)(\exists !y)D(x, y)$[5]

is provable in **T**. Rewriting "p", "m", and "v" as functions whose variable ranges over bodies and rewriting (3) as

(7) $p(x) = n \equiv m(x)v(x) = n$,

N′ = **N** + (7) is an edf of **N**, since

(8) $(x)(\exists !n)(m(x)v(x) = n)$[6]

is provable in **N**.

Notice that ι-definitions require eu-conditions. This is clearest in the relational case, in which y is to be the x which bears R to z. The defining axiom (schema) for this case is

(9) $y = (\iota x)R(x, z) \equiv (\exists x)R'(x, z)$

and the eu-condition is

(10) $(z)(\exists !x)R'(x, z)$.

This is the form of (3) thought of as an ι-definition, with "x is a momentum of z" as $R(x, z)$ and "x is a mass-velocity product of z" as $R'(x, z)$. But even extending **T** by a singular term for "the F," F monadic, has as defining axiom

(11) $y = (\iota x)Fx \equiv F'(y, x)$, where $F'(y, x) =_{df.} (Fy \;\&\; x = x)$[6a]

and eu-condition

(12) $(x)(\exists !y)F'(y, x)$

Thus, the theory of ι-definitions becomes part of the theory of edf's.

Suppose that the eu-condition for a defining axiom D to be added to **T**, is provable in **T** but that it depends essentially on at least one empirical assumption of **T**—i.e., there is not derivation of the eu-condition of D in **T** which does not make use of at least one empirical assumption of **T**. In other words, even though D makes no *new* commitment, it does make an empirical commitment of some sort. More specifically, let us call a definition (i.e., the introduction of a new term via a defining axiom) D of **T** *hypocreative* if D either has no eu-condition (where e.g., the left-hand side of D contains no identity symbol) or the eu-condition of D is deducible from the logical axioms of **T**. If D is not hypocreative, it is *pseudocreative*. Thus, definitional extensions by predicate symbols are hypocreative, since they involve no eu-condition. This is intuitively proper.

The interesting case is that in which D is non-creative but pseudocreative. The eu-condition of a hypocreative definition cannot be discovered to be false, but the eu-condition of a pseudocreative definition can. A pseudocreative definition is precisely one that embodies a factual assumption to which its home theory is already independently committed. In particular, it is a matter of inspection that (7) is, even if non-creative, pseudocreative. It makes, or rests on, a factual assertion, although one to which **N** is already committed. The discovery mentioned above that $m_v \neq m_0$ now becomes the discovery that $\neg(x)(\exists !n)m(x) = n$, which entails that the composite relation $m(x)v(x) = n$ is one-many, and hence that the eu-condition (8) for (7) is false.

Here is the philosophical moral I want to draw. The intuition behind unrevisable definitional truth, the sort of truth Quine opposes, is that the definition on which such a definitional truth rests be not only non-creative but also hypocreative.[7] It must not avoid new commitments, but it must have no consequences not deducible from the underlying logic of the home theory even if these consequences follow from the non-logical axioms. A pseudocreative definition is still a tacit factual assertion, even if it only reiterates factual assertions made elsewhere.

A reason for denying definitionality to creative definitions is a reason for denying definitionality to pseudocreative definitions. It is entailment of factual material that ruins definitionality, and this indeed varies less promiscuously than does creativity with the strength of the surrounding nonlogical principles.[8]

It is no wonder that pseudocreative definitions, and in particular ℩-definitions in a formalized version of physical theory, are vulnerable to empirical refutation. Thus "$p = mv$" is not the example of a falsified stipulative definition Quine is looking for. I suspect that this analysis generalized, and that many other ostensible or alleged examples of falsified definitions are in reality ℩-definitions or pseudocreative definitions whose eu-conditions have been falsified. It may well be the reverse danger that truly "stipulative" definitions are quite rare in since; but the bearing of this on Quine's rejection of logical necessity is not clear.

The indeterminacy of creativity for empirical theories now becomes quite understandable. It is tantamount to unclarity about whether a definition is creative, or non-creative but pseudocreative. For example, consider the colloquial definition

(13) Let "Charlie" stand for "Sam's brother."

Now *if*

(14) $(\exists ! y)$ (y is a brother of Ralph),

or

(14′) $(x)(\exists ! y)$ (y is a brother of x)

is a working assumption or "theorem" of the "background theory" to which (13) is added, (13) is pseudocreative and non-creative; if (14) is not such a working assumption, then (13) is creative as well. Since in an unformalized body of discourse it is never clear just what sentences count as antecedently known, pseudocreativity fades into creativity. Thus, the initial intuition that ℩-definitions are creative turns out to have had a measure of truth.

I am obligated to say a word about distinguishing the empirical elements of a theory from its underlying logic, for is it not Quine's very contention that this distinction is ultimately untenable? It is important, first, to recognize that the postulates of a formalized mathematical theory, like number theory or the theory of integral domains, can be construed in Church's way as part of the underlying logic of any theory which happens to use them.[9] Thus, I am not committed to construing as pseudocreative such empirically non-committal definitions as

(15) $x^y = n \equiv ((y = 0 \;\&\; n = 1) \vee (y > 0 \;\&\; n = x^{y-1} \cdot x))$,

since the eu-condition of (15) can be construed as provable from the underlying logic alone of any theory which uses number theory. The Quinean might reply that Church's method hardly serves my purposes, since it permits the incorporation of empirical postulates, like those of **N** into logic (as the definition of "a mechanics"). Thus, a prior decision is needed as to what is to count as a mathematical postulate and what an empirical one—but it is simply begging the question against Quine to assume that this can be done. My reply is that this problem is surely *much more general* than the one we have been concerned with overall, namely the embedding of a factual assumption in an ostensible definition. Even if at some extremely general level of philosophizing we find we cannot segregate empirical postulates from other assumptions we thought they should be segregable from, it hardly follows that we must immediately dismiss all definitions that use the notion of an empirical postulate.[10] The very fact that we can distinguish definitions (as of predicate symbols) which have no eu-conditions from definitions (as of function symbols) which do indicates that there is some distinction to be made between definitions which embody factual assumptions and those which do not, even if this distinction will have ultimately to be recast. In any case, Quine would not want to maintain that his rejection of necessity is *equivalent* to the claim that erstwhile definitions are sensitive to empirical refutation. He would surely want to say that *even if* a distinction between empirical and mathematical postulates is conceeded, his point that any definitional truth is vulnerable to empirical disconfirmation would still hold. And it is against this latter claim that I have been contending.

There is abundant historical evidence that at the time the ⁊-definition of "p" for "mv" was introduced, physicists explicitly subscribed to $m_v = m_0$ (or, equivalently, $(v)(v')(m_v = m_{v'})$), the assumption I have argued underlies "p" as a definite description.[11] This construal of mass was a live option up through Mach.[12] Thus, until the time of the Special Theory of Relativity, the factual assumption $m_v = m_0$ was actually made by physicists. It is laboring the obvious to say that $m_0 = m_v$ is an assumption of classical mechanics, but there is a philosophical point in insisting that classical physicists actually, historically, had this principle in mind. The point concerns the construction that can be put on "$m_0 = m_v$ is an assumption of classical mechanics." Quine could construe as Monday-morning quarterbacking the claim, made *after* the discovery that $m_0 \neq m_v$, that earlier physicists had assumed that m_0 was equal to m_v, and meant "$p = mv$" to be the complex factual claim "There is a velocity-independent inertial . . ." etc. And Wittgenstein has sensitized us to the fact that when we say "*A* meant so-and-so at t when he said *P*" we may *not* be referring to anything that went on at t.[13] In this vein, Quine has remarked that the distinction between empirical statement and definition can be made for theories, but only in the spirit of retrospective reconstruction. Sentence *S* in theory **T** is a definition only relative to *one* way, of the many possible, of "regimenting" **T**. In other regimentations (or official finished versions) of **T**, *S* will have the status of an empirical hypothesis, the term that *S* introduced in the first regimentation being introduced in this second one by another sentence. Thus definitionality exhibits the very relativity and logical insignificance that prompts Quine to demote it as an epistemological category.

Now, if it can be shown that one "reconstruction" is the one that actually describes the way the theory was thought of at the time of its presentation, we have, it would seem, non-arbitrary grounds for classifying sentences of the theory as definitional and non-definitional. We can now say more than that "$p = mv$" *can be looked at* as if it introduced an abbreviation for a definite description. We can now say that this is how "$p = mv$" was actually introduced, how it was *meant*. And if this was how it was introduced, then it was not a definition; and, finally, its subsequent rejection fails to exemplify the rejection of a definition on empirical grounds. Quine's thesis is thus not yet substantiated by an historical example.

It is possible of course to argue that even these considerations fail to disambiguate the logical status of "$p = mv$" sufficiently to warrant reinstatement of the logical/empirical distinction with respect of theories containing it. (Such a position would, I suspect, be an application of the thesis of the indeterminacy of radical translation.) But in light of the historical facts it is arbitrary *not* to allow that "p" and "$p = mv$" actually were meant as I have suggested. To insist in the face of this evidence that attributions of meaning are inevitably reconstructional is to turn away from historical evidence as relevent at all, and to fall back on some special theory of meaning.

NOTES

1. *The Ways of Paradox*, 1st ed., Random House, New York (1966), pp. 54-55.

2. I suspect Quine's diffidence is simply his consistent maintenance of an attitude of incomprehension about modalities.

3. *Thought*, Princeton University Press, Princeton, New Jersey (1973), p. 99.

4. A definition in a theory **T** is creative if it leads to theorems not involving the definiendum but not entailed by the axioms of **T**. In the standard terminology of Shoenfield, *Mathematical Logic*, Addison-Wesley, Reading, Massachusetts (1967), **T**′ is a *conservative extension* of **T** if any wff of **T** provable in **T**′ is provable in **T**. So a definition D is non-creative iff **T** + D is a conservative extension of **T**.

5. Or, if ι is available in **T**, 6′) $(x)(\exists y)(y = (\iota z)D(x,z))$. Clearly 6′) entails $(\exists !y)(y = (\iota z)D(x,z))$ and $(\exists !y)D(x,y)$.

6. Or 8′) $(x)(\exists n)(n = (\iota m)(m(x)v(x) = m))$.

6a. *Cf.* Bernays and Fraenkel, *Axiomatic Set Theory*, North Holland, Amsterdam (1968), p. 49.

7. Belnap, in "Tonk, Plonk and Plink," in Strawson (ed.), *Philosophical Logic*, Oxford University Press, Oxford (1967), has already pointed up the mischief that creative definitions work on "truth-by-definition."

8. Partial definitions of dispositional predicates by open-ended sets of reduction sentences are normally pseudocreative. The conjunction of the definiens clauses generally implies some empirical law-like generalizations. These law-like generalizations are already part of the background theory into which the dispositional predicate is introduced, and indeed rationalize the introduction of "multi-tracked" dispositions. In a suitably enlarged sense, bilateral reductions are pseudocreative.

9. *Introduction to Mathematical Logic I*, Princeton University Press, Princeton, New Jersey (1956), pp. 330-332. The gist of Church's method is to replace all primitive symbols in the closures of the postulates by free variables and explicitly define the mathematical theory as a predicate in just those free variables via the conjunction and partial prenexing of the transformed postulates. Thus, if F, G and H are the primitives of the theory of integral domains, and $A_1, \ldots, A_n$ its postulates, we can define the predicate "$id(f,g,h)$" by $id(f,g,h) \equiv A'(f,g,h)$, where A' is the assembly of the transformed A_i. A complication Church notes (footnote 544) for the important case of number theory is that while the assertion that there is *at most* one instantiation of number theory (up to isomorphism) requires no new axioms be added to theories to which Peano's postulates have been added in the indicated way, the assertion that there is *at least* one instantiation does. Whether, in light of this, the Peano postulates are a genuine addition to the underlying logic of theories which contain them is a most perplexing question.

10. *Cf.* Strawson, "Categories," in Wood and Pitcher (eds.), *Ryle*, Anchor, New York (1970), p. 190.

11. *Cf.* A. Koslow, "Changes in the concept of mass, from Newton to Einstein," Unpublished Dissertation, Columbia University, New York, 1965, pp. 53, 55.

12. *Ibid.*, pp. 176, 198.

13. *Blue and Brown Books*, Harper & Row, New York (1958), p. 39.

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