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A MODAL NATURAL DEDUCTION SYSTEM FOR S4

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Fitch-style natural deduction systems are noteworthy not only for the ease with which derivations can be constructed in general [1], but also for the relative simplicity of modal derivations in particular. Many basic texts in symbolic logic, however, omit a presentation of such a system. Further, there are often disadvantages inherent in the various available alternatives. For example, axiomatic derivations are usually difficult, and Georgacarakos' truth trees [2] are appropriate only if that sort of decision procedure is introduced prior to the consideration of modalities.

The system proposed here is a partial solution to these difficulties, and a pedagogical convenience in the sense that it is readily adaptable to numerous common texts as well as being easy for students to learn. In this regard it can be presented as an extension of the natural deduction system with which they are already familiar.

If the system of modal derivations proposed here is in fact seen as an extension of a natural deduction system for the propositional calculus, it consists of the following. First, the basic elements of the system are taken to be the basic elements of the propositional calculus, including primitive vocabulary, introduced definitions, formation rules, inference rules, and replacement rules. The extension of the propositional calculus into a modal system comes about through the introduction of new elements,

1. Primitive vocabulary:
   L Monadic operator

2. Introduced definitions:
   (Def) M $Mp =_{df} \sim L \sim p$
   (Def) ⥽ p ⥽ $q =_{df} L(p \supset q)$
   (Def) $=$ $p = q =_{df} ((p$ ⥽ $q) \,\&\, (q$ ⥽ $p))$

3. Formation rules:
   If p is a wff, then Lp and Mp are wff's.
   If p, q are wff's, then (p ⥽ q), ($p = q$) are wff's.

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4. Inference rules:

L elimination: **LE**

$$\frac{Lp}{p}\ \textbf{LE}$$

M introduction: **MI**

$$\frac{p}{Mp}\ \textbf{MI}$$

M elimination: **ME** (Strict Proof)

(**ME**) j	Mp	Assumption, $\mathbf{CP}_1$, (**ME**)
k	p	line j, **ME**
l	q	(reason α)
m	Mq	line l, **MI**
n	$Mp \supset Mq$	$\mathbf{CP}_1$, (**ME**), lines j-m

To insure that the proof remains strict, certain rules or conventions must be maintained. All *M* elimination must take place within a conditional proof that is strict for **ME**. No assumption that is not fully modalized by either *L* or *M* may be introduced into the body of the strict proof, nor can any prior line be utilized in the strict proof which is not fully modalized by either *L* or *M*. Finally, the conclusion must be fully modalized by *M* before its conditional nature can be discharged.

L introduction: **LI** (Strict Proof)

(**LI**) h	Lp	Assumption, $\mathbf{CP}_1$, (**LI**)
i	q	(reason α)
j	Lq	line i, **LI**
l	$Lp \supset Lq$	$\mathbf{CP}_1$, (**LI**), lines h-j

This proof must also have certain strictures. That is, all *L* introduction must take place within a conditional proof that is strict for **LI**. Further, no assumption that is not fully modalized by *L* can be introduced into the body of the strict proof, nor can any prior line which is not fully modalized by *L* be utilized within the body of the strict proof. Finally, the conclusion of the strict proof must be fully modalized by *L* before its conditional nature can be discharged.

This system is complete for S4. If desired, however, appropriate theorems can be added for the sake of convenience, for example $L(p \,\&\, q) \equiv (Lp \,\&\, Lq)$, or "*L* Distribution". It can also be easily adjusted to deal with other modal systems. S5 for instance can be obtained by the introduction of the axiom $Mp \supset LMp$ as an inference rule.

Finally, the system as proposed is even simpler in practice than it is in presentation. A sample demonstration is the derivation of $Lp \supset LLp$.

——$/Lp \supset LLp$

(**LI**) 1	Lp	Assumption, $\mathbf{CP}_1$, (**LI**)
2	$Lp \vee Lp$	1, Addition

3 Lp	2, Tautology
4 LLp	3, **LI**
5 $Lp \supset LLp$	$\mathbf{CP}_1$, (**LI**), 1-4

REFERENCES

[1] Gilbert, M. A., "A heuristic procedure for natural deduction derivations using reductio ad absurdum," *Notre Dame Journal of Formal Logic*, vol. XVII (1976), pp. 638-639.

[2] Georgacarakos, G. N., "Semantics for S4.04, S4.4, and S4.3.2," *Notre Dame Journal of Formal Logic*, vol. XVII (1976), pp. 297-302.

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