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S5 WITHOUT MODAL AXIOMS

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The functional adequacy of a modal analogue of the Sheffer stroke for the system S5 is discussed by Massey in [1]. Here a system S5(†) containing S5 has as modal primitive the analogue of the dagger, which enables modal axioms to be dispensed with if there are propositional calculus (**PC**) axioms.

The basic vocabulary of S5(†) is that of **PC** plus the binary infix Sim. The syntax accords with that. Defined terms include, for any formulas A,B:

D1	AB	$= A \,\&\, B$	
D2	SA	$= A$ Sim A	(inconsistency)
D3	LA	$= (\sim A)$ Sim $\sim A$	(necessity)
D4	MA	$= \sim(A$ Sim $A)$	(possibility)
D5	A Con B	$= AB$ Sim AB	(Sheffer stroke)
D6	A Kon B	$= AB$ Sim $\sim A \sim B$	(contradiction)
D7	A Com B	$= \sim(AB$ Sim $AB)$	(compatibility)
D8	$A \rightarrow B$	$= A \sim B$ Sim $A \sim B$	(strict implication)
D9	$A \leftrightarrow B$	$= A \sim B$ Sim $B \sim A$	(strict biimplication)

D7 defines Lewis' consistency operator (*cf.* [2]).

The axioms are those of **PC**. The rules of proof are those of **PC**, plus:

R1 *If* $\vdash A$ Sim B, *then* $\vdash SA$ *and* $\vdash \sim B$;
R2 *If* $\vdash A$, *then* $\vdash(\sim A)$ Sim $((SB) \,\&\, B \vee S \sim (B \supset C) \,\&\, SC \,\&\, \sim SB \vee \sim (SB \vee S \sim\sim SB))$

The rule of necessitation follows directly from R1 and R2, and the theses normally axiomatic are derivable from **PC**. For example:

(i)	$p \supset p$	
(ii)	$\sim(p \supset p)$ Sim $((SB) \,\&\, B \vee \ldots$ etc)	(i),R2
(iii)	$\sim((SB) \,\&\, B \vee \ldots$ etc)	(ii),R1
(iv)	$LB \supset B$	(iii),D2,D3, **PC**
(v)	$L(B \supset C) \supset (LB \supset LC)$	(iii),D2,D3, **PC**
(vi)	$MB \supset LMB$	(iii),D2,D3, **PC**

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S5(†) thus contains S5. A similar basis for T, S4, etc. may be formulated by altering the third disjunct in R2.

Whether S5 contains S5(†) depends on the rules of proof for derived rules. Define A Sim B in S5 as $L(A \dagger B)$ or as $L{\sim}A \,\&\, L{\sim}B$, and abbreviate the conjunction of all axioms in S5 by '**Ax**'. Given two rules for 'if', R2 is derivable thus:

I

$$\frac{\vdash A/}{\therefore \textit{if} \vdash B, \textit{ then} \vdash A}$$

II

$$\frac{\textit{if} \vdash A, \textit{ then} \vdash B \qquad \textit{if} \vdash A, \textit{ then} \vdash C}{\therefore \textit{if} \vdash A, \textit{ then} \vdash B \,\&\, C}$$

From I, for any wff B, if $\vdash B$ then $\vdash$**Ax**. Necessitation gives: if $\vdash B$, then $\vdash L$**Ax**, and also: if $\vdash B$, then $\vdash LB$. By II, if $\vdash B$, then $\vdash LB \,\&\, L$**Ax**, and by definition: if $\vdash B$ then $\vdash({\sim}B)$ Sim ${\sim}$**Ax**, which is R2. R1 is more easily derived, so with these assumptions, S5(†) is equivalent to S5. R2 cannot be derived by this route without modal axioms, since they are used in the proof.

The primitive relation is symmetric and transitive, and idempotent with respect to conjunction and alternation. Theorems include p Sim $p \equiv p$ Con p and p Sim $q \supset p$ Con q. Some of the binary operators have division laws:

T1 pq Con pq $\equiv p$ Con q
T2 pq Com pq $\equiv p$ Com q (*cf.* Theorem 18.3 in (2))
T3 pq Sim pq $\equiv p$ Con q
T4 $(p \vee q)$ Sim $(p \vee q) \equiv p$ Sim q

REFERENCES

[1] Massey, G. J., "Binary connectives functionally complete by themselves in S5 modal logic," *The Journal of Symbolic Logic*, vol. 32 (1967), pp. 91-92.

[2] Lewis, C. I. and C. H. Langford, *Symbolic Logic*, Dover Co. (1932).

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