Notre Dame Journal of Formal Logic
Volume XX, Number 3, July 1979
NDJFAM

DEFINING GENERAL STRUCTURES

JACK C. BOUDREAUX

0 *Introduction* Henkin demonstrated in [7] that an adequate semantic theory for any axiomatic higher-order functional calculi could be developed, if the class of *structures* (viz., models, interpretations, realizations, etc.) upon which the semantic theory is based is significantly "wider" than the class of all standard structures. Moreover, if the calculus contains the axiom schemas of Extensionality and Comprehension, then it is obvious that the members of the appropriate "wider" class of structures, which may be called the class of general structures, must exhibit a rather high degree of "internal" organization.

In [9], p. 324, Henkin observed that an important technical problem had remained unsolved, i.e., to give a perspicuous definition of this class of structures which is not overly dependent upon the syntactic design of the higher-order language. Andrews proposed one solution to this problem in [1] and [2]; he proved that every general structure must be "closed" with respect to certain combinatory operators. In this paper I will propose an alternative solution in which the definition of the class of general structures is given in strictly *set-theoretical* terms. Specifically, I will prove that every general structure must be "closed" with respect to a family of *Projective operations*, *cf.* Kuratowski and Mostowski [10], pp. 357-358, and a "Cut" operation, which I have adapted from Shoenfield [12], p. 230.

1 *Syntactic preliminaries* In the language of the simple theory of types, henceforth, "**ST**", every categorematic term is assigned a *type index*. For our purposes, we may assume that all of the type indices, henceforth, simply "indices", are generated by the following recursion

(1) *Indices:* "**i**" is the primitive index (i.e., the type index assigned to individuals); if B is a finite, but unempty, sequence of indices, then (**B**) is a non-primitive index.[1]

The *adicy* function on the indices is the function which maps the index "**i**" onto 0 and which maps every non-primitive index (**B**) onto the *length* of the sequence B, henceforth, "$\mathsf{lh}(B)$". The adicy of an index is its value under the adicy function.

Received November 26, 1977

The *order* function on the indices is the function OR which maps the index "**i**" onto 0 and which maps every non-primitive index (**B**) onto $\max\{\mathrm{OR}(\mathbf{B}(j)): 0 \leqslant j < \mathrm{lh}(\mathbf{B})\} + 1$. The order of an index is its value under the order function.

Let **a** be an index, then

(2) *Variables:* $u^a, v^a, w^a, x^a, y^a, z^a, \ldots,$

with and without numeric subscripts, are the variables of type **a**. Since every variable of **ST** is assigned an index, every variable is a categorematic term.

I will say that $\overline{X}$ (read: "X bar") is a B-sequence of variables if and only if B is a finite, but unempty, sequence of indices, a length of B is equal to the length of $\overline{X}$, and for all j, $0 \leqslant j < \mathrm{lh}(B)$, $\overline{X}(j)$ is a variable of type $\mathbf{B}(j)$.

In addition to the categorematic terms, **ST** recognizes five primitive *syncategorematic* terms,

(3)
- sentential connectives: "$\sim$" (negation), "$\rightarrow$" (conditional)
- universal quantifier: "$\forall$"
- punctuators: "(" (left paren), ")" (right paren)

The syntactic use of the syncategorematic terms is clearly shown in the following definition of *well-formity* in **ST**,

(4) *Well-Formed Formula, wff:* (i) if $\overline{X}$ is a B-sequence of variables, then $x^{(\mathbf{B})}(\overline{X})$ is an atomic wff;
(ii) if p is a wff, then $\sim p$ is a wff,
(iii) if p and q are wffs, then $(p \rightarrow q)$ is a wff;
(iv) if p is a wff, then $\forall x^a p$ is a wff;
(v) and nothing is an unabbreviated wff unless its being so follows from (i) through (iv).

If $\forall x^a p$ is a wff, then p is the *scope* of the quantifier $\forall x^a$. An occurrence of a variable is *bound* if and only if it is either an occurrence of the variable within a quantifier or an occurrence of the variable within the scope of a quantifier containing the same variable. An occurrence of a variable is *free* if and only if it is not bound.

A wff is *closed*; i.e., a *statement*, if and only if every occurrence of a variable in the wff is a bound occurrence. If a wff is not closed, then it is *open*. The categorematic term t^a is *free for* x^a *in* p if and only if no free occurrence of x^a is in the scope of a quantifier containing t^a.

(5) *Substitution Operation:* If p is a wff, x^a has zero or more free occurrences in p, and t^a is free for x^a in p, then $\mathbf{S}^{x^a}_{t^a} p$ is the wff obtained from p by replacing every free occurrence of x^a with an occurrence of t^a; otherwise, $\mathbf{S}^{x^a}_{t^a} p$ is identical to p.[2]

Suppose that p is a wff, $\overline{X}$ is a (possibly empty) B-sequence of variables, and $\overline{Y}$ is an unempty C-sequence of variables. Then,

(6) *Agreement:* $p/(\overline{X})\overline{Y}/$ is a *statement form* if and only if the sequences of variables $\overline{X}$ and $\overline{Y}$ are everywhere distinct and every free variable of p is a member of either the sequence $\overline{X}$ or the sequence $\overline{Y}$. If $\overline{X}$ is the empty sequence $\emptyset$, then I will write "$p/\overline{Y}/$" instead of "$p/(\emptyset)\overline{Y}/$".[3]

In order to avoid needlessly long wffs, I will accept the following standard abbreviations.

$$(7)\quad \begin{cases} \text{“}(p \vee q)\text{”} & \text{for “}(\sim p \rightarrow q)\text{”} \\ \text{“}(p \wedge q)\text{”} & \text{for “}\sim (p \rightarrow \sim q)\text{”} \\ \text{“}(p \leftrightarrow q)\text{”} & \text{for “}\sim ((p \rightarrow q) \rightarrow \sim (q \rightarrow p))\text{”} \\ \text{“}\exists x^a p\text{”} & \text{for “}\sim \forall x^a \sim p\text{”} \end{cases}$$

If a wff contains one or more of these syncategorematic signs it will be said to be an *abbreviated* wff.

2 *Standard structures* In order to develop a referential semantic theory for **ST**, the first order of business is to define a class of higher-order structures. Given the intuitive significance of the system of type indices, i.e., "**i**," is the type of individuals, "(**i**)" is the type of properties of individuals, "((**i**))" is the type of properties of properties of individuals, and so on, the most "natural" class of structures to introduce is the following one,[4]

(8) *Definition:* Let μ be either a nonzero finite ordinal or an initial ordinal (especially in the sense of von Neumann, *cf.* [10], pp. 269-273). Then S^μ is the *standard structure based* on μ if and only if

(i) $S^\mu[\mathbf{i}] = \mu$

(ii) $S^\mu[(\mathbf{B})] = \left\{f\colon \mathrm{dm}(f) = \bigtimes_{0 \leqslant j < \mathrm{lh}(\mathbf{B})} S^\mu[\mathbf{B}(j)] \wedge \mathrm{rg}(f) \subseteq \{\mathbf{t},\mathbf{f}\}\right\}$,

that is, the *typed universe* $S^\mu[(\mathbf{B})]$ is the set of all functions that map the elements of the Cartesian product $\bigtimes_{0 \leqslant j < \mathrm{lh}(\mathbf{B})} S^\mu[\mathbf{B}(j)]$ into the set of truth-values, especially "**t**" is the truth-value "true" and "**f**" is the truth-value "false."

Notational Remark: Since I will frequently have occasion to form the Cartesian product of typed universes, let me introduce the abbreviation "$\bigtimes_{(\mathbf{B})} S$" for the more cumbersome expression used in (8.ii).

A *referential*, or *Tarskian, semantic theory* can be based on this class of structures in the usual way. Since Tarski's construction is very well-known, it will be sufficient to provide only a brief sketch of it.

I will say that φ is an *assignment to* the standard structure S^μ if and only if for every variable x^a, $\varphi(x^a) \in S^\mu[\mathbf{a}]$. Every assignment is a type-preserving *into* function from the variables of **ST** to the typed universes of the structure. An assignment ψ is a $\overline{X}$*-variant* of an assignment φ if and only if for every variable y^b, except possibly for the variables in the sequence $\overline{X}$, $\psi(y^b) = \varphi(y^b)$.

The *satisfaction* relation "$\models$" is defined by induction on the *length* of wffs, i.e., on the number of quantifiers and sentential connectives that the wff contains,

(9) Let S^μ be a standard structure and φ be an assignment to S^μ, then

(i) $(S^\mu, \varphi) \models x^a(y^b \ldots z^c)$ iff $\varphi(x^a)(\varphi(y^b) \ldots \varphi(z^c)) = \mathbf{t}$
(ii) $(S^\mu, \varphi) \models \sim p$ iff not $(S^\mu, \varphi) \models p$;
(iii) $(S^\mu, \varphi) \models (p \rightarrow q)$ iff either not $(S^\mu, \varphi) \models p$ or $(S^\mu, \varphi) \models q$;
(iv) $(S^\mu, \varphi) \models \forall x^a p$ iff $(S^\mu, \psi) \models p$, if ψ is any x^a-variant of φ.

The extension of this definition to abbreviated wffs is completely determined by the abbreviational conventions stated in (7).

We pause for an obvious, but very useful, lemma,

(10) Lemma *If φ and ψ are assignments to S^μ that agree on the free variables of p,* i.e., *if x^a is a free variable of p, then $\varphi(x^a) = \psi(x^a)$, then $(S^\mu, \varphi) \models p$ iff $(S^\mu, \psi) \models p$. Consequently, if p is a statement then $(S^\mu, \varphi) \models p$ iff $(S^\mu, \psi) \models p$, for all assignments φ and ψ to S^μ.*

Proof: The first part of Lemma (10) is obtained by induction on the length of p, *cf.* Mendelson [11], p. 52 for hints. The second part follows as an immediate corollary of the first, i.e., if p is a statement, then p has no free variables; hence, all assignments to S^μ agree on the free variables of p.
Q.E.D.

I will say that the standard structure S^μ *verifies* the wff p, henceforth, "$S^\mu \models p$", if and only if $(S^\mu, \varphi) \models p$, if φ is any assignment to S^μ. On the other hand, I will say that S^μ *falsifies* p, henceforth, "$S^\mu \not\models p$", if and only if not $(S, \varphi) \models p$, if φ is any assignment to S^μ. Every statement is either verified or falsified by S^μ; and if a wff p is neither verified nor falsified, i.e., if there is an assignment φ such that $(S^\mu, \varphi) \models p$ and an assignment ψ such that not $(S^\mu, \psi) \models p$, then p must be an open wff. Suppose that p is a wff which contains $x^a, \ldots, y^b$ as free variables, then S^μ verifies p if and only if S^μ verifies the *universal closure* $\forall x^a \ldots \forall y^b p$, henceforth, "$\forall p$", of p.

(11) *Standard Validity:* A wff p is *standardly valid* if and only if p is verified by every standard structure. A wff p is *standardly invalid* if and only if p is falsified by every standard structure. A wff p is *standardly indeterminate*, or *factual*, if and only if it is verified by some standard structures and falsified by others.

Let this conclude our summary review of the standard semantics of **ST**.

3 *Higher-order theories* Even though standard validity seems to be a very "natural" interpretation for the type-theoretic language introduced in section **1**, Gödel's Incompleteness Theorem of 1931 convincingly established that the set of standardly valid wffs must always exceed the set of wffs obtainable in any *axiomatic* higher-order theory.

A set of wffs, either abbreviated or unabbreviated, is a higher-order theory if and only if it contains all instances of the axiom schemas,

FI. $(p \rightarrow (q \rightarrow p))$
FII. $((p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r)))$
FIII. $((\sim p \rightarrow \sim q) \rightarrow ((\sim p \rightarrow q) \rightarrow p))$
FIV. $(\forall x^a p \rightarrow \mathsf{S}^{x^a}_{t^a} p)$
FV. $(\forall x^a (p \rightarrow q) \rightarrow (p \rightarrow \forall x^a q))$, provided x^a has no free occurrence in p.

and it is inferentially closed under the following (primitive) *rules of inference*,

MP (*Modus Ponens*). From p and $(p \rightarrow q)$, infer q
Gen (*Generalization*). From p, infer $\forall x^a p$.

The term "inferentially closed" can be defined with great precision. Let T be a set of wffs, then a finite sequence of wffs $p_0, \ldots, p_n$ is a *derivation* in T if and only if for all j, $0 \leq j \leq n$, p_j is an instance of an axiom schema, an element of T, an immediate inference from two preceeding components of the sequence by **MP**, or an immediate inference from one preceeding component of the sequence by **Gen**. A set of wffs is *inferentially closed* if and only if the terminal member of every derivation in the set is an element of the set.

The inferential closure of FI through FV, henceforth, "**F**", is the exact higher-order analogue of the first-order quantificational calculus. A higher-order theory is *consistent* if and only if it is a proper subset of **ST**. Since **F** includes all of the classical, i.e., bivalued, calculus of propositions, we can reformulate this definition as follows: a higher-order theory is consistent if and only if there is no wff p such that both p and $\sim p$ are elements of the theory. A higher-order theory is *inconsistent* if and only if it is not consistent.

A higher-order theory is *maximally consistent* if and only if it is consistent and is not a proper subset of a consistent higher-order theory. That is, if T is maximally consistent and p is not an element of T, then the theory obtained by adding p to T is inconsistent. I presume that the reader is familiar with the higher-order analogue of Lindenbaum's Lemma, which says that every consistent higher-order theory is a subset of, i.e., can be *extended* to, a maximally consistent higher-order theory.

I will say that a higher-order theory is a *foundational system* if and only if it contains the axiom schemas of Extensionality and Comprehension, i.e.,

Ex $\forall x^a \forall y^a (\forall \overline{X}(x^a(\overline{X}) \leftrightarrow y^a(\overline{X})) \rightarrow \forall z^{(a)}(z^{(a)}(x^a) \leftrightarrow z^{(a)}(y^a)))$

K $\forall \overline{X} \exists x^a \forall \overline{Y}(x^a(\overline{Y}) \leftrightarrow p/(\overline{X})\overline{Y}/)$, *provided x^a is foreign to p.*

Let **FK** be the inferential closure of FI through FV, **Ex**, and **K**.

Remark: Some authors, e.g., Beth [4], p. 226, express **Ex** as a biconditional. But this is not necessary since every instance of the *converse* of **Ex**, i.e.,

$$\forall x^a \forall y^a (\forall z^{(a)} (z^{(a)}(x^a) \leftrightarrow z^{(a)}(y^a)) \rightarrow \forall \overline{X} (x^a(\overline{X}) \leftrightarrow y^a(\overline{X})))$$

is derivable in **FK**.[5]

Having thus precised the notion "higher-order theory", we can state Gödel's Theorem more accurately, viz., *there is no consistent foundational system whose elements are identically the standardly valid wffs.*

4 *Henkin's proposal* In his doctoral dissertation of 1947, published in part in [7], Henkin argued that what this "negative" result really establishes is that if one insists upon basing the semantic theory of **ST** on the class of standard structures, then it is impossible to solve the *characterization problem* for any foundational system, *cf.* Shoenfield [12], p. 41. But this only demonstrates that the class of standard structures is too "narrow", and, thus, in order to obtain a reasonable semantic theory for the type-theoretic language, it would have to be significantly "widened" by introducing *non-standard structures*, i.e., structures in which the higher-order typed universes would be populated by something less than all possible functions.

The general outline of this proposal is clear enough. E.g., suppose we are attempting to solve the characterization problem for the weakest foundational system, namely, **FK**, then we should proceed as follows:

Step I: define a "wider" class of structures, i.e., a class which comprehends the class of standard structures and a fair-sized supply of non-standard structures;

Step II: base the semantic theory of **ST** on the "wider" class defined in Step I; the only portion of the semantic theory which will require substantial modification is the definition of validity, and this can be rewritten as follows: a wff is valid in the "wide" sense iff it is verified by every structure in the "wider" class;

Step III: show that the characterization problem for **FK** can be solved with respect to the "wider" class, i.e., show that the class defined in Step I has the following properties:

the **FK**-*soundness property*: if a wff is derivable in **FK**, then it is valid in the "wider" sense,

the **FK**-*completeness property*: if a wff is not derivable in **FK**, then there is at least one member of the "wider" class that falsifies it.

The definition of the "wider" class in Step I is clearly the crucial stage of Henkin's proposal, but it is at just this point that his paper is very difficult to interpret.[6]

Presumably, the "wider" class to be defined in Step I will resemble the class of standard structures in at least this respect: all of the members of the "wider" class, and *a fortiori* all of their elements, i.e., the typed universes, will be *constructible sets in the field of* **ZFS**.[7] If this presumption is correct, then a general strategy for defining the "wider" class for **FK** comes rather quickly to mind, to wit:

Step Ia: define the "widest possible" class of structures, i.e., a class which is so comprehensive that only the "first-order" fragment of **FK**, namely, **F** itself, is verified by all of its members;

Step Ib: define **ZFS**-*conditions* Φ on the members of the "widest possible" class such that the "wider" class for **FK** is identical to the class of all members of the "widest possible" class which satisfy Φ.[8]

Intuitively, the **ZFS**-conditions Φ must be just strong enough to guarantee that any structure in the "widest possible" class which satisfies Φ will simultaneously verify all of the instances of the schemas **Ex** and **K**. As obvious as the proposed strategy is, it has to my knowledge never been worked out in detail. I will now correct this oversight.

5 *The minimally restricted class of structures* In [7], footnote 5, Henkin briefly mentions a class of "altogether arbitrary" structures. What he seems to have in mind is a class of structures whose typed universes are built in an entirely arbitrary, or minimally restricted, manner. I will now define a class of structures which fits this description very closely, and then I will prove, *cf.* Theorem (14) below, that it is in fact the "widest possible" class required in Step Ia.

(12) Definition: Let μ be any ordinal satisfying the antecedent of (8). Then $\mathfrak{M}$ is a member of the *minimally restricted class of structures*, henceforth, "the **MN** class", if and only if for every index **a**, $\emptyset \neq \mathfrak{M}[\mathbf{a}] \subseteq S^{\mu}[\mathbf{a}]$.

Every standard structure is a member of the **MN** class; in fact, the **MN** class is generated by the class of standard structures.[9] That is, let "$\bigcup_{\text{ind}} S^{\mu}$" abbreviate "$\bigcup\{S[\mathbf{a}]$: **a** is in index$\}$." Then,

(13) $\mathfrak{M}$ is a member of the **MN** class if and only if there is an ordinal μ satisfying the antecedent of (8) such that

(i) $\bigcup_{\text{ind}} \mathfrak{M} \in \mathcal{P}\left(\bigcup_{\text{ind}} S^{\mu}\right)$, i.e., $\bigcup_{\text{ind}} \mathfrak{M}$ is an element of the *power set* of $\bigcup_{\text{ind}} S^{\mu}$;

and

(ii) for every index **a**, $\mathfrak{M}[\mathbf{a}] \cap S^{\mu}[\mathbf{a}] \neq \emptyset$.

To re-express (13) in a metaphoric way, we can say that every standard structure is the *core* of a well-defined set of **MN** structures, and that this set is almost, i.e., very nearly, identical to $\mathcal{P}\left(\bigcup_{\text{ind}} S\right)$.

Step II in Henkin's program presents no essential difficulties. Let us agree to say that a wff p is *strongly valid* if and only if p is verified by every **MN** structure; and that p is *strongly invalid* if and only if p is falsified by every **MN** structure. If this notion of validity is accepted, then we can prove the following theorem,

(14) The characterization theorem for strong validity. *A wff p is strongly valid if and only if p is derivable from axiom schemas* FI *through* FV *alone.*

Proof: From the right to the left the proof is entirely straightforward; in fact, it differs from the Soundness Theorem for first-order only in inessential respects, *cf.* Mendelson [11], pp. 52-56. In the other direction, it will be sufficient to prove the following analogue of Henkin's Lemma,

(i) if T is a consistent extension of the higher-order calculus **F**, then there is an **MN** structure $\mathfrak{M}$ such that every element of T is verified by $\mathfrak{M}$, i.e., $\mathfrak{M}$ *models* T.

Proof of (i): Let **ST**$^+$ be the higher-order language obtained from **ST** by adding a denumerably infinite run of variables c_n^a, $0 \leqslant n < \omega$, for every index **a**. If T is consistent in **ST**, then T is consistent in **ST**$^+$. Let

(ii) $p_0/x_0^{a_0}/, \ldots, p_j/x_j^{a_j}/, \ldots$

be an enumeration of all of the wffs of **ST**$^+$ which contain precisely one free variable. Then define the jth *special constant* as follows:

(iii) $c_{\sigma(j)}^{a_j}$ is the jth special constant if and only if it is the earliest constant of type $\mathbf{a}_j$ in the lexicographic order which is (1) foreign to the wffs $p_k/x_k^{a_k}/$ in the enumeration, $0 \leqslant k \leqslant j$, and (2) not identical to the kth special constant, $0 \leqslant k \leqslant j$.

Clearly such a constant exists and is uniquely well-defined. Let

(iv) $\left(\mathbf{S}_{c_{\sigma(j)}^{a_j}}^{x_j^{a_j}}\ p_j/x_j^{a_j}/ \rightarrow \forall x_j^{a_j} p_j/x_j^{a_j}/\right)$

be the jth *special axiom*, henceforth, "^{j}**SA**". Then define the following sequence of higher-order theories by induction on j,

(v) $\begin{cases} {}^{0}T = T \\ {}^{(j+1)}T = {}^{j}T + {}^{(j-1)}\mathbf{SA} \\ {}^{\omega}T = \bigcup\limits_{0 \leqslant j \leqslant \omega} {}^{j}T \end{cases}$

Since T is consistent in **ST**$^+$, then all of the proposed extensions of T, including T, are consistent in **ST**$^+$ (*cf.* Mendelson [11], p. 66, for a detailed proof of the analogous result for first-order). Hence, by Lindenbaum's Lemma, T has a maximally consistent extension, say $({}^{\omega}T)^{\max}$, with the following property:

(vi) if p is any closed atomic wff of **ST**$^+$, then either $p \in ({}^{\omega}T)^{\max}$ or $\sim p \in ({}^{\omega}T)^{\max}$.

In order to complete the proof of (i), it is sufficient to establish the existence of a type-preserving bijective function, say ξ, from the constants of **ST**$^+$ into the type universes of S^{ω} such that for every closed atomic wff $c_n^a(c_m^b \ldots c_p^c)$,

(vii) $c_n^a(c_m^b \ldots c_p^c) \in ({}^{\omega}T)^{\max}$ iff $\xi(c_n^a)(\xi(c_m^b) \ldots \xi(c_p^c)) = \mathbf{t}$.

In fact, there are very many functions which satisfy this condition. If we define $\mathfrak{M}^{\xi}$ as the image of the constants under ξ, then it is easy to prove that $\mathfrak{M}^{\xi}$ is a **MN** structure which does indeed model T, *cf.* Mendelson [11], *ut supra*. Q.E.D.

Finally, assume that p is strongly valid and yet not derivable in **F**. Then $\mathbf{F} \cup \{\sim p\}$ is a consistent extension of **F**, and, by (i), there is a **MN** structure that models $\mathbf{F} \cup \{\sim p\}$. Hence, by definition, $\sim p$ is verified by this structure. On the other hand, since p is strongly valid, p is also verified by the same **MN** structure, which is impossible. Q.E.D.

This Theorem shows that the **MN** class is precisely the class of structures called for in Step Ia.

6 ZFS-*conditions on the* **MN** *class* At this point it is self-evident that the "wider" class for **FK** is at once a proper super-class of the class of standard structures and a proper sub-class of the **MN**-class. Hence, the definition of the "wider" class will be completed by the discovery of the **ZFS**-*conditions* on the **MN** class called for in Step Ib.

Φ is a **ZFS**-condition on the **MN** class iff Φ is a wff in the field of **ZFS** whose free variables are schematic variables of the following sorts: (i) "$\mathfrak{M}$", "$\mathfrak{N}$", etc., which are to be interpreted as variables ranging over the members of the **MN** class; (ii) "$\mathfrak{M}[\mathbf{a}]$", "$\mathfrak{M}[\mathbf{b}]$", etc., which are to be interpreted as variables ranging over the typed universes of **MN** structures; (iii) "$\mathfrak{M}[(\mathbf{B})]$", "$\mathfrak{M}[(\mathbf{C})]$", "$\mathfrak{M}[(\mathbf{BC})]$", etc., which are to be interpreted as variables ranging over the non-primitive typed universes of **MN** structures; and (iv) "$\mathop{\mathsf{X}}_{(\mathbf{B})}\mathfrak{M}$", "$\mathop{\mathsf{X}}_{(\mathbf{C})}\mathfrak{M}$", "$\mathop{\mathsf{X}}_{(\mathbf{BC})}\mathfrak{M}$", etc., which are to be interpreted as variables ranging over the Cartesian products of typed universes of **MN** structures.

Ψ is an *index-specific instance* of Φ if and only if Ψ is obtained from Φ by uniformly replacing every occurrence of an "index" variable, e.g., "**a**", "**b**", etc., with a particular index and every occurrence of an "index sequence" variable, e.g., "**B**", "**C**", etc., with a particular sequence of indices.

(15) *Satisfying a* **ZFS**-*condition* Let $\Phi(\mathfrak{M})$ be a **ZFS**-condition on the **MN** class. Then the **MN** structure $\mathfrak{N}$ *satisfies* $\Phi(\mathfrak{M})$ iff every index-specific instance of $\mathsf{S}_{\mathrm{N}}^{\mathrm{M}}\,\Phi(\mathfrak{M})$ is a true statement in the field of **ZFS**, i.e., if $\mathfrak{N}$ has the intrinsic set-theoretical property "expressed" by $\Phi(\mathfrak{M})$.

Since the **ZFS**-conditions for both **Ex** and its converse are fairly obvious, they can be used as paradigmatic examples. Consider **Ex** first.[10]

(16) *The Extensionality Condition.*

$$\forall f \forall g \left(f, g \in \mathfrak{M}[(\mathbf{B})] \rightarrow \left(\forall \eta \left(\eta \in \mathop{\mathsf{X}}_{(\mathbf{B})} \mathfrak{M} \rightarrow f(\eta) = g(\eta)\right) \leftrightarrow f = g\right)\right)$$

Suppose that $\mathfrak{N}$ is a **MN** structure, then $\mathfrak{N}$ satisfies (16) if and only if every index-specific instance of (16) is a true statement in **ZFS**. E.g., let us replace "**B**" with the sequence "**i**, **i**", then the relevant instance of (16) is

(16*) $\forall f \forall g \left(f, g \in \mathfrak{N}[(\mathbf{ii})] \to \left(\forall \eta \left(\eta \in \underset{(\mathbf{ii})}{\mathbf{X}} \mathfrak{N} \to f(\eta) = g(\eta)\right) \leftrightarrow f = g\right)\right)$

Assume that f and g are both elements of $\mathfrak{N}[(\mathbf{ii})]$ and that for every $\eta \in \underset{(\mathbf{ii})}{\mathbf{X}} \mathfrak{N}$, $f(\eta) = g(\eta)$. Then according to (16*), $f = g$, i.e., f and g are set-theoretically identical. Now suppose that h is any element of $\mathfrak{N}[((\mathbf{ii}))]$, then it must be the case that $h(f) = h(g)$; that is,

(17) $\forall h(h \in \mathfrak{N}[((\mathbf{ii}))] \to h(f) = h(g))$.

Hence, the relevant instance of **Ex** is verified by $\mathfrak{N}$. Since the same line of argument can be repeated for all other index-specific instances of (16), we may now infer that if $\mathfrak{N}$ satisfies (16), then $\mathfrak{N}$ verifies every instance of **Ex**, which was to be proven.

Next, consider the converse of **Ex**.[11]

(18) *The Normality Condition*

$$\forall f \forall g(f, g \in \mathfrak{M}[\mathbf{a}] \to (\forall h(h \in \mathfrak{M}[(\mathbf{a})] \to h(f) = h(g)) \leftrightarrow f = g))$$

Once again, it will be sufficient to prove that if a **MN** structure $\mathfrak{N}$ satisfies every index-specific instance of (18), then $\mathfrak{N}$ verifies every instance of the converse of **Ex**. Since the proof is closely analogous to the one just given, I will omit the details.

Remark: If an **MN** structure satisfies both (16) and (18), then I will say that it is a *normal structure*. Since both **Ex** and its converse are theorem schemas of **FK**, only the class of normal structures will be given any amount of attention in the sequel. As will soon be evident, restricting the **MN** class in this way has several important advantages, *cf.* footnotes 10 and 11 below.

Having thus given two illustrations of the general method, I will now consider the more difficult problem posed by the axiom schema of Comprehension. First, I will need to introduce some additional terminology.

(19) Definition: Let $\mathfrak{M}$ be a normal structure. Then:

(i) A is a *homogeneous* subset of $\mathfrak{M}$ iff A is a subset of $\underset{(\mathbf{B})}{\mathbf{X}} \mathfrak{M}$, i.e., A is an element of the power set $\mathcal{P}\left(\underset{(\mathbf{B})}{\mathbf{X}} \mathfrak{M}\right)$.

(ii) A is a *characterized* subset of $\mathfrak{M}$ iff $\mathfrak{A} \in \mathcal{P}\left(\underset{(\mathbf{B})}{\mathbf{X}} \mathfrak{M}\right)$, i.e., A is a homogeneous subset of $\mathfrak{M}$, and there is an $h^* \in \mathfrak{M}[(\mathbf{B})]$ such that for every $\eta \in \underset{(\mathbf{B})}{\mathbf{X}} \mathfrak{M}$,

$$h^*(\eta) = \begin{cases} \mathsf{t}, \text{ if } \eta \in A \\ \mathsf{f}, \text{ if } \eta \notin A \end{cases}$$

in this case, h^* is the *characteristic function* of A in $\mathfrak{M}$, henceforth, "$h^* = \chi_{\mathfrak{M}}(A)$".

(iii) A is a *definable* subset of $\mathfrak{M}$ iff $A \in \mathcal{P}\left(\underset{(\mathbf{B})}{\mathbf{X}} \mathfrak{M}\right)$ and there is a statement form $p/\overline{Y}/$, where $\overline{Y}$ is a B-sequence of variables, such that for every $\eta \in \bigtimes_{(\mathbf{B})} \mathfrak{M}$, $\eta \in A$ iff for every assignment φ to $\mathfrak{M}$, if $\varphi \restriction \overline{Y} = \eta$, then $(\mathfrak{M}, \varphi) \models p/\overline{Y}/$; in this case, A is the *($\mathfrak{M}$)-graph* of $p/\overline{Y}/$.

(iv) A is a *parametrically definable* subset of $\mathfrak{M}$ iff $A \in \mathscr{P}\left(\bigtimes_{(\mathbf{B})} \mathfrak{M}\right)$ and there is a statement form $q/(\overline{X})\overline{Y}/$, where $\overline{X}$ is a C-sequence of variables and $\overline{Y}$ is a B-sequence of variables, and there is a $\gamma \in \bigtimes_{(\mathbf{C})} \mathfrak{M}$ such that for all $\eta \in \bigtimes_{(\mathbf{B})} \mathfrak{M}$, $\eta \in A$ iff for every assignment φ to $\mathfrak{M}$, if $\varphi \restriction \overline{X} = \gamma$ and $\varphi \restriction \overline{Y} = \eta$, then $(\mathfrak{M}, \varphi) \models q/(\overline{X})\overline{Y}/$; in this case, A is the *($\mathfrak{M}$, γ)-graph* of $q/(\overline{X})\overline{Y}/$.

Remark: Lemma (10) alone is sufficient to guarantee that every statement form $p/\overline{Y}/$ has a unique ($\mathfrak{M}$)-graph and that every statement form $q/(\overline{X})\overline{Y}/$ has a unique ($\mathfrak{M}$, γ)-graph, for all appropriate values of γ. However, the assumption that $\mathfrak{M}$ is a normal structure is required in order to insure that if a homogeneous subset of $\mathfrak{M}$ has a characteristic function in $\mathfrak{M}$—and it may not—then it has at most one such function.[12]

We can now state an "extrinsic" condition which a normal structure must satisfy if it is to verify every instance of the Comprehension schema,

(20) The Comprehension Condition, first version *Let $\mathfrak{M}$ be a normal structure. Then $\mathfrak{M}$ verifies every instance* of **K** *iff every parametrically definable subset of $\mathfrak{M}$ is a characterized subset of $\mathfrak{M}$, and conversely.*

Proof: If a homogeneous subset of $\mathfrak{M}$ is definable, then it is parametrically definable, as the reader can easily confirm for himself. Hence, no separate provision must be made for the definable subsets of $\mathfrak{M}$. Moreover, if A is a characterized subset of $\mathfrak{M}$ then it is automatically a parametrically definable subset of $\mathfrak{M}$. That is, suppose that $A \in \mathscr{P}\left(\bigtimes_{(\mathbf{B})} \mathfrak{M}\right)$ and there is an $h^* \in \mathfrak{M}[(\mathbf{B})]$ such that $h^* = \chi_{\mathfrak{M}}(A)$. Let $x^{(\mathbf{B})}(\overline{Y})$ be an atomic wff whose variables are everywhere distinct, then it is self-evident that A is the ($\mathfrak{M}$, h^*)-graph of the statement form $x^{(\mathbf{B})}(\overline{Y})/(x^{(\mathbf{B})})\overline{Y}/$, which means that A is parametrically definable. Therefore, in order to prove the Theorem it will be both necessary and sufficient to prove that

(i) *$\mathfrak{M}$ verifies every instance of* **K** *iff every parametrically definable subset of $\mathfrak{M}$ is a characterized subset of $\mathfrak{M}$.*

In one direction, assume that $\mathfrak{M}$ verifies every instance of **K** and that $A \in \mathscr{P}\left(\bigtimes_{(\mathbf{B})} \mathfrak{M}\right)$ is a parametrically definable subset of $\mathfrak{M}$. Then there is a statement form $p/(\overline{X})\overline{Y}/$, where $\overline{X}$ is a C-sequence of variables and $\overline{Y}$ is a B-sequence of variables, such that for some $\gamma \in \bigtimes_{(\mathbf{C})} \mathfrak{M}$, A is the ($\mathfrak{M}$, γ)-graph of $p/(\overline{X})\overline{Y}/$. Given our initial assumption, we know that $\mathfrak{M}$ verifies

(ii) $\forall \overline{X} \exists x^{(\mathbf{B})} \forall \overline{Y}(x^{(\mathbf{B})}(\overline{Y}) \leftrightarrow p/(\overline{X})\overline{Y}/)$

Hence, if φ is any assignment to $\mathfrak{M}$ such that $\varphi \restriction \overline{X} = \gamma$, then there is an $x^{(\mathbf{B})}$-variant of φ, say ψ, such that

(iii) $(\mathfrak{M}, \psi) \models \forall \overline{Y}(x^{(\mathbf{B})}(\overline{Y}) \leftrightarrow p/(\overline{X})\overline{Y}/)$.

But if this is the case, then $\psi(x^{(\mathbf{B})})$ must be the characteristic function of the $(\mathfrak{M}, \gamma)$-graph of $p/(\overline{X})\overline{Y}/$, which is identical to A. This completes the proof in one direction. In the other direction, assume that every parametrically definable subset of $\mathfrak{M}$ is a characterized subset of $\mathfrak{M}$ and that

(iv) $\forall \overline{X} \exists y^{(\mathbf{B})} \forall \overline{Y}(y^{(\mathbf{B})}(\overline{Y}) \leftrightarrow q/(\overline{X})\overline{Y}/)$

is an arbitrary instance of **K**, where $\overline{X}$ is a C-sequence of variables and $\overline{Y}$ is a B-sequence of variables. Then for every assignment φ to $\mathfrak{M}$, there is one and only one $\gamma \in \mathop{\mathsf{X}}\limits_{(\mathbf{C})} \mathfrak{M}$ such that $\varphi \upharpoonright \overline{X} = \gamma$. Since the $(\mathfrak{M}, \gamma)$-graph of $q/(\overline{X})\overline{Y}/$ is a parametrically definable subset of $\mathfrak{M}$, our first assumption entitles us to conclude that this subset of $\mathfrak{M}$ has a characteristic function $h^* \in \mathfrak{M}[(\mathbf{B})]$. If we let ψ be the $y^{(\mathbf{B})}$-variant of φ which maps $y^{(\mathbf{B})}$ onto h^*, then it is self-evident that

(v) $(\mathfrak{M}, \psi) \models \forall \overline{Y}(y^{(\mathbf{B})}(\overline{Y}) \leftrightarrow p/(\overline{X})\overline{Y}/)$,

which is quite sufficient to guarantee that $\mathfrak{M}$ verifies (iv). Since (iv) is an arbitrary instance of **K**, the general result that $\mathfrak{M}$ verifies every instance of **K** follows immediately. Q.E.D.

This Theorem is saying, in effect, that a normal structure will verify every instance of **K** iff its typed universes are "sufficiently rich" to contain a characteristic function for every parametrically definable—and, hence, every definable—subset of the structure. What remains to be done is to explicate the notion "sufficiently rich" in terms of a precisely defined list of **ZFS**-conditions. I intend to prove that the typed universes are "sufficiently rich" only if (i) the "membership" relation is a characterized subset of the structure, and (ii) that the structure's typed universes are closed under a family of set-theoretical *operations*. All of the operations with the exception of the last, *cf.* point (27) below, are the "natural" higher-order correlates of operations in the theory of *projective sets, cf.* [10], p. 357 *et sq.*

(21) Membership *Let* $\mathfrak{M}$ *be a normal structure, then*

$$E^{((\mathbf{B})\mathbf{B})} = \left\{\langle f, \eta\rangle : \langle f, \eta\rangle \in \mathop{\mathsf{X}}\limits_{((\mathbf{B})\mathbf{B})} \mathfrak{M} \wedge f(\eta) = \mathbf{t}\right\}$$

is a definable subset of $\mathfrak{M}$.

Proof: Let $x^{(\mathbf{B})}(\overline{Y})$ be an atomic wff whose variables are everywhere distinct. Then $E^{((\mathbf{B})\mathbf{B})}$ is the $(\mathfrak{M})$-graph of the statement form $x^{(\mathbf{B})}(\overline{Y})/x^{(\mathbf{B})}\overline{Y}/$. (*Note*: In this proof, and in all subsequent proofs in this series, I will include only the essential information, especially the appropriate statement form. The reader can easily supply the missing details for himself.) Q.E.D.

(22) Complement *Let* $\mathfrak{M}$ *be a normal structure and* $(\mathbf{B})$ *be any non-primitive index. Then for every* $f \in \mathfrak{M}[(\mathbf{B})]$,

$$\neg f = \{\eta : \sim f(\eta) = \mathbf{t}\}$$

is a parametrically definable subset of $\mathfrak{M}$.

Proof: Let $x^{(\mathbf{B})}(\overline{Y})$ be any atomic wff whose variables are everywhere distinct. Then $\neg f$ is the $(\mathfrak{M}, f)$-graph of the statement form $\sim x^{(\mathbf{B})}(\overline{Y})/(x^{(\mathbf{B})})\overline{Y}/$. $\neg f$ is an element of $\mathscr{P}\left(\underset{(\mathbf{B})}{\mathsf{X}}\mathfrak{M}\right)$. Q.E.D.

(23) Free Union *Let* $\mathfrak{M}$ *be a normal structure and let* $(\mathbf{B})$ *and* $(\mathbf{C})$ *be any non-primitive indices. Then for all* $f \in \mathfrak{M}[(\mathbf{B})]$ *and all* $g \in \mathfrak{M}[(\mathbf{C})]$,

$$f \oplus g = \{\langle \eta, \gamma \rangle : f(\eta) = \mathbf{t} \vee g(\gamma) = \mathbf{t}\}$$

is a parametrically definable subset of $\mathfrak{M}$.

Proof: Let $x^{(\mathbf{B})}(\overline{Y})$ and $y^{(\mathbf{C})}(\overline{Z})$ be atomic wffs whose variables are everywhere distinct. Then $f \oplus g$ is the $(\mathfrak{M}, \langle f, g\rangle)$-graph of the statement form

$$(x^{(\mathbf{B})}(\overline{Y}) \vee y^{(\mathbf{C})}(\overline{Z}))/(x^{(\mathbf{B})}y^{(\mathbf{C})})\overline{YZ}/.$$

$f \oplus g$ is an element of $\mathscr{P}\left(\underset{(\mathbf{BC})}{\mathsf{X}}\mathfrak{M}\right)$. Q.E.D.

(24) Permutation *Let* $\mathfrak{M}$ *be a normal structure, let* $(\mathbf{B})$ *be any non-primitive index and let* π *be a permutation on the numbers* $0, \ldots, n = \mathrm{lh}(\mathbf{B}) - 1$. *Then for every* $f \in \mathfrak{M}[(\mathbf{B})]$,

$$^{\pi}\mathrm{Pm}(f) = \{\langle g_{\pi(0)}, \ldots, g_{\pi(n)}\rangle : f(g_0 \ldots g_n) = \mathbf{t}\}$$

is a parametrically definable subset of $\mathfrak{M}$.

Proof: Let $x^{(\mathbf{B})}(Y)$ be an atomic wff whose variables are everywhere distinct; moreover, let "$^{\pi}B$" abbreviate "$B(\pi(0)), \ldots B(\pi(n))$" and "$^{\pi}\overline{Y}$" abbreviate "$\overline{Y}(\pi(0)), \ldots, \overline{Y}(\pi(n))$." Then $^{\pi}\mathrm{Pm}(f)$ is the $(\mathfrak{M}, f)$-graph of the statement form $x^{(\mathbf{B})}(\overline{Y})/(x^{(\mathbf{B})})^{\pi}\overline{Y}/$. $^{\pi}\mathrm{Pm}(f)$ is an element of $\mathscr{P}\left(\underset{(^{\pi}\mathbf{B})}{\mathsf{X}}\mathfrak{M}\right)$. Q.E.D.

(25) Projection *Let* $\mathfrak{M}$ *be a normal structure and let* $(\mathbf{BC})$ *be any non-primitive index such that* $n = \mathrm{lh}(\mathbf{B}) - 1$. *Then for every* $f \in \mathfrak{M}[(\mathbf{BC})]$,

$$^{n}\mathrm{Pj}(f) = \left\{\eta : \exists\gamma\left(\gamma \in \underset{(\mathbf{B})}{\mathsf{X}}\mathfrak{M} \wedge f(\gamma\eta) = \mathbf{t}\right)\right\}$$

is a parametrically definable subset of $\mathfrak{M}$.

Proof: Let $x^{(\mathbf{BC})}(\overline{ZY})$ be an atomic wff whose variables are everywhere distinct and let $\overline{Z}$ be a B-sequence of variables and $\overline{Y}$ a C-sequence of variables. Then $^{n}\mathrm{Pj}(f)$ is the $(\mathfrak{M}, f)$-graph of the statement form $\exists\overline{Z}(x^{(\mathbf{BC})}(\overline{ZY}))/(x^{(\mathbf{BC})})\overline{Y}/$. Hence, $^{n}\mathrm{Pj}(f)$ is an element of $\mathscr{P}\left(\underset{(\mathbf{C})}{\mathsf{X}}\mathfrak{M}\right)$. Q.E.D.

(26) Identification *Let* $\mathfrak{M}$ *be a normal structure, let* $(\mathbf{B})$ *be any non-primitive index such that for some* j *and* k, $0 \leq j < k \leq n = \mathrm{lh}(\mathbf{B}) - 1$. *Then for every* $f \in \mathfrak{M}[(\mathbf{B})]$,

$$^{j,k}\mathrm{Id}(f) = \{\langle g_0, \ldots, g_{k-1}, g_{k+1}, \ldots, g_n\rangle : f(g_0 \ldots g_{k-1}g_jg_{k+1} \ldots g_n) = \mathbf{t}\}$$

is a parametrically definable subset of $\mathfrak{M}$.

Proof: Let $x^{(\mathbf{B})}(\overline{Y})$ be an atomic wff whose variables are everywhere distinct; moreover, let "${}^{-k}\overline{Y}$" abbreviate "$\overline{Y}(0), \ldots \overline{Y}(k-1), \overline{Y}(k+1), \ldots \overline{Y}(n)$" and "${}^{-k}B$" abbreviate "$B(0), \ldots, B(k-1), B(k+1), \ldots, B(n)$". Then ${}^{j,k}\mathsf{Id}(f)$ is the $(\mathfrak{M}, f)$-graph of the statement form $\mathbf{S}^{\overline{Y}(k)}_{\overline{Y}(j)}(x^{(\mathbf{B})}(\overline{Y}))/(x^{(\mathbf{B})})^{-k}\overline{Y}/$. Hence, ${}^{j,k}\mathsf{Id}(f)$ is an element of $\mathscr{P}\left(\underset{(^{-k}\mathbf{B})}{\mathsf{X}}\mathfrak{M}\right)$. Q.E.D.

Finally, I need another set-theoretical operation, which, so far as I know, was first introduced by Shoenfield in the context of second-order arithmetic, *cf.* [12], p. 230,

(27) Cut *Let* $\mathfrak{M}$ *be a normal structure and let* $(\mathbf{CB})$ *be any non-primitive index. Then for every* $\eta \in \underset{(\mathbf{C})}{\mathsf{X}}\mathfrak{M}$ *and every* $f \in \mathfrak{M}[(\mathbf{C}|\mathbf{B})]$,

$$\mathsf{Cut}(\eta f) = \left\{\gamma : \gamma \in \underset{(\mathbf{B})}{\mathsf{X}}|\mathfrak{M} \wedge f(\eta\gamma) = \mathbf{t}\right\}$$

is a parametrically definable subset of $\mathfrak{M}$.

Proof: Let $x^{(\mathbf{CB})}(\overline{XY})$ be an atomic wff whose variables are everywhere distinct; moreover, let $\overline{X}$ be a C-sequence of variables and $\overline{Y}$ be a B-sequence of variables. Then $\mathsf{Cut}(\eta f)$ is the $(\mathfrak{M}, \langle\eta f\rangle)$-graph of the statement form $x^{(\mathbf{CB})}(\overline{XY})/(\overline{X}x^{(\mathbf{CB})})\overline{Y}/$. Hence, $\mathsf{Cut}(\eta f)$ is an element of $\mathscr{P}\left(\underset{(\mathbf{B})}{\mathsf{X}}\mathfrak{M}\right)$. Q.E.D.

Having introduced all of the operations that will be needed, I can now present the long awaited definition of Henkin's "wider" class for **FK**,

(28) Definition: Let $\mathfrak{M}$ be a normal structure. Then $\mathfrak{M}$ is a *general structure* iff $\mathfrak{M}$ satisfies the following **ZFS**-conditions on the **M|N** class:

CI $\chi_{\mathfrak{M}}(E^{((\mathbf{B})\mathbf{B})}) \in \mathfrak{M}[(\mathbf{B})]$
CII $\forall f(f \in \mathfrak{M}[(\mathbf{B})] \to \chi_{\mathfrak{M}}(\neg f) \in \mathfrak{M}[(\mathbf{B})]$
CIII $\forall f \forall g((f \in \mathfrak{M}[(\mathbf{B})] \wedge g \in \mathfrak{M}[(\mathbf{C})]) \to \chi_{\mathfrak{M}}(f \oplus g) \in \mathfrak{M}[(\mathbf{BC})])$
CIV $\forall f(f \in \mathfrak{M}[(\mathbf{B})] \to \chi_{\mathfrak{M}}({}^{\pi}\mathsf{Pm}(f)) \in \mathfrak{M}[({}^{\pi}\mathbf{B})])$
CV $\forall f(f \in \mathfrak{M}[(\mathbf{BC})] \to \chi_{\mathfrak{M}}({}^{n}\mathsf{Pj}(f)) \in \mathfrak{M}[(\mathbf{C})])$
CVI $\forall f(f \in \mathfrak{M}[(\mathbf{B})] \to \chi_{\mathfrak{M}}({}^{j,k}\mathsf{Id}(f)) \in \mathfrak{M}[({}^{-k}\mathbf{B})])$
CVII $\forall \eta \forall f\left(\left(\eta \in \underset{(\mathbf{B})}{\mathsf{X}}\mathfrak{M} \wedge f \in \mathfrak{M}[(\mathbf{BC})]\right) \to \chi_{\mathfrak{M}}(\mathsf{Cut}(\eta f)) \in \mathfrak{M}[(\mathbf{C})]\right)$

provided that the superscripts on "${}^{\pi}\mathsf{Pm}$", "${}^{n}\mathsf{Pj}$", and "${}^{j,k}\mathsf{Id}$" satisfy all of the conditions in the antecedent of (24), (25), and (26), respectively.

Remark: Since $\mathfrak{M}$ is a normal structure, the characteristic functions listed above are unique elements of the indicated typed universes of $\mathfrak{M}$. Hence, if $\mathfrak{M}$ is a general structure, and only in this case, we may construe the operations as functions from the typed universes of $\mathfrak{M}$ into the typed universes of $\mathfrak{M}$. This allows us to avoid what is in effect the unnecessary repetition of "$\chi_{\mathfrak{M}}(\ldots)$". Thus, e.g., if $\mathfrak{M}$ is a general structure and $f \in \mathfrak{M}[(\mathbf{B})]$, then I will write "$\neg f \in \mathfrak{M}[(\mathbf{B})]$" instead of "$\chi_{\mathfrak{M}}(\neg f) \in \mathfrak{M}[(\mathbf{B})]$".

Having proposed a definition of the "wider" class for **FK**, I must now show that it is in fact the class of structures that we have been looking for. Specifically, I must prove that if a normal structure satisfies **ZFS**-conditions CI through CVII, then it does in fact verify every instance of **K**.

The proof of this result is not especially difficult, but it is rather long. Therefore, it seems to me that the most reasonable line of attack is to shorten the proof of the main result as much as possible by distributing some of the work to a series of preliminary lemmas. The first part of the series will increase the number of set-theoretical operations under which the typed universes of general structures must be closed.

(28) Free Intersection *Let* $\mathfrak{M}$ *be a general structure, let* $(\mathbf{B})$ *and* $(\mathbf{C})$ *be any non-primitive indices, and for every* $f \in \mathfrak{M}[(\mathbf{B})]$ *and every* $g \in \mathfrak{M}[(\mathbf{C})]$, *let*

$$f \odot g = \{\langle \eta\gamma \rangle : f(\eta) = \mathbf{t} \wedge g(\gamma) = \mathbf{t}\}$$

then $\mathfrak{M}$ *satisfies the* **ZFS**-*condition:*

CVIII $\forall f \forall g((f \in \mathfrak{M}[(\mathbf{B})] \wedge g \in \mathfrak{M}[(\mathbf{C})]) \rightarrow \chi_{\mathfrak{M}}(f \odot g) \in \mathfrak{M}[(\mathbf{BC})])$.

Proof: Since $\mathfrak{M}$ is a general structure, $\neg(\neg f \oplus \neg g) \in \mathfrak{M}[(\mathbf{BC})]$. All that needs to be proven is that $\neg(\neg f \oplus \neg g)$ is the characteristic function of $f \odot g \in \mathcal{P}\left(\underset{(\mathbf{BC})}{\mathsf{X}} \mathfrak{M}\right)$; that is, for all $\eta \in \underset{(\mathbf{B})}{\mathsf{X}} \mathfrak{M}$ and for all $\gamma \in \underset{(\mathbf{C})}{\mathsf{X}} \mathfrak{M}$,

(i) $\neg(\neg f \oplus \neg g)(\eta\gamma) = \mathbf{t}$ *iff* $\langle \eta\gamma \rangle \in f \odot g$.

Since the proof of this point is an elementary exercise in the field of **ZFS**, I will omit the details (Note: Since $\mathfrak{M}$ is a general structure and since $f \odot g$ has a characteristic function in $\mathfrak{M}$, I will write "$f \odot g \in \mathfrak{M}[(\mathbf{BC})]$" instead of "$\chi_{\mathfrak{M}}(f \odot g) \in \mathfrak{M}[(\mathbf{BC})]$". Henceforth, a notational agreement of this kind will be tacitly assumed.) Q.E.D.

(29) Boolean Operators *Let* $\mathfrak{M}$ *be a general structure, let* $(\mathbf{B})$ *be any non-primitive index, and for every* $f, g \in \mathfrak{M}[(\mathbf{B})]$, *let*

$$f \cup g = \{\eta : f(\eta) = \mathbf{t} \vee g(\eta) = \mathbf{t}\}$$
$$f \cap g = \{\eta : f(\eta) = \mathbf{t} \wedge g(\eta) = \mathbf{t}\}.$$

Then $\mathfrak{M}$ *satisfies the* **ZFS**-*conditions*

CIX $\forall f \forall g(f, g \in \mathfrak{M}[(\mathbf{B})] \rightarrow \chi_{\mathfrak{M}}(f \cup g) \in \mathfrak{M}[(\mathbf{B})])$
CX $\forall f \forall g(f, g \in \mathfrak{M}[(\mathbf{B})] \rightarrow \chi_{\mathfrak{M}}(f \cap g) \in \mathfrak{M}[(\mathbf{B})])$

Proof: If $f, g \in \mathfrak{M}[(\mathbf{B})]$, then $f \oplus g \in \mathfrak{M}[(\mathbf{BB})]$. Hence, the characteristic function for $f \cup g \in \mathcal{P}\left(\underset{(\mathbf{B})}{\mathsf{X}} \mathfrak{M}\right)$ may be obtained by repeatedly applying the Identification operation to $f \oplus g$, as in [10], p. 358; i.e., if $n = \mathrm{lh}(\mathbf{B})$,

(i) $\chi_{\mathfrak{M}}(f \cup g) = {}^{0,n}\mathrm{Id}(\ldots({}^{n-1,2n-1}\mathrm{Id}(f \oplus g))\ldots)$.

In an analogous way, the characteristic function for $f \cap g$ may be obtained by repeatedly applying the Identification operation to $f \odot g \in \mathfrak{M}[(\mathbf{BB})]$. Q.E.D.

(30) Boolean One and Zero *Let* $\mathfrak{M}$ *be a general structure, let* $(\mathbf{B})$ *be any non-primitive index, and let*

$$1^{(\mathbf{B})} = \left\{\eta : \eta \in \underset{(\mathbf{B})}{\mathsf{X}} \mathfrak{M}\right\}$$
$$0^{(\mathbf{B})} = \left\{\eta : \eta \in \underset{(\mathbf{B})}{\mathsf{X}} \mathfrak{M} \wedge \sim \eta \in \underset{(\mathbf{B})}{\mathsf{X}} \mathfrak{M}\right\}.$$

Then $\mathfrak{M}$ *satisfies the* **ZFS**-*conditions:*

CXI $\chi_{\mathfrak{M}}(1^{(\mathbf{B})}) \in \mathfrak{M}[(\mathbf{B})]$
CXII $\chi_{\mathfrak{M}}(0^{(\mathbf{B})}) \in \mathfrak{M}[(\mathbf{B})]$

Proof: By CI, $E^{((\mathbf{B})\mathbf{B})} \in \mathfrak{M}[((\mathbf{B})\mathbf{B})]$; thus, by CV, ${}^0\mathsf{P}^1_1(E^{((\mathbf{B})\mathbf{B})}) \in \mathfrak{M}[(\mathbf{B})]$. Hence, given CII and CIX,

(i) $\chi_{\mathfrak{M}}(0^{(\mathbf{B})}) = {}^0\mathsf{P}^1_1(E^{((\mathbf{B})\mathbf{B})}) \cup \neg{}^0\mathsf{P}^1_1(E^{((\mathbf{B})\mathbf{B})})$
(ii) $\chi_{\mathfrak{M}}(0^{(\mathbf{B})}) = {}^0\mathsf{P}^1_1(E^{((\mathbf{B})\mathbf{B})}) \cap \neg{}^0\mathsf{P}^1_1(E^{((\mathbf{B})\mathbf{B})})$,

which was to be proven. Q.E.D.

These additional operations are only a small sample of the set-theoretical operations under which the typed universes of general structures are closed, see section **7** below for some others. The principal reason for introducing them at this point is to facilitate the proofs of the final part of the series of preliminary lemmas.

(31) Superfluous Variables *Let* $\mathfrak{M}$ *be a general structure and let* p *be any wff whose free variables are identical to the variables in the* B-*sequence* $\overline{X}$ *of everywhere distinct variables. Then if the* $(\mathfrak{M})$-*graph of the statement form* $p/\overline{X}/$ *is a characterized subset of* $\mathfrak{M}$, *so is the* $(\mathfrak{M})$-*graph of the statement form* $p/\overline{XY}/$, *where* $\overline{Y}$ *is any finite sequence of everywhere distinct variables, all of which are foreign to* p.

Proof: Assume that the $(\mathfrak{M})$-graph of $p/\overline{X}/$ is a characterized subset of $\mathfrak{M}$. Then we can prove the desired result by induction on the length of the sequence $\overline{Y}$.

Case 0: If the length of $\overline{Y}$ is 0, then $p/\overline{XY}/$ is identical to $p/\overline{X}/$ and there is nothing to prove.

Inductive Hypothesis: Assume that the Lemma has been confirmed for all sequences of variables whose length is less than or equal to n.

Case $n+1$: Let $\overline{Y} = \overline{Z}y^a$ be a sequence of length $n+1$; moreover, let $\overline{Z}$ be a C-sequence of variables. Then, by the Inductive Hypothesis, the $(\mathfrak{M})$-graph of $p/\overline{XZ}/$, say A, has a characteristic function $h^* \in M[(\mathbf{BC})]$. Now assume that $B \in \mathcal{P}\left(\bigtimes_{(\mathbf{BCa})} \mathfrak{M}\right)$ is the $(\mathfrak{M})$-graph of $p/\overline{XZ}y^a/ = p/\overline{XY}/$. Then since y^a is foreign to p, it must be the case that

(i) for all $\langle \eta f\rangle \in \bigtimes_{(\mathbf{BCa})} \mathfrak{M}$, $\langle \eta f\rangle \in B$ iff $\langle\eta\rangle \in A$, that is, $h^*(\eta) = \mathbf{t}$;

and thus,

(ii) for all $\langle \eta f\rangle \in \bigtimes_{(\mathbf{BCa})} \mathfrak{M}$, $\langle \eta f\rangle \in B$ iff $h^*(\eta) = \mathbf{t}$ and $1^{(\mathbf{a})}(f) = \mathbf{t}$.

Hence, $\chi_{\mathfrak{M}}(B) = h^* \odot 1^{(\mathbf{a})}$, which completes the inductive argument. Q.E.D.

(32) Permuting Variables *Let* $\mathfrak{M}$ *be a general structure and let* $p/\overline{X}/$ *be any statement form. Then if the* $(\mathfrak{M})$-*graph of* $p/\overline{X}/$ *is a characterized subset of* $\mathfrak{M}$, *so is the* $(\mathfrak{M})$-*graph of* $p/{}^{\pi}\overline{X}/$, *where* π *is a permutation on the numbers* $0, \ldots, n = \mathrm{lh}(\overline{X}) - 1$ *and* "${}^{\pi}\overline{X}$" *is defined as in* (24).

Proof: Assume that $\overline{X}$ is a B-sequence of everywhere distinct variables and that the $(\mathfrak{M})$-graph of $p/\overline{X}/$ does have a characteristic function $h^* \in \mathfrak{M}[(\mathbf{B})]$. Then the characteristic function of the $(\mathfrak{M})$-graph of $p/{}^{\pi}\overline{X}/$ is ${}^{\pi}\mathsf{Pm}\,(h^*) \in \mathfrak{M}[({}^{\pi}\mathbf{B})]$. Q.E.D.

(33) Identifying Variables *Let $\mathfrak{M}$ be a general structure and let p be a wff whose free variables are identical to the variables in the B-sequence $\overline{X}$ of everywhere distinct variables; moreover, assume that every variable which has at least one free occurrence in p has at most one free occurrence in p. Then if the $(\mathfrak{M})$-graph of the statement form $p/\overline{X}/$ is a characterized subset of $\mathfrak{M}$, then so is the $(\mathfrak{M})$-graph of $\mathsf{S}^{\overline{Y}(k)}_{\overline{Y}(j)}\, p/{}^{-k}\overline{Y}/$, where $0 \leq j < k < \mathsf{lh}(\overline{Y})$, $\overline{Y}(j)$ and $\overline{Y}(k)$ are variables of the same type, and "${}^{-k}\overline{Y}$" is defined as in* (26).

Proof: Assume that the $(\mathfrak{M})$-graph of $p/\overline{X}/$ has a characteristic function $h^* \in \mathfrak{M}[(\mathbf{B})]$. Then the characteristic function of the $(\mathfrak{M})$-graph of $\mathsf{S}^{\overline{Y}(k)}_{\overline{Y}(j)}\, p/{}^{-k}\overline{Y}/$ is ${}^{j,k}\mathsf{Id}\,(h^*) \in \mathfrak{M}[({}^{-k}\mathbf{B})]$. Q.E.D.

(34) Parametrizing Variables *Let $\mathfrak{M}$ be a general structure and let $p/\overline{XY}/$ be a statement form such that $\overline{X}$ is a B-sequence of variables and $\overline{Y}$ is a C-sequence of variables. Then if the $(\mathfrak{M})$-graph of $p/\overline{XY}/$ is a characterized subset of $\mathfrak{M}$, then for all $\eta \in \bigtimes_{(\mathbf{B})} \mathfrak{M}$, the $(\mathfrak{M}, \eta)$-graph of $p/(\overline{X})\overline{Y}/$ is a characterized subset of $\mathfrak{M}$.*

Proof: Assume that the $(\mathfrak{M})$-graph of $p/\overline{XY}/$ has a characteristic function $h^* \in \mathfrak{M}[(\mathbf{BC})]$ and that $\eta \in \bigtimes_{(\mathbf{B})} \mathfrak{M}$. Then the characteristic function of the $(\mathfrak{M}, \eta)$-graph of $p/(\overline{X})\overline{Y}/$ is $\mathsf{Cut}(\eta h^*) \in \mathfrak{M}[(\mathbf{C})]$. Q.E.D.

This concludes the series of preliminary lemmas. I will now state and prove the main theorem.

(35) The Comprehension Condition, final version *Let $\mathfrak{M}$ be a normal structure. Then $\mathfrak{M}$ verifies every instance of* **K** *iff $\mathfrak{M}$ is a general structure.*

Proof: In one direction, assume that $\mathfrak{M}$ verifies every instance of **K**. Then by the first version of the Comprehension Condition Theorem, every parametrically definable, and, hence, every definable, subset of $\mathfrak{M}$ has a characteristic function in $\mathfrak{M}$. Hence, $\mathfrak{M}$ will obviously satisfy **ZFS**-conditions CI through CVIII which means that $\mathfrak{M}$ is a general structure. In the other direction, assume that $\mathfrak{M}$ is a general structure. According to the first version of the Comprehension Condition Theorem, $\mathfrak{M}$ will verify every instance of **K** iff

(i) every parametrically definable subset of $\mathfrak{M}$ is a characterized subset of $\mathfrak{M}$,

or, equivalently,

(ii) for every statement form $p/(\overline{X})\overline{Y}/$, where $\overline{X}$ is a B-sequence of variables and $\overline{Y}$ is a C-sequence of variables, and for every $\eta \in \underset{(B)}{\mathsf{X}}\mathfrak{M}$, the $(\mathfrak{M}, \eta)$-graph of $p/(\overline{X})\overline{Y}/$ is a characterized subset of $\mathfrak{M}$.

Given preliminary Lemmas (31), (32), (33), and (34), it is possible to greatly reduce the number of cases to be considered; that is, point (ii) can be shown to be equivalent to

(iii) for every statement form $p/\overline{X}/$ such that:

(1) the free variables of p are identical to the variables in the B-sequence $\overline{X}$;
(2) every variable having at least one free occurrence in p has at most one free occurrence in p;

and

(3) the order of the variables in the sequence $\overline{X}$ is identical to the order of their first occurrence in p, the $(\mathfrak{M})$-graph of $p/\overline{X}/$ is a characterized subset of $\mathfrak{M}$.

To demonstrate the equivalence of (ii) and (iii), let $p/(\overline{X})\overline{Y}/$ be any statement form. Then, by Lemma 34, if the $(\mathfrak{M})$-graph of $p/\overline{XY}/$ is a characterized subset of $\mathfrak{M}$, then so is the $(\mathfrak{M}, \eta)$-graph of $p/(\overline{X})\overline{Y}/$, for all $\eta \in \underset{(B)}{\mathsf{X}}\mathfrak{M}$.

Suppose that p contains more than one occurrence of the same free variable. Then there is a wff p^* such that p^* satisfies condition (iii.2) and p is a substitution instance of p^*. Let $\overline{Z}$ be the set of all of the free variables of p^* which are not free variables of p. Then, by Lemma (33), if the $(\mathfrak{M})$-graph of $p^*/\overline{XYZ}/$ is a characterized subset of $\mathfrak{M}$, then so is the $(\mathfrak{M})$-graph of $p/\overline{XY}/$. Let π be a permutation on the sequence of variables $\overline{XYZ}$ such that every variable which has a free occurrence in p^* preceeds every variable which does not have a free occurrence in p^*; moreover, let π order the variables of the first kind, say, the variables in the sequence $\overline{U}$, in the order of their first occurrence in p^*, and let π order the variables of the second kind, say, the variables in the sequence $\overline{V}$, in the order of their occurrence in the sequence $\overline{XYZ}$. Then, by Lemma (32), if the $(\mathfrak{M})$-graph of $p^*/\overline{UV}/$ is a characterized subset of $\mathfrak{M}$, then so is the $(\mathfrak{M})$-graph of $p^*/\overline{XYZ}/$. Finally, since no free variables of p^* are in the sequence $\overline{V}$, then by Lemma (31), if the $(\mathfrak{M})$-graph of $p^*/\overline{U}/$ is a characterized subset of $\mathfrak{M}$, then so is the $(\mathfrak{M})$-graph of $p^*/\overline{UV}/$. Hence, $p^*/\overline{U}/$ is a statement form which satisfies all of the conditions of (iii) and which has the following property: if the $(\mathfrak{M})$-graph of $p^*/\overline{U}/$ is a characterized subset of $\mathfrak{M}$, then for every $\eta \in \underset{(B)}{\mathsf{X}}M$, the $(\mathfrak{M}, \eta)$-graph of $p/(\overline{X})\overline{Y}/$ is a characterized subset of $\mathfrak{M}$. Since the proof that (ii) implies (iii) is trivial, this concludes the demonstration that (ii) is equivalent to (iii). This in turn implies that $\mathfrak{M}$ will verify every instance of **K** iff

(iv) if $p/\overline{X}/$ is a statement form satisfying all of the conditions of (iii), then the $(\mathfrak{M})$-graph of $p/\overline{X}/$ is a characterized subset of $\mathfrak{M}$.

It remains to be proven that $\mathfrak{M}$ does satisfy (iv), and this can now be done by induction on the length of the wff p.

Case 0: Let p be the atomic wff $x^{(\mathbf{B})}(\overline{Y})$ and let $\overline{x}^{(\mathbf{B})}(\overline{Y})/x^{(\mathbf{B})}\,\overline{Y}/$ satisfy the antecedent of (iv), then the $(\mathfrak{M})$-graph of $x^{(\mathbf{B})}(\overline{Y})/x^{(\mathbf{B})}\,\overline{Y}/$ is $E^{((\mathbf{B})\mathbf{B})}$, which, by CI, is a characterized subset of $\mathfrak{M}$.

Inductive Hypothesis Assume that (iv) is satisfied by all statement forms $p/\overline{X}/$ such that $p/\overline{X}/$ satisfies the antecedent of (iv) and the length of p is less than or equal to n.

Case $n + 1$: Let $p/\overline{X}/$ be a statement form which satisfies the antecedent of (iv) and let p be a wff of length $n + 1$. Moreover, let $\overline{X}$ be a B-sequence of variables. Then there are precisely three subcases to be considered, namely: (i) $p/\overline{X}/$ is $\sim q/\overline{X}/$, (ii) $p/\overline{X}/$ is $(q \rightarrow r)/\overline{X}/$, and (iii) $p/\overline{X}/$ is $\forall y^b q/\overline{X}/$.

Subcase (i): If $p/\overline{X}/$ is $\sim q/\overline{X}/$, then the length of q is less than $n + 1$. Since $q/\overline{X}/$ satisfies the antecedent of (iv), the Inductive Hypothesis implies that the $(\mathfrak{M})$-graph of $q/\overline{X}/$ has a characteristic function $h^* \in \mathfrak{M}[(\mathbf{B})]$. Hence, $\neg h^* \in \mathfrak{M}[(\mathbf{B})]$ is the characteristic function of the $(\mathfrak{M})$-graph of $p/\overline{X}/$.

Subcase (ii): If $p/\overline{X}/$ is $(q \rightarrow r)/\overline{X}/$, then the length of both q and r is less than $n + 1$. Thus, the Inductive Hypothesis, and, if necessary, Lemmas (31) and (32), enable us to infer that the $(\mathfrak{M})$-graph of $q/\overline{X}/$ has a characteristic function $f^* \in \mathfrak{M}[(\mathbf{B})]$ and that the $(\mathfrak{M})$-graph of $r/\overline{X}/$ has a characteristic function $g^* \in \mathfrak{M}[(\mathbf{B})]$. Hence, $\neg f^* \cup g^* \in \mathfrak{M}[(\mathbf{B})]$ is the characteristic function of the $(\mathfrak{M})$-graph of $p/\overline{X}/$.

Subcase (iii): If $p/\overline{X}/$ is $\forall y^b q/\overline{X}/$, then the length of q is less than $n + 1$. Thus, the Inductive Hypothesis, and, if necessary, Lemmas (31) and (32), enable us to infer that the $(\mathfrak{M})$-graph of $q/y^b X/$ has a characteristic function $h^* \in \mathfrak{M}[(\mathbf{bB})]$. Hence, $\neg^0 \mathsf{P}_1^{\mathsf{i}}(\neg h^*) \in \mathfrak{M}[(\mathbf{B})]$ is the characteristic function of $p/\overline{X}/$. Since the desired conclusion has been obtained in each subcase, Case $n+1$—and with it the inductive proof of (iv)—is now complete.

Therefore, if $\mathfrak{M}$ is a general structure, then $\mathfrak{M}$ verifies every instance of **K**, which was proven. Q.E.D.

Having completed the proof of the main theorem, I have now satisfied in every respect the specifications set forth in Step Ib, *cf.* section **4** above. In order to prove that the class of general structures as defined in (28) is in fact Henkin's "wider" class for **FK**, i.e., in order to complete Step II and Step III of Henkin's proposal, it will be both necessary and sufficient to solve the Characterization Problem for **FK** with respect to the following notion of validity: a wff p is *generally valid* iff p is verified by every general structure; a wff p is *generally invalid* iff p is falsified by every general structure.

If this modified notion of validity is accepted, then we can prove

(36) The Characterization Theorem for General Validity *A wff p is generally valid iff p is derivable in* **FK**.

Proof: The proof of (36) is very nearly an exact replica of the proof of Theorem (14). I will omit the details. (Note: The analogue of Henkin's Lemma, *cf.* (14.i), used in the proof of this Theorem reads as follows

(i) *if T is a consistent extension of the higher-order calculus* **FK**, *then there is a general structure $\mathfrak{M}$ such that $\mathfrak{M}$ models T.*

As before, the general structure $\mathfrak{M}$ constructed in the course of the proof is obtained from the standard structure S^{ω} and all of its typed universes are either uniformly finite or uniformly denumerable.) Q.E.D.

7 *Some concluding observations* Suppose that $\mathfrak{M}$ is a general structure and that $\mathbf{a}$ is a non-primitive index, then it is an immediate consequence of CII and CIX through CXII that $\mathfrak{A}(\mathfrak{M}, \mathbf{a}) = \langle \mathfrak{M}[\mathbf{a}], \cup, \cap, \neg, 1^{\mathbf{a}}, 0^{\mathbf{a}} \rangle$ is a *Boolean Algebra*, henceforth, "**BA**".

(37) Observation *If $\mathfrak{M}$ is a general structure and* $\mathbf{a}$ *is a non-primitive index, and if $\mathfrak{A}(\mathfrak{M}, \mathbf{a})$ is an atomic* **BA**, i.e., *for every $\eta \in \underset{\mathbf{a}}{\mathsf{X}}\mathfrak{M}$, let*

$$\mathbf{1}\eta = \left\{\gamma: \gamma \in \underset{\mathbf{a}}{\mathsf{X}}\mathfrak{M} \wedge \eta = \gamma\right\},$$

then $\mathfrak{M}$ satisfies the **ZFS**-*condition:*

CXIII $\forall\eta \left(\eta \in \underset{\mathbf{a}}{\mathsf{X}}\mathfrak{M} \rightarrow \chi_{\mathfrak{M}}(\mathbf{1}\eta) \in \mathfrak{M}[\mathbf{a}]\right)$

Proof: Assume that $\mathbf{a} = (\mathbf{b}_0 \ldots \mathbf{b}_n)$. Then for every $\mathbf{b}_j$, $0 \leqslant j \leqslant n$, let q_j be the wff

(i) $\forall z^{(\mathbf{b}_j)}(z^{(\mathbf{b}_j)}(x^{b_j}) \leftrightarrow z^{(\mathbf{b}_j)}(y^{b_j}))$

Since $\mathfrak{M}$ satisfies the Normality Condition, if is any assignment to $\mathfrak{M}$ such that $(\mathfrak{M}, \varphi) = q_j$, then, necessarily, $\varphi(x^{b_j}) = \varphi(y^{b_j})$. Hence, if q is the conjunction $q_0 \wedge \ldots \wedge q_n$, then $\mathbf{1}\eta$ is the $(\mathfrak{M}, \eta)$-graph of the statement form

(ii) $q/(x^{b_0}, \ldots, x^{b_n})y^{b_0}, \ldots, y^{b_n}/$

that is, if $\varphi \upharpoonright x^{b_0}, \ldots, x^{b_n} = \eta$, then $(\mathfrak{M}, \varphi) \models q$ iff $\varphi \upharpoonright y^{b_0}, \ldots, y^{b_n} = \gamma$ and $\eta = \gamma$. Therefore, $\mathbf{1}\eta$ is a parametrically definable subset of $\mathfrak{M}$, and, by Theorem (35), $\chi_{\mathfrak{M}}(\mathbf{1}\eta) \in \mathfrak{M}[\mathbf{a}]$, as required by CXIII. Q.E.D.

The fact that $\mathfrak{A}(\mathfrak{M}, \mathbf{a})$ is an atomic **BA** for every non-primitive index $\mathbf{a}$ has a very interesting consequence, namely:

(38) Corollary *Every finitary general structure is identical to a finitary standard structure up to isomorphism.*

Proof: If $\mathfrak{M}$ is a finitary general structure, i.e., if all of $\mathfrak{M}$'s typed universes are finite sets, then every element of $\mathcal{P}\left(\underset{\mathbf{a}}{\mathsf{X}}\mathfrak{M}\right)$ is a finite set. Since $\mathfrak{M}$ is closed under the formation of "unit sets" and under all applications of the union operation, then every element of $\mathcal{P}\left(\underset{\mathbf{a}}{\mathsf{X}}\mathfrak{M}\right)$ has a characteristic function in $\mathfrak{M}[\mathbf{a}]$. Therefore, it is an easy exercise to confirm that $\mathfrak{M}$ is a "standard" structure in the sense of footnote 10. Q.E.D.

(39) Observation *If* $\mathfrak{M}$ *is a general structure and* **a** *is a non-primitive index, then* $\mathfrak{A}(\mathfrak{M}, \mathbf{a})$ *is a pseudo-complete* **BA**, i.e., *for every* $h \in \mathfrak{M}[(\mathbf{a})]$, *let*

$$\sqcup h = \left\{\eta \colon \eta \in \mathop{\mathsf{X}}_{\mathbf{a}} \mathfrak{M} \wedge \exists f(h(f) = \mathbf{t} \wedge f(\eta) = \mathbf{t})\right\}$$

then $\mathfrak{M}$ *satisfies the* **ZFS**-*Condition*

CXIV $\forall h(h \in \mathfrak{M}[(\mathbf{a})] \rightarrow \chi_{\mathfrak{M}}(\sqcup h) \in \mathfrak{M}[\mathbf{a}])$

Proof: The set $\sqcup h \in \mathcal{P}\left(\mathop{\mathsf{X}}_{\mathbf{a}} \mathfrak{M}\right)$ is the $(\mathfrak{M}, h)$-graph of the statement form

(i) $\exists x^a(z^{(\mathbf{a})}(x^a) \wedge x^a(\overline{Y}))/(z^{(\mathbf{a})})\overline{Y}/$,

which means that $\sqcup h$ is parametrically definable. Therefore, by Theorem (35), $\chi_{\mathfrak{M}}(\sqcup h) \in \mathfrak{M}[\mathbf{a}]$, as required by CXIV. Q.E.D.

Let $A \in \mathcal{P}(\mathfrak{M}[\mathbf{a}])$ and let

(40) $\bigcup A = \left\{\eta \colon \eta \in \mathop{\mathsf{X}}_{\mathbf{a}} \mathfrak{M} \wedge \exists f(f \in A \wedge f(\eta) = \mathbf{t})\right\}$

then $\chi_{\mathfrak{M}}\left(\bigcup A\right)$, if it exists, is the *least upper bound*, henceforth, "l.u.b.", of A. Observation (39) can now be reformulated as follows,

(41) *Let* $A \in \mathcal{P}(\mathfrak{M}[\mathbf{a}])$, *then the* l.u.b. *of* A *exists only if there is an* $h \in \mathfrak{M}[(\mathbf{a})]$ *such that* $h = \chi_{\mathfrak{M}}(A)$, *in which case* $\sqcup h = \chi_{\mathfrak{M}}\left(\bigcup A\right)$, *that is, every characterized subset of* $\mathfrak{M}$ *has a* l.u.b.

The stronger claim, namely, that every homogeneous subset of $\mathfrak{M}$ has a l.u.b. is equivalent to the claim that $\mathfrak{A}(\mathfrak{M}, \mathbf{a})$ is a complete **BA** for every non-primitive index **a**, *cf.* Sikorski [13], p. 65. The stronger claim is true iff $\mathfrak{M}$ is identical to a standard structure up to isomorphism.

NOTES

1. This system of type indices is based on a single primitive index. Church's system, described in [5], which was used both by Henkin in [7] and Andrews in [1] and [2], is based on two primitive indices, namely: "i" the type of *individuals* and "o" the type of *sentences*, and it makes use of the "monadic predicate only" device invented by Shoenfinkel. Despite the elegance of Church's system, I have decided to use the slightly more popular one-based system with indices for n-adic predicates because, for my purposes, this system has a more "natural" set-theoretical interpretation.

2. The last clause of this definition is intended to insure that the substitution operation is defined for all wffs and for all choices of t^a, even if there is no free occurrence of x^a in the wff or t^a is not free for x^a.

3. It should be emphasized that statement forms are not simply open wffs. Specifically, infinitely many different statement forms can be generated from one and the same open wff. The criteria for identity of statement forms is as follows: $p/(\overline{X})\overline{Y}/$ is identical to $q/(\overline{U})\overline{V}/$ iff p is identical to q, $\overline{X}$ is identical to $\overline{U}$, i.e., $\mathrm{lh}(\overline{X}) = \mathrm{lh}(\overline{Y})$ and for all j, $0 \leqslant j \leqslant \mathrm{lh}(\overline{X})$, $\overline{X}(j) = \overline{U}(j)$, and $\overline{Y}$ is identical to $\overline{V}$. Thus, the notion "statement form" as used in this paper is *not*

equivalent to the notion "propositional form" as used by Church in [6]; in his sense, any open wff is a propositional form. Finally, if $p/(\overline{X})\overline{Y}/$ is a statement form, then the parenthesized variables will be said to be *parameters*, for reasons which will be obvious if the reader consults the axiom schema of Comprehension in section **3**.

4. The structures defined in (8) are given many different descriptive titles in the literature, viz. "intended (primary) interpretations," "full models," "principle realizations," etc. It is often said, *cf.* Church [6], p. 315, that these studies are the proper interpretations of the type theoretic language. However, it seems to me that the "propriety" of standard structures has been greatly exaggerated. In fact, I believe that it would be very difficult to establish that, e.g., the universe of "properties" of individuals, in a strict sense, forms complete atomic Boolean Algebra! But whether this claim is true or not, it is certainly not self-evident.

5. In the field of **FK**, **Ex** is strong enough to guarantee the truth of *Leibniz's Principles*. That is, suppose that p is any wff, u^a is a free variable of p, x^a and y^a are foreign to p, and $\forall^*$ is a block of universal quantifiers that bind all of the free variables of p other than u^a, then the schemas

   L I. $\forall x^a \forall y^a \left(\forall \overline{X}(x^a(\overline{X}) \leftrightarrow y^a(\overline{X})) \leftrightarrow \forall^*\left(\mathsf{S}^{u^a}_{x^a} p \leftrightarrow \mathsf{S}^{u^a}_{y^a} p\right)\right)$

   L II. $\forall x^a \forall y^a \left(\forall z^{(\mathbf{a})}(z^{(\mathbf{a})}(x^a) \leftrightarrow z^{(\mathbf{a})}(y^a)) \leftrightarrow \forall^*\left(\mathsf{S}^{u^a}_{x^a} p \leftrightarrow \mathsf{S}^{u^a}_{y^a} p\right)\right)$

   are both theorem schemas of **FK**.

6. Henkin's instructions are given in [7], footnote 5. He says, in effect, that each method of compounding a wff, e.g., adding a negation sign, applying an existential quantifier, etc., has an operation associated with it on the typed universes of the structure, e.g., complement, projection, etc., and that the typed universes must be closed under all of these operations. In a strict sense, this observation is at best only partially correct. What we have to guarantee is that the typed universes are closed under the operations associated with all ways of constructing *statement forms*, *cf.* footnote 3 above are definition (28).

7. The universe of individuals of every standard structure is an ordinal in the sense of von Neumann, and, hence, a constructible **ZFS**-set. Let us agree that the truth value "**f**" designates the von Neumann ordinal 0, i.e., the null set, and that "**t**" designates the von Neumann ordinal 1, i.e., the set $\{\phi\}$. Then since the universe of constructible **ZFS**-sets is closed under the formation of finite Cartesian products and under set-exponentiation, every typed universe of every standard structure is a constructible **ZFS**-set. Moreover, since the finite ordinals can be *effectively* mapped onto the type indices, the Axiom (Schema) of Replacement in **ZFS**, *cf.* [10], p. 54, guarantees that every standard structure is a constructible **ZFS**-set. It might be objected at this point that this way of defining standard structures omits an immense variety of constructions which are fully entitled to this honor, viz., let $M^2[\mathbf{i}] = \{2n{:}0 \leqslant n < \omega\}$, let $M^2[(\mathbf{B})]$ be the set of *all* functions from $\underset{(\mathbf{B})}{\mathsf{X}} M^2$ into $\{\mathbf{t},\mathbf{f}\}$, i.e., $\{\{\phi\},\phi\}$, etc. I am certainly willing to conceed this point, but it should be noted that every structure which is entitled to be called a standard structure, like $\mathbf{M}^2$, is *essentially identical* to a standard structure, that is, is identical to a standard structure *up to isomorphism*. Specifically, there is a type-preserving bijection θ from $\bigcup\{M^2[\mathbf{a}]\text{: }\mathbf{a}\text{ is an index}\}$ *onto* $\bigcup\{S^\omega[\mathbf{a}]\text{: }\mathbf{a}\text{ is an index}\}$ such that for all $f \in M^2[(\mathbf{B})]$ and all $\langle g_0, \ldots, g_n\rangle \in \underset{(\mathbf{B})}{\mathsf{X}} M^2$, $f(g_0 \ldots g_n) = \theta(f)(\theta(g_0) \ldots \theta(g_n))$. Any function θ which satisfies these conditions is an isomorphism from $\mathfrak{M}^2$ to S^ω. I leave it to the reader to establish that such a function exists. This result can obviously be generalized. Let x be any unempty set, then define the "standard" structures S^x as follows:

   (i) $S^x[\mathbf{i}] = x$

   (ii) $S^x[(\mathbf{B})] = \left\{f\text{: } \mathrm{dm}(f) = \underset{(\mathbf{B})}{\mathsf{X}} S^x \wedge \mathrm{rg}(f) \subseteq \{\mathbf{t},\mathbf{f}\}\right\}$

then for any sets y and z, if y is equinumerous with z, i.e., if there is a bijection θ from y *onto* z, then θ can be "extended" to an isomorphism θ^* from S^y to S^z. Since isomorphic structures are identical to each other *in all semantically important respects* and since, given the Axiom of Choice, every set is equinumerous to a finite or initial ordinal, it seems to me that the omission of these "standard" structures is an altogether reasonable move.

8. The notion "**ZFS**-condition," which I am here using in an intuitive way, will be accurately explained at the beginning of section **6**.

9. **MN** structures do have rather "odd" properties, e.g., if $\mathfrak{M}$ is a non-standard **MN** then there must be at least one index (**B**) such that

   (i) $\forall f(f \in \mathfrak{M}[(\mathbf{B})] \leftrightarrow \underset{(\mathbf{B})}{\mathsf{X}}\mathfrak{M} \subset \mathrm{dm}(f))$

   The reader should be able to convince himself that (i) will not give rise to any special difficulties, i.e., since every element of $\mathfrak{M}[(\mathbf{B})]$ maps every element of $\underset{(\mathbf{B})}{\mathsf{X}}\mathfrak{M}$ onto either **t** or **f**, the fact that $\underset{(\mathbf{B})}{\mathsf{X}}\mathfrak{M}$ is a proper subset of the domain of the elements $\mathfrak{M}[(\mathbf{B})]$ is really of no essential importance. Even so, the reader might think that a more "reasonable" definition of the non-standard structures would look like this, (ii) *Frames* Let μ be an ordinal satisfying the antecedent of (8). Then

   $$\begin{cases} 1.\ \mathfrak{M}[\mathbf{i}] = \mu \\ 2.\ \phi \neq \mathfrak{M}[(\mathbf{B})] \subseteq \left\{f\colon \mathrm{dm}(f) = \underset{(\mathbf{B})}{\mathsf{X}}\mathfrak{M} \wedge \mathrm{rg}(f) \subseteq \{\mathbf{t},\mathbf{f}\}\right\} \end{cases}$$

   However, it is rather easy to show that the class of all frames is not the "widest possible" class of structures. Specifically, the reader should be able to confirm that every frame verifies every instance of **Ex**, whereas it is very easy to construct **MN** structures which falsify every instance of **Ex**. In any case, since **Ex** is obviously not a theorem schema of **F**, the "frame" construal of the "widest possible" class simply won't do.

10. This **ZFS**-condition is a little stronger than it has to be to get the job done. I could have introduced a much weaker condition, i.e.,

    (i) $\forall f \forall g(f,g \in \mathfrak{M}[(\mathbf{B})] \rightarrow \forall \eta(\eta \in \underset{(\mathbf{B})}{\mathsf{X}}\mathfrak{M} \rightarrow f(\eta) = g(\eta)) \rightarrow \forall h(h \in \mathfrak{M}[((\mathbf{B}))] \rightarrow h(f) = h(g)))$

    If an **MN** structure satisfies (i), then it will verify every instance of **Ex**. However, it seems to me that in this case the comparative weakness of (i) is not especially advantageous. First, it can be proven that if $\mathfrak{M}$ is any **MN** structure satisfying (i), then there is an **MN** structure $\mathfrak{N}$ satisfying (16) such that $\mathfrak{M}$ is *elementarily equivalent* to $\mathfrak{N}$, i.e., for every wff p, $\mathfrak{M}$ verifies p iff $\mathfrak{N}$ verifies p. Secondly, it can be proven that the class of **MN** structures satisfying (16) is identical to the class of **MN** structures that are isomorphic to frames, *cf.* footnote 9 above. Both of these facts very clearly show the advantages to be gained by accepting the stronger **ZFS**-condition.

11. This condition can also be weakened, i.e.,

    (i) $\forall f \forall g(f,g \in \mathfrak{M}[\mathbf{a}] \rightarrow (\forall h(h \in \mathfrak{M}[(\mathbf{a})] \rightarrow h(j) = h(g)) \rightarrow \forall \eta(\eta \in \underset{(\mathbf{B})}{\mathsf{X}}\mathfrak{M} \rightarrow f(\eta) = g(\eta))))$

    However, the superiority of (18) to (i), can be established by much the same line of argument that was used in footnote 10 above. It should be noted that **ZFS**-conditions (16) and (18) are independent of one another, i.e., there are **MN** structures which satisfy (16) but not (18), and conversely. Since these constructions are not especially difficult, I omit the details.

12. If either of the "weakened" conditions, *cf.* footnote 10, point (i); footnote 11, point (i), had been accepted, then it would not have been possible to guarantee the uniqueness of the characteristic functions. This is yet another argument in favor of (16) and (18).

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The Catholic University of America
Washington, District of Columbia