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SENTENTIAL NOTATIONS: UNIQUE DECOMPOSITION

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Two notations for sentential logic are compared: that of Chapter II of *Logic: Techniques of Formal Reasoning* (New York, 1964) by Donald Kalish and Richard Montague and a parenthesis-free variant presented in Chapter VIII*. These notations, *SC* and *SC** respectively, are set out in section **1**, said to be unambigous in section **2**, and in sections **3** and **4** shown to be unambiguous; lastly, and briefly, in section **5** comments are made on their relative merits.

1 *The two notations* Symbols: sentence letters P through Z with or without subscripts, sentential connectives $\sim$, $\rightarrow$, $\vee$, $\wedge$, and $\leftrightarrow$, and in the case of *SC* left- and right-parentheses.

The set of sentences of *SC* is the smallest set such that: (1) Sentence letters are members of *SC*. (2) If ϕ and ψ are members of *SC*, then so are, $\sim\phi$, $(\phi \rightarrow \psi)$, $(\phi \vee \psi)$, $(\phi \wedge \psi)$, and, $(\phi \leftrightarrow \psi)$.

The set of sentences of *SC** is the smallest set such that: (1) Sentence letters are members of *SC**. (2) If ϕ and ψ are members of *SC**, then so are $\sim\phi$, $\rightarrow\phi\psi$, $\vee\phi\psi$, $\wedge\phi\psi$, and, $\leftrightarrow\phi\psi$.

The *SC*-counterpart of an *SC**-sentence is reached by successive applications of the rule:

> Where ϕ is an *SC**-sentence or sequence of *SC*-symbols and the left-most occurrence in ϕ of a binary connective is an occurrence of δ, replace the left-most occurrence in ϕ of an *SC**-sentence of the form,
>
> $$\delta\psi\chi,$$

*The lemma for Section **4** is entailed by Theorem 1 of Chapter IV, *The Elements of Mathematical Logic*, Paul Rosenbloom (New York, 1950), p. 154; see also "Bibliographical and Other Remarks," p. 205. Theorems similar to that of Section **3** are proved in *Introduction to Mathematical Logic*, Alonzo Church (Princeton, 1956), pp. 92 and 122; and in section 2 of "Proof by Cases in Formal Logic," S. C. Kleene, *Annals of Mathematics*, vol. 35 (1934), wherein can be found, see 2I, p. 531 an inductive proof for the lemma of our Section **3**. I owe these references to Alisdair Urquhart. None (I confess) were known to me before completion of this paper.

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(ψ and χ *SC**-sentences) by the sequence,

$$(\psi\delta\chi).$$

The *SC**-counterpart of an *SC*-sentence is reached by successive applications of the rule:

> Where ϕ is an *SC*-sentence or sequence of *SC*-symbols, replace an occurrence in ϕ of a longest *SC*-sentence in ϕ of the form,
>
> $$(\psi\delta\chi),$$
>
> (δ a binary connective, ψ and χ *SC*-sentences) by the sequence,
>
> $$\delta\psi\chi.$$

That *SC*- and *SC**-sentences have *unique* *SC**- and *SC*-counterparts is a corrollary of the unique decomposition property of *SC*- and *SC**-sentences described and proved in following sections.

2 *Unique decomposition: statement* The sentence 'Helen will attend if she can and she has been invited' is ambiguous: it could be a conditional (Add emphasis to, or place a comma before, 'if'.) or a conjunction (Add emphasis to, or place a comma before, 'and'). No sentence of *SC* or *SC** is similarly ambiguous. Each decomposes *uniquely* into component sentences. More precisely: if ϕ is a sentence of *SC*, then, *exclusive* disjunction, either,

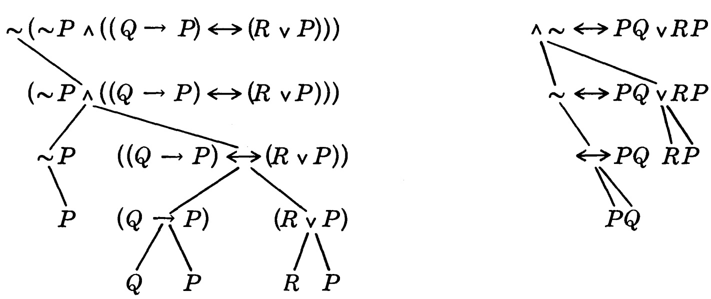
(1) ϕ is a sentence letter,
(2) $\phi = \sim\psi$, where ψ is a sentence of *SC*,

or,

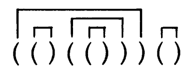
(3) there is exactly one triple (δ,ψ,χ) such that (i) δ is a binary connective and ψ and χ are sentences of *SC*, and (ii) $\phi = (\psi\delta\chi)$.

Similarly for *SC**

Each sentence of *SC* or *SC** decomposes uniquely into components that decompose uniquely into components and so on to sentence letters, simple components. *SC* and *SC** sentences have unique decomposition or structural diagrams as here illustrated:



3 *Unique decomposition: demonstration for SC* Two terms descriptive of sequences: An ordered sequence of parentheses *nests* under a one-to-one pairing $\mathscr{P}$ iff each left-hand parenthesis is paired in $\mathscr{P}$ with a right-hand parenthesis to its right and, if left-hand parenthesis i and j are paired with right-hand parentheses n and m respectively, i is to the left of j, and j is to the left of n, then m is to the left of n. A sequence nests under a pairing iff paired parentheses can be marked with non-intersecting brackets as here:



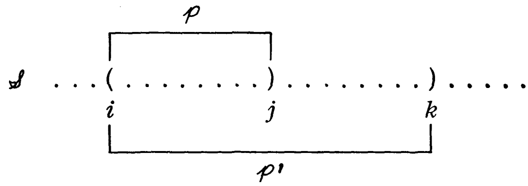
Count the empty sequence as nesting trivially. A sequence of parentheses $\mathscr{J}$ forms a *bounded* nest under a pairing $\mathscr{P}$ iff $\mathscr{J}$ nests under $\mathscr{P}$ and the first and last constituents in $\mathscr{J}$ are paired in $\mathscr{P}$.

If ϕ is a sentence of *SC*, then either ϕ is a sentence letter, contains an initial occurrence of $\sim$, or contains an initial parenthesis. Unique decomposition holds without argument in the first two cases. We turn to the third. Let ϕ be of the form, $(\psi\delta\chi)$, (ψ and χ sentences of *SC* and δ a binary connective). Let $\mathscr{J}^{\phi}$, $\mathscr{J}^{\psi}$, and $\mathscr{J}^{\chi}$ be the (perhaps empty) sequences of parentheses in ϕ, ψ, and χ respectively. Then there exists a pairing $\mathscr{P}$ for $\mathscr{J}^{\phi}$ under which the parentheses in $\mathscr{J}^{\phi}$, $\mathscr{J}^{\psi}$, and $\mathscr{J}^{\chi}$ form bounded nests, for parentheses enter sentences of *SC* in bounding pairs. Now *suppose* that ϕ is *also* of the form, $(\psi'\delta'\chi')$, (ψ' and χ' sentences of *SC* and δ' a binary connective such that $(\delta',\psi',\chi') \neq (\delta,\psi,\chi)$): that is, *suppose* unique decomposition fails for ϕ—if it does it must fail in this way for manifestly ϕ is not a sentence letter or sentence with an initial occurrence of $\sim$. Without loss of generality suppose further that δ' stands in ϕ to the left of δ. Then ϕ has the form (The underlines, for later use, are not part of the form.),

$$(\psi'\delta'\zeta\underline{(}\theta\delta\pi\underline{\underline{)}}\lambda)$$

where ψ', θ, and π are sentences of *SC*, ζ and λ are sequences of *SC*-symbols, $\zeta(\theta\delta\pi)\lambda$ is identical with the *SC*-sentence χ', and the displayed occurrences of δ in $(\psi\delta\chi)$ and $(\psi'\delta'\zeta(\theta\delta\pi)\lambda)$ are identical. And there exists a pairing $\mathscr{P}'$ for $\mathscr{J}^{\phi}$ under which the parentheses in $\mathscr{J}^{\phi}$, $\mathscr{J}^{\psi'}$, and $\mathscr{J}^{\chi'}$ form bounded nests, and in which the singly-underlined parenthesis is paired with a parenthesis, namely the doubly-underlined one, that stands to the *right* of the displayed occurrence of δ. But the singly-underlined parenthesis is, in $\mathscr{P}$, paired with a parenthesis to the *left* of the displayed occurrence of δ: for the parentheses in $\mathscr{J}^{\psi}$ form a bounded nest under $\mathscr{P}$. Thus $\mathscr{P} \neq \mathscr{P}'$. Briefly, on the *hypothesis* that ϕ, an *SC*-sentence with an initial parenthesis, is of two distinct forms, $(\psi\delta\chi)$ and $(\psi'\delta'\chi')$, it follows that there exist two distinct pairings under which the parentheses in ϕ form bounded nests. *But this is impossible*. Indeed *no* sequence of parentheses can have distinct *nesting* pairings: proof of this lemma is presented below. Rejecting the hypothesis we conclude that unique decomposition holds for all sentences of *SC*.

Proof of the lemma: There exists for a sequence $\mathscr{J}$ of parentheses at most one pairing under which $\mathscr{J}$ nests. Suppose there exists for a sequence two distinct pairings $\mathscr{P}$ and $\mathscr{P}'$ under which $\mathscr{J}$ nests. Then there is a left-hand parenthesis, let it be the ith parenthesis in $\mathscr{J}$, that is in $\mathscr{P}$ paired with say the jth parenthesis and in $\mathcal{P}'$ with say the kth, $j \neq k$: without loss of generality we assume the jth parenthesis is to the left of the kth and display our several part supposition thus—



Let L^{ij} be the number of left-hand parentheses in the interval i through j *exclusive* of i and j and R^{ij} the number of right-hand parentheses in this interval; understand L^{jk} and R^{jk} similarly. Then,

(1) $$L^{ij} = R^{ij}\,.$$

For the ith and jth parentheses are paired in $\mathcal{P}$ and so no parenthesis in the interval bounded by these parentheses can be paired in $\mathcal{P}$ with a parenthesis outside it on pain of breaking the nest made by $\mathcal{P}$. Similarly,

(2) $$L^{ij} + L^{jk} = R^{ij} + 1 + R^{jk},$$

for the ith and kth parentheses are paired in $\mathcal{P}'$. Further, though the sequence in the interval bounded by but not including parentheses j and k could begin with a "run" of right-hand parentheses, since each left-hand parenthesis in this interval must be paired in $\mathcal{P}'$ with a right-hand parenthesis in it (on pain of breaking the nest made by $\mathcal{P}'$ in which i is paired with k), we have,

(3) $$L^{jk} \leqslant R^{jk}.$$

But, subtracting (1) from (2),

(4) $$L^{jk} = 1 + R^{jk},$$

and thus,

(5) $$L^{jk} > R^{jk}.$$

(5) contradicts (3) and concludes the proof of the lemma.

4 *Unique decomposition: demonstration for SC** If ϕ is a sentence of SC^*, then either ϕ is a sentence letter, contains an initial occurrence of '$\sim$', or contains an initial occurrence of a binary connective. Unique decomposition holds without argument for the first two cases. We turn to the third. Let ϕ be an SC^*-sentence of the form, $\delta\psi\chi$, (δ a binary connective, ψ and χ SC^*-sentences). *Suppose* ϕ is also of the form, $\delta\psi'\chi'$, (ψ' and χ' SC^*-sentences distinct from ψ and χ respectively): that is, suppose unique decomposition fails for ϕ—if it does it must fail in this way. Without loss of generality suppose ψ' is of greater length than ψ. Then, $\psi' = \psi\theta$, (θ a sequence of SC^*-symbols): that is, ψ', an SC^*-sentence has as an initial proper segment an SC^*-sentence, namely ψ. *But this is impossible:* no SC^*-sentence has as an initial proper segment an SC^*-sentence—proof of this lemma is presented below. We conclude that unique decomposition holds for SC^*-sentences.

Proof of the lemma: Let the length of a sentence ϕ be the number of occurrences of symbols in ϕ.
Basis—the lemma holds for sentences of length-1: the lemma holds for sentence letters.
Inductive step—hypothesis: the lemma holds for sentences of lengths $\leq n$. Let ϕ be of length-$(n + 1)$. There are two cases to consider:

(i) $\phi = \sim\psi$, ψ an SC^*-sentence of length-n.
(ii) $\phi = \delta\psi\chi$, δ a binary connective and ψ and χ SC^*-sentences of length $< n$.

Case (i): Suppose the lemma fails for ϕ. Then, $\phi = \Delta\theta$, (Δ an SC^*-sentence, θ a non-empty sequence of SC^*-symbols), $\Delta = \sim\psi'$, (ψ' an SC^*-sentence) *and*, $\psi = \psi'\theta$, *that is*, an SC^*-sentence ψ of length n has an SC^*-sentence, ψ', as an initial proper segment. This contradicts the inductive hypothesis.
Case (ii): Suppose the lemma fails for ϕ in this case. Then, $\phi = \Delta\theta$, (Δ an SC^*-sentence, θ a non-empty sequence of SC^*-symbols), $\Delta = \delta\psi'\chi'$, and so, $\phi = \delta\psi'\chi'\theta$, ($\psi'$ and χ' SC^*-sentences). There are three cases to consider [under Case (ii)] regarding the relative lengths of ψ' and ψ. *First* case, ψ' is shorter than ψ. In this case, contrary to the inductive hypothesis, an SC^*-sentence of length $< n$, namely ψ, has as an initial proper segment an SC^*-sentence, namely ψ'. *Second* case, ψ' is of the same length as ψ. In this case, $\chi = \chi'\theta$, and, contrary to the inductive hypothesis, an SC^*-sentence of length $< n$, namely χ, has as an initial proper segment an SC^*-sentence, namely χ'. *Third case*, ψ' is longer than ψ, and contrary to the inductive hypothesis, an SC^*-sentence of length $< n$, namely ψ', has as an initial proper segment an SC^*-sentence, namely ψ.

5 *SC and SC*—relative merits* The SC-notation has, at least for some purposes, certain advantages. Consider the counterparts ϕ,

$$(\sim((P \vee Q) \leftrightarrow T) \rightarrow ((P \wedge \sim R) \vee S))$$

and ϕ^*,

$$\rightarrow\sim\leftrightarrow \vee PQT \vee\wedge P \sim RS.$$

Suppose the context is that of a derivation. Discerning the bounded nest of parentheses, one can straight-away find ϕ's major connective and, finding it, identify antecedent and consequent without *inter alia* doing all that is required to determine the total structure of either. (So one knows what is needed, for example, for *modus ponens*, and what it yields. More *detailed* information regarding ϕ's structure is not required for this purpose—may not be required for any purpose in the context.) In contrast, were ϕ^* given, though one could determine its major connective straight-away, one could *not* determine its antecedent and consequent without *inter alia* performing enough thought-operations to also determine the total structure of the antecedent: setting aside the initial occurrence of '$\rightarrow$', one would search for the shortest SC^*-sentence that follows it examining in turn initial segments of increasing length and deciding of each whether or not it is an *SC**-sentence—no more is required, and no less is sufficient, to identify ϕ^*'s antecedent and consequent *and* no more is needed than this segment-by-segment examination to determine the total structure of the antecedent.

An advantage of the *SC*-notation lies in this: to read an *SC*-sentence one performs a number of discrete operations that provide information of progressively greater detail and the process can be stopped, with results useful at least in inference-contexts, at many *more* points than can its *SC**-counterpart. Much more needs saying if the widespread preference for parenthesis notations is to be fully explained. But what remains consists, I think, mainly of psychological analyses of such things as pattern-discernment, scanning techniques, 'record-keeping', etc.

If the choice is of a notation to *use*, most persons will choose *SC*. But if the issue is what notation to *develop* and take as 'official' (with others perhaps brought in as informal variants) or what notation to *discuss* and, for example, show to be unambiguous, then legibility will matter less, relative simplicity and economy more, and *SC** may be preferred. (Thus Kalish and Montague present a parenthesis-free notation as 'official' in their general grammar for first-order theories, Chapter VIII of *Logic*. And the argument of section **4** of this note was more easily found and is perhaps more easily followed than that of section **3**.) Choice of notation should depend upon purposes to be served. Often, of course, it depends in fact largely on taste and tradition.

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