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THE BINARY REPRESENTATION OF m-VALUED LOGIC WITH APPLICATIONS TO UNIVERSAL DECISION ELEMENTS

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1 *Introduction* It has been shown [1] that the variable P of m-valued logic may be represented by the n 2-valued variables $P_1, \ldots, P_n (2^{n-1} < m \leqslant 2^n)$. For example, in 3-valued logic P may be represented by the 2-valued variables P_1, P_2 as follows:

P	P_1	P_2
1	F	F
2	T	F
3	F	T
3	T	T

The representation of any formula $\Phi(P, Q, \ldots)$ may then be achieved by finding the corresponding 2-valued formulae $(\Phi(P, Q, \ldots))_j$ $(j = 1, \ldots, n)$ in terms of the variables $P_1, \ldots, P_n, Q_1, \ldots, Q_n, \ldots$.

If we assume that the logical constants **t**, **f** are available then in addition to the assertium functor $F_1(\ )$, the unary functors $F_2(\ ), \ldots, F_\nu(\ )$ are also available by virtue of the binary representation, since we may make $(F_i(P))_j$ equal to any member of the set $\{P_1, \ldots, P_n, \mathbf{t}, \mathbf{f}\}$. In the 3-valued case Rose [2] has shown that the additional unary functors $F_2(\ )$, $F_3(\ )$, $F_4(\ )$ are available, the truth tables of $F_i(P)$ and $(F_i(P))_j$ $(i = 1, 2, 3;$ $j = 1, 2)$ being as follows:

P	P_1	P_2	$F_2(P)$	$(F_2(P))_1$	$(F_2(P))_2$	$F_3(P)$	$(F_3(P))_1$	$(F_3(P))_2$	$F_4(P)$	$(F_4(P))_1$	$(F_4(P))_2$
1	F	F	1	F	F	1	F	F	2	T	F
2	T	F	1	F	F	1	F	F	2	T	F
3	F	T	2	T	F	3	F	T	3	T	T
3	T	T	2	T	F	3	F	T	3	T	T

These functors are available since we may make $(F_i(P))_j$ equal to any member of the set $\{P_1, P_2, \mathbf{t}, \mathbf{f}\}$. However, the functors thus formed are only valid if the last two entries in the truth table both correspond to the same truth values. Also some of the valid combinations will correspond

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to trivial functors. Thus the three functors mentioned above are represented by the equations:

$$(F_2(P))_1 = P_2,\ (F_2(P))_2 = \mathbf{f};$$
$$(F_3(P))_1 = \mathbf{f},\ \ (F_3(P))_2 = P_2;$$
$$(F_4(P))_1 = \mathbf{t},\ \ (F_4(P))_2 = P_2.$$

Any other possible combination will lead to invalid functors, duplicates of those already found, or trival functors.

In this paper we will consider the determination of ν for various values of m and then consider the implications of a binary representation of m-valued logic in finding universal decision elements. Throughout we will denote the m values of m-valued logic by the integers $1, \ldots, m$. For the section on universal elements it will be useful to introduce the notion of the description number of a unary functor: if the unary functor $F(\ )$ of m-valued logic is such that $F(P)$ takes the truth value x_j when P takes the truth value j, then the description number, i, of $F(\ )$ is given by

$$i = \sum_{j=1}^{m} (x_j - 1) \cdot m^{m-j} + 1 \ .$$

Where convenient the unary functor whose description number is i will be denoted by $\Psi_i(\ )$. We will assume that in the general case the representation of the m truth values will follow the pattern given for the 3-valued case. More precisely, if we replace **F**, **T** by 0, 1 respectively then associated with the 2^n possible assignments of 0, 1 to the variables $P_1, \ldots, P_n$ we may define an assignment number j, given by

$$j = \sum_{i=1}^{m} x_i \cdot 2^{i-1} + 1 \ ,$$

where P_i takes the truth value x_i. Then assignment j represents the truth value j if $j \leq m$ and represents the truth value m if $j > m$.

2 *Determination of* ν If $m = 2^n$ we may make $(F_i(P))_j$ equal to any member of the set $\{P_1, \ldots, P_n, \mathbf{t}, \mathbf{f}\}$ $(i = 1, \ldots, \nu;\ j = 1, \ldots, n)$ and obtain a valid functor, and thus there will be $(n + 2)^n$ distinct functors. However, m of these will correspond to the logical constants $1, \ldots, m$. Thus for $m = 2^n$

$$\nu = (n + 2)^n - m \ . \tag{1}$$

For example, if $m = 4$ we have $\nu = 12$ and the truth tables for the functors $F_2(\ ), \ldots, F_{12}(\ )$ are as follows:

P	P_1	P_2	$F_2(P)$	$F_3(P)$	$F_4(P)$	$F_5(P)$	$F_6(P)$	$F_7(P)$	$F_8(P)$	$F_9(P)$	$F_{10}(P)$	$F_{11}(P)$	$F_{12}(P)$
1	**F**	**F**	1	1	2	3	1	1	1	1	2	3	1
2	**T**	**F**	3	2	4	4	3	1	1	1	2	3	1
3	**F**	**T**	1	1	2	3	1	2	3	2	4	4	4
4	**T**	**T**	3	2	4	4	4	4	3	2	4	4	4

Their binary representations are given by the equations:

$(F_2(P))_1 = \mathbf{f}$, $(F_2(P))_2 = P_1$; $(F_3(P))_1 = P_1$, $(P_3(P))_2 = \mathbf{f}$; $(F_4(P))_1 = \mathbf{t}$, $(F_4(P))_2 = P_1$, $(F_5(P))_1 = P_1$; $(F_5(P))_2 = \mathbf{t}$; $(F_6(P))_1 = P_1$, $(F_6(P))_2 = P_1$; $(F_7(P))_1 = P_2$, $(F_7(P))_2 = P_1$, $(F_8(P))_1 = \mathbf{f}$, $(F_8(P))_2 = P_2$; $(F_9(P))_1 = P_2$, $(F_9(P))_2 = \mathbf{f}$; $(F_{10}(P))_1 = \mathbf{t}$, $(F_{10}(P))_2 = P_2$, $(F_{11}(P))_1 = P_2$, $(F_{11}(P))_2 = \mathbf{t}$; $(F_{12}(P))_1 = P_2$, $(F_{12}(P))_2 = P_2$.

We may generalize the result given in equation (1) to the case $m = 2^n - 2^{k-1} + 1$, $k \in \{1, \ldots, n\}$, by the following theorem.

Theorem 1 *If we represent* m*-valued logic by the* n 2-valued variables $P_1, \ldots, P_n$ *and* $m = 2^n - 2^{k-1} + 1$, $k \in \{1, \ldots, n\}$, *then the number of unary functors* $F_1(\ ), \ldots, F_\nu(\ )$ *available is given by*

$$\nu = A^n - A^{k-1} \cdot B^C - 2^{k-1}(2^C - 1) + B^C(n + 2)^{k-1} - C(2^C - 1)\{(n + 2)^{k-1} - (n + 1)^{k-1}\} - (n + 2)^{k-1}$$

where, $A = n - k + 3$, $B = n - k + 2$, $C = n - k + 1$.

Proof: We may consider the unary functors $F_1(\ ), \ldots, F_\nu(\ )$ as consisting of two discrete sets:

(I) those functors $F_p(\ )$ such that $F_p(P)$ does not take the truth value m when P takes the truth value m;
(II) those functors $F_q(\ )$ such that $F_q(P)$ takes the truth value m when P takes the truth value m, p, $q \in \{1, \ldots, \nu\}$.

We will refer to these as type I and type II functors respectively. To form valid functors of type I we require that $F_i(P)$ takes the truth value $k (k \neq m)$ when P takes the truth value m irrespective of the particular configuration of truth values taken by $P_1, \ldots, P_n$ which correspond to the truth value m. This requires that the last 2^{k-1} rows in the table of 2-valued functors $(F_i(\ ))_j$ are identical since there are 2^{k-1} different representations of the truth value m when $m = 2^n - 2^{k-1} + 1$. Thus to form such functors all $(F_i(P))_j$ must be taken from the set $\{P_k, \ldots, P_n, \mathbf{t}, \mathbf{f}\}$. However, unless $(F_i(P))_j = \mathbf{f}$ for at least one $j \in \{k, \ldots, n\}$ the functor will be of type II. Also, if $(F_i(P))_j = \mathbf{f}$ or $\mathbf{t}$ for all j then the resulting functor will correspond to one of the logical constants $1, \ldots, m - 1$.

The number of ways of substituting for $(F_i(P))_{j_1}$ $(j_1 = k, \ldots, n)$ from the set $\{P_k, \ldots, P_n, \mathbf{t}, \mathbf{f}\}$ such that $\mathbf{f}$ occurs at least once is given by $(n - k + 3)^{n-k+1} - (n - k + 2)^{n-k+1}$. The number of ways of substituting for $(F_i(P))_{j_2}$ $(j_2 = 1, \ldots, k - 1)$ from the set $\{P_k, \ldots, P_n, \mathbf{t}, \mathbf{f}\}$ is $(n - k + 3)^{k-1}$ Thus there are

$$\{(n - k + 3)^{n-k+1} - (n - k + 2)^{n-k+1}\} \cdot (n - k + 3)^{k-1}$$

type I functors. This number, however, includes the functors corresponding to logical constants, the number of these being $2^{k-1}(2^{n-k+1} - 1)$. Thus the total number of non-trivial type I functors is

$$(n - k + 3)^n - (n - k + 2)^{n-k+1} \cdot (n - k + 3)^{k-1} - 2^{k-1}(2^{n-k+1} - 1).$$

To form valid type II functors we require that the last 2^{k-1} truth values of $(F_i(P))_{j_1}$ $(j_1 = k, \ldots, n)$ should all correspond to the truth value true. Thus we may substitute for $(F_i(P))_{j_1}$ from the set $\{P_k, \ldots, P_n, \mathbf{t}\}$. Since the last 2^{k-1} truth values of $(F_i(P))_{j_2}$ $(j_2 = 1, \ldots, k-1)$ may be true or false we may substitute for $(F_i(P))_{j_2}$ from the set $\{P_1, \ldots, P_n, \mathbf{t}, \mathbf{f}\}$. Thus the total number of type II functors is given by $(n - k + 2)^{n-k+1} \cdot (n + 2)^{k-1}$. However, this number includes those substitutions which yield the logical constant m and further not all the remaining substitutions will yield distinct functors. Now any substitution which is such that $(F_i(P))_{j_1} = \mathbf{t}$ will yield the logical constant m regardless of the substitution made for $(F_i(P))_{j_2}$. The total number of these is $(n + 2)^{k-1}$. With regard to duplicate functors we will first prove the following lemma.

Lemma *Consider a substitution for* $(F_i(P))_j$ $(j = 1, \ldots, n)$ *such that* $(F_i(P))_{j_1}$ $(j_1 = k, \ldots, m)$ *is substituted from the set* $\{P_q, \mathbf{t}\}$, $q \in \{k, \ldots, n\}$, *so that* P_q *occurs at least once, and* $(F_i(P))_{j_2}$ $(j_2 = 1, \ldots, k-1)$ *is substituted from the set* $\{P_1, \ldots, P_n, \mathbf{t}, \mathbf{f}\}$ *so that* $\mathbf{f}$ *occurs at least once. Denote the set of substitutions satisfying the above conditions by* S. *Denote the set of all other substitutions yielding type* II *functors by* S'. *Then for every substitution in the set* S *there exists one and only one substitution in the set* S' *which will yield the same unary functor.*

Proof: Consider the table of truth values of $(F_i(P)_j$ $(j = 1, \ldots, n)$ for a substitution from the set S. The truth values in the kth to nth columns are all true whenever P_q takes the truth value true and thus for these rows and only these rows the resulting truth value of $F_i(P)$ will be m. The truth values in columns 1 to $k - 1$ may be changed only in these rows if the formula $F_i(P)$ is to remain unchanged. Thus if $(F_i(P))_{j_2} = \mathbf{f}$ for some $j_2 \in \{1, \ldots, k-1\}$ we may make $(F_i(P))_{j_2} = P_q$ without altering the formula $F_i(P)$. Further, we may not replace $(F_i(P))_{j_2}$ by any other variable since this would imply alteration in truth values in rows other than those where P_q takes the truth value true, which in turn implies that the formula $F_i(P)$ would be changed. Similarly, if $(F_i(P))_{j_2} \neq \mathbf{f}$ then there is no alternative substitution for $(F_i(P))_{j_2}$ which will yield the same functor. Thus if every occurrence of $\mathbf{f}$ is replaced by P_q, the resulting substitution is the only one in S' which yields the same functor. Clearly this holds for all the substitutions in the set S.

The number of substitutions that satisfy the conditions of the Lemma for a particular q is given by

$$\{(n + 2)^{k-1} - (n + 1)^{k-1}\} \cdot (2^{n-k+1} - 1)$$

and since $q \in \{k, \ldots, n\}$ the total number of duplications is given by

$$\{(n + 2)^{k-1} - (n + 1)^{k-1}\} \cdot (2^{n-k+1} - 1) \cdot (n - k + 1).$$

Thus the total number of non-trivial functors of type II is

$$(n - k + 2)^{n-k+1} \cdot (n + 2)^{k-1} - (n + 2)^{k-1}$$
$$- \{(n + 2)^{k-1} - (n + 1)^{k-1}\} \cdot (2^{n-k+1} - 1) \cdot (n - k + 1),$$

and the result then follows.

The value of ν for the cases $m = 3, \ldots, 16$ are shown below, these having been determined explicitly. The cases covered by the above theorem are asterisked.

m	n	k	ν
3	2	2	4*
4	2	1	12*
5	3	3	21*
6	3	-	24
7	3	2	56*
8	3	1	117*
9	4	4	144*
10	4	-	132
11	4	-	136
12	4	-	150
13	4	3	322*
14	4	-	324
15	4	2	648*
16	4	1	1280*

Universal decision elements of νth degree An Nth order universal decision element of the rth degree has been defined in [3] but for completeness the definition is given below:

A Functor $\Phi(\ , \ldots, \ )$ of n argument places corresponds to an Nth order universal decision element of the rth degree if we may define all non-trivial functors of m-valued logic with N or less argument places, solely by substitution of the formulae $F_1(P_1), \ldots, F_r(P_1), \ldots, F_1(P_n), \ldots, F_r(P_n)$, $(F_1(P) =_T P)$ and the logical constants $1, \ldots, m$ in its argument places, the functor $\Phi(\ , \ldots, \ )$ being used only once in the definiens.

If a binary representation of m-valued logic is employed then the unary functors $F_1(\ ), \ldots, F_\nu(\ )$ are available without the use of any additional hardware. Thus we may consider the problem of finding universal decision elements of the νth degree. The 3- and 4-valued cases with $N = 1$ have been considered in detail elsewhere [4] and we will here consider the first order case for $m = 5$. For the 5-valued case $\nu = 21$ and the truth tables of $F_2(P), \ldots, F_{21}(P)$ are given below.

P	$F_2(P)$	$F_3(P)$	$F_4(P)$	$F_5(P)$	$F_6(P)$	$F_7(P)$	$F_8(P)$	$F_9(P)$	$F_{10}(P)$	$F_{11}(P)$
1	1	1	3	1	1	1	3	1	1	1
2	4	2	4	3	1	1	3	3	1	1
3	1	1	3	2	4	2	4	1	3	1
4	4	2	4	4	4	2	4	3	3	1
5	5	5	5	5	5	5	5	5	5	5

P	$F_{12}(P)$	$F_{13}(P)$	$F_{14}(P)$	$F_{15}(P)$	$F_{16}(P)$	$F_{17}(P)$	$F_{18}(P)$	$F_{19}(P)$	$F_{20}(P)$	$F_{21}(P)$
1	1	1	3	3	1	2	2	2	2	4
2	1	1	3	3	1	4	2	2	2	4
3	1	1	3	3	1	2	4	2	2	4
4	1	1	3	3	1	4	4	2	2	4
5	4	2	5	4	3	5	5	5	4	5

Their binary representations are given by the following equations:

$(F_2(P))_1 = P_1$, $(F_2(P))_2 = P_1$, $(F_2(P))_3 = P_3$; $(F_3(P))_1 = P_1$, $(F_3(P))_2 = P_3$,
$(F_3(P))_3 = P_3$; $(F_4(P))_1 = P_1$, $(F_4(P))_2 = \mathbf{t}$, $(F_4(P))_3 = P_3$; $(F_5(P))_1 = P_2$,
$(F_5(P))_2 = P_1$, $(F_5(P))_3 = P_3$; $(F_6(P))_1 = P_2$, $(F_6(P))_2 = P_2$, $(F_6(P))_3 = P_3$;
$(F_7(P))_1 = P_2$, $(F_7(P))_2 = P_3$, $(F_7(P))_3 = P_3$; $(F_8(P))_1 = P_2$, $(F_8(P))_2 = \mathbf{t}$,
$(F_8(P))_1 = P_3$, $(F_9(P))_1 = P_3$, $(F_9(P))_2 = P_1$, $(F_9(P))_3 = P_3$; $(F_{10}(P))_1 = P_3$,
$(F_{10}(P))_2 = P_2$, $(F_{10}(P))_3 = P_3$; $(F_{11}(P))_1 = P_3$, $(F_{11}(P))_2 = P_3$, $(F_{11}(P))_3 = P_3$;
$(F_{12}(P))_1 = P_3$, $(F_{12}(P))_2 = P_3$, $(F_{12}(P))_3 = \mathbf{f}$; $(F_{13}(P))_1 = P_3$, $(F_{13}(P))_2 = \mathbf{f}$,
$(F_{13}(P))_3 = \mathbf{f}$; $(F_{14}(P))_1 = P_3$, $(F_{14}(P))_2 = \mathbf{t}$, $(F_{14}(P))_3 = P_3$; $(F_{15}(P))_1 = P_3$,
$(F_{15}(P))_2 = \mathbf{t}$, $(F_{15}(P))_3 = \mathbf{f}$; $(F_{16}(P))_1 = \mathbf{f}$, $(F_{16}(P))_2 = P_3$, $(F_{16}(P))_3 = \mathbf{f}$;
$(F_{17}(P))_1 = \mathbf{t}$, $(F_{17}(P))_2 = P_1$, $(F_{17}(P))_3 = P_3$; $(F_{18}(P))_1 = \mathbf{t}$, $(F_{18}(P))_2 = P_2$,
$(F_{18}(P))_3 = P_3$; $(F_{19}(P))_1 = \mathbf{t}$, $(F_{19}(P))_2 = P_3$, $(F_{19}(P))_3 = P_3$; $(F_{20}(P))_1 = \mathbf{t}$,
$(F_{20}(P))_2 = P_3$, $(F_{20}(P))_3 = \mathbf{f}$; $(F_{21}(P))_1 = \mathbf{t}$, $(F_{21}(P))_2 = \mathbf{t}$, $(F_{21}(P))_3 = P_3$.

The number of valid substitutions from the set $\{F_1(P), \ldots, F_r(P), 1, \ldots, 5\}$ for the argument places of the n-place functor $\Phi(P_1, \ldots, P_n)$ is given by $(5 + r)^n - 5^n$, which gives the number 17576 when $n = 3$ and $r = 21$. Since it is not necessary to define the logical constants or the functors $F_1(\ ), \ldots, F_{21}(\ )$, the number of unary functors to be defined is $5^5 - 5 - 21 = 3099$. Using the method described in [4] and starting with an arbitrary ternary functor it was found that initially 1412 unary functors were undefined. This number was reduced to 688 but at this point the search was abandoned in view of the amount of computer time being used.

Consider again the number of valid substitutions, $(5 + r)^n - 5^n$. If $n = 4$ and $r \geqslant 3$ then the number of substitutions exceeds the number of unary functors to be defined. Thus a 4-place functor $\Phi(\ ,\ ,\ ,\ )$ corresponding to a first order universal decision element of 3rd or higher degree may exist. Using the same method as before a substitution set with $r = 11$ was considered, the 11 unary functors being $F_1(\ )$, $F_2(\ )$, $F_3(\ )$, $F_4(\ )$, $F_5(\ )$, $F_{10}(\ )$, $F_{13}(\ )$, $F_{15}(\ )$, $F_{16}(\ )$, $F_{18}(\ )$, and $F_{20}(\ )$. This proved successful and a formula $\Phi(P_1, P_2, P_3, P_4)$ satisfying the conditions was found. We may represent the truth table of $\Phi(P_1, P_2, P_3, P_4)$ in a modified form as follows: Let

$$\Phi(P_1, P_2, P_3, P_4) =_{\mathrm{T}} [P_1, \Lambda_1(P_2, P_3, P_4), \Lambda_2(P_2, P_3, P_4), \Lambda_3(P_2, P_3, P_4), \Lambda_4(P_2, P_3, P_4), \Lambda_5(P_2, P_3, P_4), P_1]$$

where

$$\Lambda_i(P_2, P_3, P_4) =_{\mathrm{T}} [P_2, \theta_{i1}(P_3, P_4), \theta_{i2}(P_3, P_4), \theta_{i3}(P_3, P_4), \theta_{i4}(P_3, P_4), \theta_{i5}(P_3, P_4), P_2], \quad i = 1, \ldots, 5$$

and

$$\theta_{ij}(P_3, P_4) =_T [P_3, \Psi_{k_{ij1}}(P_4), \Psi_{k_{ij2}}(P_4), \Psi_{k_{ij3}}(P_4), \Psi_{k_{ij4}}(P_4), \Psi_{k_{ij5}}(P_4), P_3],$$
$$k_{ijp} \in \{1, \ldots, 3124\},$$

and $[\ ,\ldots,\ ]$, the generalized conditioned disjunction functor, is such that $[P, Q_1, \ldots, Q_m, P]$ takes the truth value of Q_i when P takes the truth value i. Then the modified truth table of the formula $\Phi(P_1, P_2, P_3, P_4)$ found above is as follows:

i	j	k_{ij1}	k_{ij2}	k_{ij3}	k_{ij4}	k_{ij5}	i	j	k_{ij1}	k_{ij2}	k_{ij3}	k_{ij4}	k_{ij5}
1	1	690	2225	121	1820	3000	3	4	812	673	441	129	345
1	2	423	959	2070	1237	1681	3	5	446	675	1084	1086	2144
1	3	1428	1932	2999	2967	989	4	1	3021	2973	1245	2576	123
1	4	2925	3064	1934	817	978	4	2	3096	750	2350	801	1102
1	5	112	3058	698	1027	1851	4	3	2008	1276	251	632	1184
2	1	2648	2414	2046	1680	2436	4	4	2002	3027	1567	2543	2830
2	2	283	738	1541	1393	3016	4	5	2531	1987	138	1273	1627
2	3	2537	39	2277	2818	2000	5	1	1899	2874	3111	101	456
2	4	133	323	2483	962	433	5	2	350	2067	999	654	675
2	5	1332	2860	1170	3094	1095	5	3	1110	2222	69	2787	3002
3	1	1990	1891	1652	1761	2681	5	4	2001	2748	463	463	2283
3	2	1987	1234	1045	2096	2066	5	5	1373	8	1895	1519	1237
3	3	2530	2045	2765	428	1765							

The number of unary functors in the substitution set could probably have been reduced. However, this line of investigation was not pursued since in the context of binary representation there is no basic difference between first order universal decision elements of degree 5, 11, or 21.

Consider now the formula $\Phi_1(P, Q, R, S)$ of the form

$$\Phi_1(P, Q, R, S) =_T [P, \Lambda(Q, R, S), H_1(\Lambda(Q, R, S)), H_2(\Lambda(Q, R, S)),$$
$$H_3(\Lambda(Q, R, S)), H_4(\Lambda(Q, R, S)), P],$$

where $H_1(\ )$, $H_2(\ )$, $H_3(\ )$, and $H_4(\ )$ are arbitrary unary functors. Choosing $H_1(P) =_T {\sim}P$, $H_2(P) =_T {\sim\sim}P$, $H_3(P) =_T {\sim\sim\sim}P$, and $H_4(P) =_T {\sim\sim\sim\sim}P$, where $\sim$ corresponds to the cyclic negation functor of Post [5], then using the same method as above a formula $\Phi_1(P, Q, R, S)$ corresponding to a first order universal decision element of degree 11 was found, the substitution set being the same as in the first example. The modified truth table of the formula $\Lambda(Q, R, S)$ is as follows:

i	k_{i1}	k_{i2}	k_{i3}	k_{i4}	k_{i5}
1	1065	2225	121	1820	3000
2	423	959	2070	1237	1681
3	1428	1935	2995	1217	989
4	3000	3018	1882	817	2734
5	562	2274	3068	2922	1730

The binary representation of $\Phi_1(P, Q, R, S)$ may be arrived at as follows: We have,

$$\Phi_1(P, Q, R, S) =_{\mathsf{T}} [P, \Lambda(Q, R, S), {\sim}\Lambda(Q, R, S), {\sim\sim}\Lambda(Q, R, S), {\sim\sim\sim}\Lambda(Q, R, S), {\sim\sim\sim\sim}\Lambda(Q, R, S), P],$$

where,

$$\Lambda(Q, R, S) =_{\mathsf{T}} [Q, \theta_1(R, S), \theta_2(R, S), \theta_3(R, S), \theta_4(R, S), \theta_5(R, S), Q]$$

and

$$\theta_1(R, S) =_{\mathsf{T}} [R, \Psi_{1065}(S), \Psi_{2225}(S), \Psi_{121}(S), \Psi_{1820}(S), \Psi_{3000}(S), R]$$
$$\theta_2(R, S) =_{\mathsf{T}} [R, \Psi_{423}(S), \Psi_{959}(S), \Psi_{2070}(S), \Psi_{1237}(S), \Psi_{1681}(S), R]$$
$$\theta_3(R, S) =_{\mathsf{T}} [R, \Psi_{1428}(S), \Psi_{1935}(S), \Psi_{2995}(S), \Psi_{1217}(S), \Psi_{989}(S), R]$$
$$\theta_4(R, S) =_{\mathsf{T}} [R, \Psi_{3000}(S), \Psi_{3018}(S), \Psi_{1882}(S), \Psi_{817}(S), \Psi_{2734}(S), R]$$
$$\theta_5(R, S) =_{\mathsf{T}} [R, \Psi_{562}(S), \Psi_{2274}(S), \Psi_{3068}(S), \Psi_{2922}(S), \Psi_{1730}(S), R].$$

The truth tables of $\theta_1(R, S)$, $(\theta_1(R, S))_j$, $j = 1, 2, 3$ are shown below. Those entries in the tables which are starred indicate that at these points either **T** or **F** could have been chosen; the choice given was dictated by the simplicity of the resulting formulae. Corresponding formulae for $(\theta_1(R, S))_j$ are as follows:

$$(\theta_1(R, S))_1 =_{\mathsf{T}} R_3 \vee (([S_1, R_1, S_2] \equiv R_2) \not\supset S_3),$$
$$(\theta_1(R, S))_2 =_{\mathsf{T}} R_3 \vee ((R_1 \vee (S_1 \vee S_2 \not\supset R_2)) \not\supset S_3),$$
$$(\theta_1(R, S))_3 =_{\mathsf{T}} (R_3 \,\&\, S_3) \vee [S_1 \supset S_2, R_3, [R_2 \supset R_1, S_3, [R_2 \not\supset R_1, R_2 \equiv S_2, R_1 \,\&\, S_1]]].$$

Similarly, we may construct the truth tables for $(\theta_i(R, S))_j$, $i = 2, \ldots, 5$, and one set of suitable formulae is given below.

$\theta_1(R,S)$	1	2	3	4	5	S
1	2	4	3	3	5	
2	4	3	4	5	5	
3	1	1	5	5	1	
4	3	5	3	4	5	
5	5	4	5	5	5	
R						

		$(\theta_1(R,S))_1$									
			F	T	F	T	F	T	F	T	S_1
			F	F	T	T	F	F	T	T	S_2
			F	F	F	F	T	T	T	T	
F	F	F	T	T	F	F	F*	F*	F*	F*	
T	F	F	T	F	T	F*	F*	F*	F*	F*	
F	T	F	F	F	T*	T*	F	F	F	F	
T	T	F	F	T*	F	T	F*	F*	F*	F*	
F	F	T	T*	T	T*	T*	T*	T*	T*	T*	
T	F	T	T*	T	T*	T*	T*	T*	T*	T*	
F	T	T	T*	T	T*	T*	T*	T*	T*	T*	
T	T	T	T*	T	T*	T*	T*	T*	T*	T*	
R_1	R_2	R_3									

$(\theta_1(R,S))_2$											
			F	T	F	T	F	T	F	T	S_1
			F	F	T	T	F	F	T	T	S_2
			F	F	F	F	T	T	T	T	S_3
F	F	F	F	T	T	T	F*	F*	F*	F*	
T	F	F	T	T	T	T*	F*	F*	F*	F*	
F	T	F	F	F	F*	F*	F	F	F	F	
T	T	F	T	T*	T	T	F*	F*	F*	F*	
F	F	T	T*	T	T*	T*	T*	T*	T*	T*	
T	F	T	T*	T	T*	T*	T*	T*	T*	T*	
F	T	T	T*	T	T*	T*	T*	T*	T*	T*	
T	T	T	T*	T	T*	T*	T*	T*	T*	T*	
R_1	R_2	R_3									

$(\theta_1(R,S))_3$											
			F	T	F	T	F	T	F	T	S_1
			F	F	T	T	F	F	T	T	S_2
			F	F	F	F	T	T	T	T	S_3
F	F	F	F	F	F	F	T	T	T	T	
T	F	F	F	F	F	T	T	T	T	T	
F	T	F	F	F	T	T	F	F	F	F	
T	T	F	F	T	F	F	T	T	T	T	
F	F	T	T	F	T	T	T	T	T	T	
T	F	T	T	F	T	T	T	T	T	T	
F	T	T	T	F	T	T	T	T	T	T	
T	T	T	T	F	T	T	T	T	T	T	
R_1	R_2	R_3									

$$(\theta_2(R,S))_1 =_{\mathsf{T}} [S_1/S_3, R_3, [R_1, S_3, (R_2 \equiv S_2)/(R_1 \equiv S_1)]]$$
$$(\theta_2(R,S))_2 =_{\mathsf{T}} [(S_1 \not\supset S_2) \not\supset S_3, R_3, [R_2, S_3, [S_1, R_1 \equiv R_2, [S_1 \not\equiv S_2, R_1, S_1, S_2]]]]$$
$$(\theta_2(R,S))_3 =_{\mathsf{T}} [R_1 \not\supset R_2, S_3, [S_1 \not\equiv S_2, R_1, S_1 \,\&\, S_2] \not\supset (R_1 \not\equiv R_2)] \not\supset R_3.$$
$$(\theta_3(R,S))_1 =_{\mathsf{T}} [S_1 \supset S_3, R_3, [R_1, S_3, R_2 \vee [S_1 \equiv S_2, R_1, S_1 \not\supset S_2]]]$$
$$(\theta_3(R,S))_2 =_{\mathsf{T}} [S_3 \vee S_1, R_3, [{\sim}R_1, S_3, [R_1 \supset S_2, R_2, {\sim}S_1]]]$$
$$(\theta_3(R,S))_3 =_{\mathsf{T}} [S_3 \downarrow (S_2 \supset S_1), R_3, [R_1 \not\equiv R_2, S_3, R_2 \,\&\, [S_1 \not\supset S_2, R_1, {\sim}S_1]]]$$
$$(\theta_4(R,S))_1 =_{\mathsf{T}} R_3 \vee [R_2, S_3, [S_1, S_2, S_1 \not\supset (R_1 \equiv R_2)]]$$
$$(\theta_4(R,S))_2 =_{\mathsf{T}} [S_3, R_3, [{\sim}R_2, S_3, [R_2 \equiv R_1, R_2 \equiv S_2, [S_1 \downarrow R_1, R_2, S_1]]]]$$
$$(\theta_4(R,S))_3 =_{\mathsf{T}} [S_3 \downarrow S_1, R_3, [R_1 \downarrow R_2, S_3, [{\sim}S_2, R_1, S_1 \supset S_2] \not\supset R_2]].$$
$$(\theta_5(R,S))_1 =_{\mathsf{T}} [(S_1 \not\equiv S_2, R_3, [R_2 \supset R_1, S_3, (R_2 \equiv S_2) \not\equiv (R_1 \,\&\, R_2)]]$$
$$(\theta_5(R,S))_2 =_{\mathsf{T}} [{\sim}S_2, R_3, [R_1 \not\equiv R_2, S_3, S_2 \not\equiv R_1]]$$
$$(\theta_5(R,S))_3 =_{\mathsf{T}} [R_3, S_3, [S_1 \not\supset S_2, R_3, [S_2 \equiv R_1, S_1, R_2 \supset S_2]]].$$

Consider now the formula $G(U, V, W, X, Y, Z) =_{\mathsf{T}} [U, V, W, X, Y, Z, U]$. We may represent this formula in binary form by the following equations:

$$(G(U, V, W, X, Y, Z))_i =_{\mathsf{T}} (V_i \,\&\, \Omega_1(U_1, U_2, U_3)) \vee (W_i \,\&\, \Omega_2(U_1, U_2, U_3)) \vee (X_i \,\&\, \Omega_3(U_1, U_2, U_3)) \vee (Y_i \,\&\, \Omega_4(U_1, U_2, U_3)) \vee (Z_i \,\&\, \Omega_5(U_1, U_2, U_3)),\ i = 1, 2, 3$$

where the formulae $\Omega_j(U_1, U_2, U_3)$ are such that $\Omega_j(U_1, U_2, U_3)$ takes the truth value true if and only if U takes the truth value j. Given the binary representation of 5-valued logic, we may define the formulae $\Omega_j(U_1, U_2, U_3)$ as follows:

$$\Omega_1(U_1, U_2, U_3) =_{df} {}^{\sim}U_1 \ \& \ {}^{\sim}U_2 \ \& \ {}^{\sim}U_3, \quad \Omega_2(U_1, U_2, U_3) =_{df} U_1 \ \& \ {}^{\sim}U_2 \ \& \ {}^{\sim}U_3,$$
$$\Omega_3(U_1, U_2, U_3) =_{df} {}^{\sim}U_1 \ \& \ U_2 \ \& \ {}^{\sim}U_3, \quad \Omega_4(U_1, U_2, U_3) =_{df} U_1 \ \& \ U_2 \ \& \ {}^{\sim}U_3,$$
$$\Omega_5(U_1, U_2, U_3) =_{df} U_3.$$

Thus replacing U, V, W, X, Y, Z by Q, $\theta_1(R, S)$, $\theta_2(R, S)$, $\theta_3(R, S)$, $\theta_4(R, S)$, $\theta_5(R, S)$ respectively, we have now achieved the binary representation of the formula $\Lambda(Q, R, S)$. To complete the binary representation of the formula $\Phi_1(P, Q, R, S)$, we could apply the above result again and replace U, V, W, X, Y, Z by P, $\Lambda(Q, R, S)$, ${}^{\sim}\Lambda(Q, R, S)$, ${}^{\sim\sim}\Lambda(Q, R, S)$, ${}^{\sim\sim\sim}\Lambda(Q, R, S)$, ${}^{\sim\sim\sim\sim}\Lambda(Q, R, S)$ respectively. Alternatively we may find the binary representation of the formula

$$K(U, V) =_{T} [U, V, {}^{\sim}V, {}^{\sim\sim}V, {}^{\sim\sim\sim}V, {}^{\sim\sim\sim\sim}V, U]$$

and then make the appropriate substitutions. The truth tables for $K(U, V)$, $(K(U, V))$ $i = 1, 2, 3$ are given below together with formulae for $(K(U, V))_i$.

$K(U,V)$	1	2	3	4	5	V
1	1	2	3	4	5	
2	2	3	4	5	1	
3	3	4	5	1	2	
4	4	5	1	2	3	
5	5	1	2	3	4	
U						

	$(K(U,V))_1$		F	T	F	T	F	T	F	T	V_1
			F	F	T	T	F	F	T	T	V_2
			F	F	F	F	T	T	T	T	V_3
F	F	F	F	T	F	T	T*	T*	T*	T*	
T	F	F	T	F	T	F*	F	F	F	F	
F	T	F	F	T	T*	F	T	T	T	T	
T	T	F	T	F*	F	T	F	F	F	F	
F	F	T	T*	F	T	F	T	T	T	T	
T	F	T	T*	F	T	F	T	T	T	T	
F	T	T	T*	F	T	F	T	T	T	T	
T	T	T	T*	F	T	F	T	T	T	T	
U_1	U_2	U_3									

$(K(U, V))_2$											
			F	T	F	T	F	T	F	T	V_1
			F	F	T	T	F	F	T	T	V_2
			F	F	F	F	T	T	T	T	V_3
F	F	F	F	F	T	T	T*	T*	T*	T*	
T	F	F	F	T	T	T*	F	F	F	F	
F	T	F	T	T	F*	F	F	F	F	F	
T	T	F	T	T*	F	F	T	T	T	T	
F	F	T	T*	F	F	T	T	T	T	T	
T	F	T	T*	F	F	T	T	T	T	T	
F	T	T	T*	F	F	T	T	T	T	T	
T	T	T	T*	F	F	T	T	T	T	T	
U_1	U_2	U_3									

$(K(U, V))_3$											
			F	T	F	T	F	T	F	T	V_1
			F	F	T	T	F	F	T	T	V_2
			F	F	F	F	T	T	T	T	V_3
F	F	F	F	F	F	F	T	T	T	T	
T	F	F	F	F	F	T	F	F	F	F	
F	T	F	F	F	T	F	F	F	F	F	
T	T	F	F	T	F	F	F	F	F	F	
F	F	T	T	F	F	F	F	F	F	F	
T	F	T	T	F	F	F	F	F	F	F	
F	T	T	T	F	F	F	F	F	F	F	
T	T	T	T	F	F	F	F	F	F	F	
U_1	U_2	U_3									

$$(K(U, V))_1 =_{\mathrm{T}} [V_1 \supset V_3,\ U_3,\ [{\sim}U_1,\ V_3,\ (U_2 \,\&\, V_2) \equiv (U_1 \equiv V_1)]]$$
$$(K(U, V))_2 =_{\mathrm{T}} [V_3 \vee (V_1 \equiv V_2),\ U_3,\ [U_1 \equiv U_2,\ V_3,\ (U_2 \equiv V_2) \supset ((U_1 \,\&\, V_1) \not\supset U_2)]]$$
$$(K(U, V))_3 =_{\mathrm{T}} [(V_1 \downarrow V_2) \not\supset V_3,\ U_3,\ [U_1 \downarrow U_2,\ V_3,\ [U_2 \,\&\, (U_1 \downarrow V_1),\ U_2 \equiv V_2,\ U_1 \,\&\, V_1]]].$$

REFERENCES

[1] Rose, A., "Many valued logical machines," *Proceedings of the Cambridge Philosophical Society*, vol. 54 (1958), pp. 307-321.

[2] Rose, A., "Sur les éléments universels trivalent de décision," *Comptes Rendus Hebdomadires des Séances de l'Académie des Sciences, Paris*, vol. 269 (1969), pp. 1-3.

[3] Loader, J., "Second order and higher order universal decision elements in *m*-valued logic," *Proceedings of 1975 International Symposium of Multiple-Valued Logic*, pp. 53-57.

[4] Loader, J., "An alternative concept of the universal decision element in *m*-valued logic," *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, vol. 21 (1975), pp. 369-375.

[5] Post, E. L., "Introduction to a general theory of elementary propositions," *American Journal of Mathematics*, vol. 43 (1921), pp. 163-185.

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