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DOMAIN RESTRICTIONS IN STANDARD DEDUCTIVE LOGIC

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Introduction Although this paper draws upon logical evidence associated with Gödel-type undecidability arguments, its concern is independent of current work in the philosophy of mathematics. The issue to be examined is the conceptualization of a system of deductive logic as applicable to a domain, both infinite and comprehensive, of individual classes or sets of numbers. Historically, the notion of an infinite domain was introduced into deductive logic for the purpose of accommodating number theory and thus establishing a logical foundation for the conventions of mathematical practice. Yet subsequent developments in mathematical logic have made it evident that a host of formal difficulties attend any naive understanding of the infinity hypothesis. Moreover, it is no longer certain that the notion of a comprehensive ontological domain for classical logic is a primary conceptual need. The focus of attention has shifted from the importance of standard logic for mathematical theory to the impact of mathematical analysis upon our understanding of what constitutes a logical system. Questions of logical domain seem less relevant to mathematics proper than to issues involving the nature and ontological status of a formalized language.

In this paper the evidence both of the class contradiction and of Gödel's undecidability proof will be used as evidence against the assumption of an ontologically total domain for either a typed postulational logic or an untyped logic with the class membership predicate as primitive. The notion most essential to my argument concerns the status of individual or predicative constants when logical schemata are supplemented by actual class or property names. If we say that a logical calculus has the capacity to represent formally an infinite number of classes or properties, we are committed to the view that the formalized language in some sense contains an infinite supply of interpreted constant symbols which can serve as replacements for variables. However, any formula in which such constants are employed will contain only symbols that are discoverable within some finite list of the expressions of the language. It is this characteristic of a

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formalized language that justifies the determinate application of a system of Gödel encoding.

As Finsler [2] pointed out some years ago, the analysis of mathematical proofs as consisting of finite concatenations of available signs will lead to the conclusion that not all definable binary sequences are finitely definable. In this paper a similar argument is focused upon the formal representation, by an individual class or property name, of complex but determinate conditions upon a quantifiable variable. In an untyped logic with a primitive membership predicate, a system such as Quine's **ML**, the restrictive theorem '$\sim(\exists y)(x)(x \epsilon y \equiv \sim(x \epsilon x))$' can be derived from basic axioms. This theorem can be understood to mean that no class name selected to represent the condition '$\sim(x \epsilon x)$' could in fact be the name of the requisite class of individuals—by virtue of the contradiction which would follow. A comparison of the **ML** situation to that of Richards' Paradox, the inspiration for Finsler's account, can be instructive. If the phrase 'the least number not formally designated' is taken to be a definitive means for designating a number formally, the result is an obvious contradiction. Similarly, the rule that all complex conditions upon x can be represented by an '$x \epsilon y$' expression, with y the name of the determined class, would lead to an **ML** contradiction. The significance of Richards' Paradox is the evidence it provides of the need to discriminate formal from informal names or designations of numbers. The effect of the restrictive theorem upon **ML** logic is to show that not all polyadic conditions upon a variable can be represented as monadic conditions [12]. If **ML** schemata are given ontological implication, the implication is that no 'existent' class of individuals determined by '$\sim(x \epsilon x)$' can fall within the semantic domain of the formal system.

In the case of undecidability arguments, when the mathematical calculus is a typed system modeled upon *Principia,* this mode of analysis leads to the conclusion that metamathematical arguments of the Gödel type are insufficient to overturn the assumption of decidability for the calculus in question. It cannot be demonstrated that such a system is decidable, but it can be shown that the assumption of decidability is enough to cast in question the basis for existing undecidability proofs—namely the notion that all computable mathematical functions are formally representable by a propositional function of the calculus. Though technically outside the formal system, the recursively defined diagonalization function upon which the undecidability argument depends can be compared, in its semantic role, to the expression 'the least natural number not formally designated' in Richards' Paradox and to the determinate condition '$\sim(x \epsilon x)$' in **ML** logic. In the face of the hypothesis of decidability, the rationale for assuming the representability of such a recursive function can only be the view that a mathematical calculus is ontologically comprehensive—that such a calculus can contain names for all sets of natural numbers, including those that are recursively definable.

It appears to be the case that the above argument does not apply to a mathematical calculus in which numerical variables and constants are not supplemented by class or property names, for example the system to which Tarski and Robinson [13] apply their undecidability analysis. In such a system there is no gap between computation of the diagonalization function and the assumption of formal representation by a class or property symbol. In Tarski and Robinson the effect of predicate structure is achieved not by the use of a membership predicate, nor by the juxtaposition of symbols of distinct type, but by the incorporation into the formal system of symbols for mathematical calculation. The domain restriction with which this paper is concerned is a notational restriction that applies to systems with quantifiable individual variables and in which sets or properties of individuals can be formally named. In the case of an untyped logic for set theory the existence of such a restriction is directly demonstrable; in the case of a typed mathematical calculus any proof that such a restriction pertains must be filtered through a coding methodology such as a system for Gödel numbering. The two arguments for a domain restriction are fully parallel only when the distinction between provable and unprovable formulas is given a status equivalent to that between truth and falsity. Such a status appears to fail for basic number theory, but contrary to Gödel's 1931 claim it will succeed for a typed *Principia* logic, provided that the set or property terms of such a system are understood to obtain their semantic value through an initial ontological interpretation of the system as a whole.

The results of this paper are thus directly applicable to questions involving the ontological status of a general logic for set theory or of a mathematical calculus which names sets or properties of numbers. There is no inconsistency in the claim that a deductive logic admits an infinite domain of individuals and supplies conditions for the generation of an infinite number of classes or properties of individuals. However, it can be seen that restrictions apply to the inclusion of names for all such infinite totalities. We can indicate the ontological existence of more sets or properties than are subject to formal representation within the deductive system.

1 The failure of efforts to create an equivalent of the Liar's Paradox in a deductive logic with sentential variables is not usually taken as evidence of a notational restriction within such a logic. Yet the difference between natural and formal expressions of the Paradox can be highly instructive from this point of view. Given appropriate utterance conditions, there is no formal obstacle to our taking the eidetic phrase 'this sentence' in the sentence 'This sentence is false' as referring to the natural sentence of which it is a part. The assignation of value to such a phrase is not unlike the assignation of arbitrary value to the functional variables of mathematics upon the basis of practical need. That is to say, the specified use of such a phrase as 'this sentence' will vary from situation to situation, even as the phrase retains the status of a variable name.

However, in a system of deductive logic which lacks eidetic signs, and in which the discrimination between a variable and a constant or proper name is enforced, such an assignation of value according to the situation of use is no longer possible. The following definition of the Liar sentence can be immediately recognized as ill formed.

(1) *S = df. S is false*

We can explain this ill formedness as a violation of the language-metalanguage discrimination. But even if the expressions of the formal language can contain the names of such expressions, it is evident that the constant-symbol '*S*' cannot be the name of a sentence containing that same symbol, since substitution rules will lead to the creation of an infinite regress. If we take '*S*' to be a constant-symbol which has been given a referential meaning according to an initial interpretation of the symbols of the formal deductive system, then it is evident that '*S*' cannot have been assigned such a value. The restriction in question applies only to the choice of '*S*' as the name of that particular sentence and does not extend to the choice of some other symbol as naming the sentence in question—at least within the hypothesis that the language contains names for its own sentences.

The limited explanatory power of the language-metalanguage distinction becomes evident when the problem of finding an appropriate name for sentences of the language is transformed, by way of quantification theory, into the class contradiction difficulty. That there exists a formal parallel between the two issues of deductive logic is evident in the following examples of eidetic reference combined with a quantification over sentence names and with a quantification over classes.

(2) *(X) (If X is the name of this sentence, then X is false)*
(3) *(z) (If z is the class x determined by this class matrix, then z fails to be included within z)*

In the case of (2) the choice of any name for the sentence in question will lead to the Liar contradiction, but the eidetic expression 'this sentence' is evidence of imperfect formalization, so that the contradiction is not one of deductive logic. However, the application of the same strategy to the names of classes determined by sentences, not the names of the sentences themselves, leads to the formulation of (3), where the eidetic reference can be removed without affecting meaning.

(4) $(\exists y)(x)(x \in y \equiv (z)(z = x \supset \sim(z \in z))$

Thus (4) expresses the claim that there can exist a class y such that objects x belong to y if and only if they satisfy the condition '$(z)(z = x \supset \sim(z \in z))$', which expresses the sense of (3) in logical language. In (4) we have a nonstandard but effective version of the class contradiction, since for any y taken to be x, both '$y \in y \supset \sim(y \in y)$' and '$\sim(y \in y) \supset y \in y$' must hold.

The threat of the class contradiction in an untyped logic is usually explained by giving '$\sim(\exists y)(x)(x \in y \equiv \sim(x \in x))$' as a derivation from tentative acceptance of '$(\exists y)(x)(x \in y \equiv \emptyset)$' as an axiom for class abstraction. But the version given as (4), where the class matrix is in the form of a quantification statement, helps to explain why no class name can be generated from the matrix '$\sim(x \epsilon x)$'. The explanation lies in the distinction between single and two place conditions upon a variable, or '$F(x)$' sentences and sentences that like '$\sim(x \epsilon x)$' are of '$G(x,x)$' form. (See [12] for a discussion of the '$G(x,x)$' status of '$x \epsilon x$' in an untyped logic with 'ϵ' a primitive dyadic predicate). Any sentence of '$G(x,x)$' form will be truth equivalent with a universal quantification over y which contains 'x' as a free variable:

(5) $G(x,x) \equiv (y)(y = x \supset G(x,y))$.

The equivalency given as (5) is a formalization of the fact that '$G(x,x)$' cannot be reduced to a sentence of $F(x)$ form without implying a universal quantification over y, and in an untyped logic such a reduction can lead to contradiction. The sense of (5) can also be conveyed by (7), which is derived directly from (6), in a non-empty logical universe.

(6) $(y)(G(y,y) \equiv G(y,y))$
(7) $(y)(\exists x)(G(x,y) \equiv G(x,x))$,

But (7) is identical with (8)

(8) $\sim(\exists y)(x)(G(x,y) \equiv \sim G(x,x))$,

which yields (9) as a theorem in untyped logic with 'ϵ' primitive.

(9) $\sim(\exists y)(x)(x \epsilon y \equiv \sim(x \epsilon x))$.

The derivation of (9) from (6) is an explicit denial of the claim that there can exist a class name y such that '$x \epsilon y \equiv \sim(x \epsilon x)$' holds for all x. The existence of such a theorem as (9) also indicates that because of the truth identity given as (5) there can exist no '$F(x)$' (or '$x \epsilon y$') sentence to which '$\sim G(x,x)$' is truth equivalent, provided G is taken to be the class membership relation. There must exist at least one multiplace condition upon x in such a logic which cannot be represented as of '$F(x)$' form.

The standard interpretation of (9) as a theorem of untyped logic is that there can be no class y that is determined by the class matrix '$\sim(x \epsilon x)$'. But a more precise interpretation, where the domain of the logic is taken to be natural classes, is that if such a class exists it cannot be included within the logical domain. In contemporary theory the implication of the class contradiction for questions of logical domain has been obscured by the failure of logicians to recognize any difficulty in the notion that the variables of a deductive logic can have an infinite totality of classes as their effective range. As long as class existence is defined within the formalism of the logic—as in Quine's definition of existence as being the value of a bound variable—there is no way to prove the restriction to a finite domain. But in Quine's own *Mathematical Logic* (**ML**), the direct consequence of the infinity assumption is formal proof within the **ML** system of the existence of at least one class that is a non-element, or a class that can belong to no other class. This is a consequence of Quine's emendation of the naive class abstraction rule '$(\exists y)(x)(x \epsilon y \equiv \emptyset)$' by compounding $\emptyset$ with the requirement of elementhood '$x \epsilon V$', which is definable as '$(\exists z)(x \epsilon z)$'. For any class y such that '$x \epsilon y \equiv : x \epsilon V. \sim(x \epsilon x)$' holds is a class for which '$(z)(\sim(y \epsilon z))$' is also true. However, Quine's need to assume the existence of a class that is a nonelement is explained more effectively by the hypothesis of a finitary limit upon class representation in **ML** or upon the effective domain of its quantifiable variables. Such a limitation can be expressed as the notion that for any finite list of class values supplied to such variables, there may exist a natural class that cannot belong to such a list. The finitary restriction does not prohibit the infinite extension of such a list of possible values; it merely informs us that no such list can include a class that is in fact the class of all the included values that do not include themselves. The existence of such a restriction cannot be demonstrated within **ML** logic itself, but its consequences are manifest in the incorporation of the theorem '$\sim(\exists y)(x)(x \epsilon y \equiv \sim(x \epsilon x))$' and in the fact that the matrix '$\sim(x \epsilon x)$' is irreducible to '$F(x)$' form.

A close natural parallel to the domain restriction upon an untyped logic such as **ML** is the bibliography paradox. We know that there can exist no natural bibliography of all bibliographies that do not include themselves. But if the existence of classes of bibliographies is defined only with regard to constructible bibliographies, then it also follows that there can be no natural class of bibliographies that do not include themselves. We can preserve the concept of such a class by isolating it from the notion of a constructible list—a list which if constructed would constitute another bibliography in need of listing. Similarly, to preserve the concept of an existent class of self-exclusive classes we must separate such a concept from that of a deductive logic with a domain that includes such a class. In short, we may reject Quine's definition of existence and posit at least one existent class that falls outside the domain of an untyped logic.

It can be seen that this notion of a finitary restriction upon the values of quantifiable variables is parallel to the generalization of the Liar Paradox given as (2) above where the restriction is against the choice of a constant-symbol with a specified referential value. In a system of logical schemata such as **ML**, the proof of a restriction against an '$F(x)$' equivalent for '$\sim(x \epsilon x)$' can be formulated only as proof that no value exists for y such that '$x \epsilon y \equiv \sim(x \epsilon x)$'. But if names of individual classes are added to such a system, the restriction can be stated as the existence of at least one dyadic condition upon x, namely '$\sim(x \epsilon x)$' for which no '$x \epsilon a$' or '$x \epsilon b$' or '$x \epsilon c$', . . ., equivalent exists, where 'a', 'b', 'c', etc. are actual names of classes. This lack of a proper name in such a logic for the class of classes that satisfy '$\sim(x \epsilon x)$', if such a natural class exists, can also be expressed as a finitary limit upon any ordered series of classes. The class of all such classes in the series will be a class within the series, and it will be a member of the class of all classes that do not include themselves. But if the class of all classes in the series is determined by a compound expression $\emptyset$ which contains $\sim(x \epsilon x)$, it will prove to be a nonelement that cannot belong to any further class and in this formal respect will bring the series to a finitary end.

2 In order to apply the evidence of Gödel's Proof to issues involving the domain of deductive logic it is necessary to focus attention upon the formal nature of those systems to which a methodology of Gödel numbering can be applied. Such a methodology depends upon the initial assignation of numerical values by application of a metamathematical function, both to primitive expressions of the system and to well formed expressions defined as concatenations of such expressions. It is a common-place of mathematical practice to assign specific recursive values to functional variables in such a way that the notational status of the variable *as a variable* is not affected. But a predicative variable of a deductive calculus, ranging over sets of numbers indeterminately, cannot retain its status as a primitive variable of the deductive system if it is given a determinate value. In an interpreted deductive system the variable-constant discrimination must apply at all type levels; the distinction is enforced by the customary definition of a propositional function as any sentence containing a free variable. In typed logic the sentence '$F(x)$', where 'F' and 'x' are both variables, remains devoid of truth value as long as either variable is unreplaced by a constant of the appropriate type. In a standard logic to which a system of Gödel numbering is applied, the constant-variable distinction is doubly reinforced in that no such '$F(x)$' sentence can have a determinate value if its Gödel number is computed upon the basis of a number assigned to 'F' in its formal status as a predicate variable. In addition the constant-variable distinction at the level of primitive signs must be understood to imply that in an interpreted deductive calculus each predicative constant will possess a determinate value that is prior to any use of the calculus. If 'A' is a predicative constant and '3' a numeral, then the truth or falsity of the sentence '$A(3)$' will be independent of any effort to assign 'A' an ad hoc predicative value.

To make this critical assessment of what is notationally implied for a standard deductive logic subject to Gödel numbering is to give the essence of the analysis which follows. For certain accepted formulations of Gödel's Proof are characterized by the use of predicative variables or constants as if they were like the functional variables of mathematics in receiving values upon the basis of ad hoc definition. For example, the deductive system to which Gödel [3] applies his original undecidability argument is a typed logic with variables of type n assigned Gödel numbers p, where p is a prime number >13. To the variables of type 1, or numerical variables, there will correspond ordinal numerals of the same type, either zero or zero concatenated with one or more applications of the successor function. However, Gödel lists no predicative constants that correspond to variables of types n, when $n > 1$. As a result, in the schematic sentences or propositional functions of the language, numerical variables can be replaced by numerical constants, but there are no such replacements possible for variables of a higher type. However, any formulation of an undecidable sentence upon which the Proof depends will require the creation of a truth functional sentence containing a predicate symbol of type 2 with a definitive value—a value that is recursively defined by a mathematical process. Strictly speaking, no such sentence can be formulated in a deductive system with no predicative constants, since any variable of the requisite type cannot surrender its formal status as a symbol with the Gödel number of a variable with an indeterminate meaning. It is possible to add predicative constants to such a mathematical calculus and revise the numbering function accordingly. But the difficulty now emerges that such constants cannot be assigned recursive values arbitrarily. To formulate an undecidable sentence based on recursive calculations it is necessary to use a dummy constant letter that is defined as standing for a constant with the requisite value, assuming that such a constant is contained within the supply of constants among the primitive signs of the formal language. Such an assumption rests upon the hypotheses not merely of an infinite supply of numerals for the deductive logic but of an infinite and comprehensive supply of predicates of the requisite type. However, the conditions of the undecidability argument suffice to cast question upon the validity of such an hypothesis. For upon the assumption of consistency *and* decidability for the mathematical calculus, it follows that the predicate necessary for formulation of the undecidable sentence cannot be found within the supply of predicates for the calculus. It is in this context that the undecidability proof can be compared to the class contradiction, and both difficulties taken as evidence against the hypothesis of a total domain for the deductive system in question.

Using the informal exposition of Gödel's Proof by Nagel and Newman [6] we can demonstrate the feasibility of this alternative interpretation. By applying a process of metamathematical reasoning, the authors formulate the following diagonalization function.

(10) $(x) \sim Dem\ (x,\ Sub\ (y,13,y))$.

Such an expression "belongs to the arithmetical calculus," the authors state, even as it conveys the information that for every x there exists no sequence of formulas with Gödel number x that constitutes a proof of the formula obtained from the formula with Gödel number y by replacing any appearance of the variable with Gödel number 13 (the variable 'y') with the numeral for y. Since the logical equivalent of such a formula "has a Gödel number that can actually be calculated," we can replace 'y' by the numeral for such a number, and in this way obtain a formula that occurs within the arithmetical calculus and expresses the claim that it is itself not demonstrable. Since no consistent system can incorporate a proof either for such a formula or for its negation, it follows that if the system in which such a formula appears is consistent, the formula itself must be undecidable.

The source of notational difficulty in such an argument can be located in the claim that an expression conveying the sense of (10) belongs to the arithmetical calculus, a claim which can be interpreted to mean that an expression of the type (11) can be formulated within the calculus.

(11) $(x)(\sim A(x,y))$

In (11) the symbol 'A' stands for some actual predicate of the calculus that is satisfied by x and y only if the arithmetical relationship indicated by '*Dem* $(x,$ *Sub* $(y,13,y))$' in fact holds for x and y. Nagel and Newman include no such predicate in their formal language and follow Gödel in assigning predicative values to second-level variables. From enforcement of the variable-constant distinction it follows that if the mathematical function (10) is representable in the formal calculus there must exist such a predicate as 'A' with the requisite value. However, if this is the case, then the expression (11), containing only 'y' as a free variable, will have a definitive Gödel number that can be designated as p. If we replace 'x' by the numeral for p, then the undecidable sentence (12) can be obtained.

(12) $(x)(\sim A(x,p))$

Such a sentence states that it is itself undemonstrable; thus (12) cannot be false without contradiction and must be true though undecidable. However, the formulation of such a sentence depends upon the existence of such a predicate as 'A', with the requisite predicative value. Given the nature of such a predicate, we cannot prove its existence in the formal calculus except upon the hypothesis of a domain of predicates both infinite in number and unlimited in its capacity to represent all recursively definable sets.

The question which is thus raised is whether or not the arithmetical calculus can contain a comprehensive supply of interpreted predicative constants. This question concerns the nature of deductive systems and cannot be answered by metamathematical analysis. But it can be shown that if the hypothesis of decidability is preferred to that of a total domain, then the proof that (12) is undecidable can be taken as proof that (12) is ill formed within the deductive system—or as proof that no predicate with the value given to 'A' can be included in the formal calculus. The argument that no such predicate exists can be understood to imply that no number p can be the Gödel number of (11). Since the predicative constants of the system will each have a Gödel number, we can arrange such predicates according to the order of their Gödel numbers. Beginning with the first such predicate 'A_1', we find that upon the assumption of consistency and decidability such a predicate cannot be the predicate that represents the requisite arithmetical function. The same argument applies to 'A_2', to 'A_3', and so on for all the predicates in the formal system.

The impossibility of finding an adequate representation of the diagonal function that is crucial to Gödel's Proof is explained by the fact that the Gödel number of any such predicate must enter into the computational process that is indicated by (10). The effect for recursive arithmetic is comparable to a self-contradictory quantification over its own domain, as in the case of the Class Paradox. However, such an indeterminacy does not affect the computational process *per se*, since the diagonalization function will yield a determinate set of numbers for any finite extension of the list of predicates of the formal system. This is because we need not assume, for any finite list of Gödel numbers of predicates, that such a list will contain the Gödel number of the predicate that represents a function which is computed upon the basis of such a list. Since any infinite extension of the list of formal predicates must run parallel to the infinite generation of natural numbers, we have in the conditions of the Gödel argument an effective counter-example to the assumption that in the generation of any recursively enumerable function, the Gödel number for a representing predicate will be subject to computation. Even if a process of computing numbers inductively is infinitely extended, the numbers in question can only contain Gödel numbers for a finite list of formal predicates. But for any such list of predicates the undecidability argument constitutes evidence that it cannot contain a representing predicate for at least one recursively enumerable function—the diagonal function in question. This means that even upon the assumption that the desired predicate is contained within an infinite supply of predicates, its Gödel number will never be subject to calculation, and it can never be determined through calculation whether or not the number computed upon the basis of such a predicate does or does not satisfy the diagonal function. In this respect the assumption of determinacy for the recursive process is tied to the hypothesis of a finite domain for any calculus to which it is related by a process of Gödel numbering. The contrary assumption of an infinite domain permits the conclusion that even if an undecidable sentence in fact exists in the formal logic, its Gödel number cannot be subject to finitary computation.

The above argument suggests the presence of a restriction against the actual formulation of an undecidable sentence in a typed deductive logic, a restriction that is independent of the more general question of whether such a sentence can exist. Yet, the notion that an undecidable sentence must exist in a deductive system is an empty notion if it implies both that such a sentence cannot be in fact formulated and that its Gödel number can never be inductively generated. In fact the effective application of a system of Gödel numbering to a deductive calculus clearly presupposes that if a predicate exists among the primitive predicates of such a logic, it can be found in some finite list of the predicates of the system, for example the list composed of its Gödel number plus predicates with a lesser Gödel number. From this point of view the restriction upon the domain of a deductive calculus can be defined as the fact that well formed expressions of the calculus can contain no expression that cannot be contained within a finite list. As a consequence the domain of the calculus is restricted to those individuals and sets of individuals whose formal representation can be found within some finite list of constants of a given type. The restriction is thus implicit in the concept of truth functional sentences that are deduced from abstraction schemata by the replacement of variables by constants. By contrast, the concept of recursive enumeration permits the 'inductive' representation (by assigning a recursive value arbitrarily to a variable) of sets of numbers that advance beyond any finite list of 'deductive' representations of sets of numbers. There is of course no given set of numbers that eludes deductive representation, since a deductive system can be interpreted so as to contain a constant-symbol with the requisite value. But if an inductive process is tied to a deductive system, as in the case of Gödel enumeration, it is possible to define recursively a set of numbers that is arithmetically definitive only upon the assumption of a finitary limit to the domain of the associated deductive system.

3 To nineteenth-century logicians such as De Morgan and Peirce, a deductive logic that accepts the law of Excluded Middle was regarded as committed to a finite domain. When the "universe of thought" is unlimited, De Morgan writes, "contrary names are of little effective use: not-man, a class containing everything except man, whether seen or thought, is almost useless" ([1], p. 180). To Peirce the requirement of a finite domain was associated with the concept of an enumerable collection, which he defined as a collection subject to a principle of well ordering but which contains a last unit ([7], 4.106). Peirce distinguished the domain of logic from that of mathematics by associating the former with collections incorporating a last unit, and the latter with collections that submit to a principle of ordering but may lack a finite termination. For example, he accepted for logic De Morgan's syllogisms of Transposed Quantity, which include the following:

> Some X is Y
> For every X there is something neither Y nor Z
> Hence, something is neither X nor Z.

Such a syllogism expresses the fact that if there is an exact pairing between X's and something neither Y nor Z, and if the number of X's is finite, then the fact that some X is Y will imply that for at least one X there is no X with which it can be paired. Thus the pairing must be with something that is not Z and yet cannot be X. If applied to the number series, the syllogism yields an odd-even paradox.

> Some odd numbers are prime
> Every odd number has for its square a number not even nor prime
> Hence, some numbers not even are not odd.

The paradox fails upon the recognition that the pairing of odd numbers with their squares is an endless process; only upon the assumption of a last odd number does it follow that such a number must have a square that is neither even nor odd.

De Morgan's syllogism is of interest to contemporary students of logic in that a correlation can be made between its assumption of a finitary limit upon enumerable collections and the restriction that is implied for deductive logic by the practice of generating classes from multiplace conditions upon a single variable. It can be easily shown that some classes must satisfy the criterion by which they are generated. But if the series of classes satisfying the criterion is of infinite length, then no class can be singled out as representing all the classes in such a series. For example, the matrix '$\sim(x \epsilon x)$' can have no '$x \epsilon y$' equivalent if the class y in question must satisfy '$\sim(x \epsilon x)$' and be followed in the series of '$\sim(x \epsilon x)$' classes by some class that includes y as well as any class that y contains. To add the criterion of elementhood to the matrix '$\sim(x \epsilon x)$', as in **ML**, is to legislate the existence of a class satisfying '$\sim(x \epsilon x)$' that contains all other classes in the series of '$\sim(x \epsilon x)$' classes but cannot be followed by any class in which it is itself contained. Such a class will satisfy Peirce's notion of a last unit that defines the collection of '$\sim(x \epsilon x)$' classes as of finite length and in this sense an enumerable collection of classes.

To speak of a limit upon the range of quantifiable variables in a deductive logic is to imply the existence of a means for distinguishing one class value from another, so that an ordered series of such values can in principle be created. In the case of deductive schemata we cannot order values according to lists of class names. But we are able to obtain proof in the untyped schemata of **ML** that no value for y can exist which satisfies the condition '$(x)(x \epsilon y \equiv \sim(x \epsilon x))$'. In the case of Gödel's Proof the definitive condition for the predicate 'A' that represents such a diagonal function as '($\sim$*Dem* (x, *Sub* (y,13,y))' is not stated within the framework of the system, so there is no way to prove absolutely that such a predicate cannot exist. But the status of the system as a typed mathematical calculus with numerical constants will guarantee the existence within the system of a series of interpreted predicate-symbols—a series that according to Gödel methodology must be subject to precise enumeration. Otherwise, predicative claims about specific numerals will lack a determinate truth value. Instead of a technique for distinguishing the condition that the class of self-exclusive classes must satisfy, we have implicit in the system a method for arranging the predicative functions of the type '$(x)(\sim A(x,y))$'—or whatever type is appropriate to the particular undecidability demonstration —in an ordered series that corresponds to the ascending Gödel numbers of the 'A' predicates. The expression of this type that would represent the diagonal function '$\sim$*Dem* (x, *Sub* (y,13,y))' has the same paradoxical relationship to such a list of predicative functions that the name of the class of all self-exclusive classes would have in an ordered list of the names of such classes. If either list is taken to be of infinite length, then for any expression we take to be the requisite expression it can be shown that the expression is not the one that was sought. As a consequence, the choice of any predicate 'A' as the representing predicate proves to contradict the hypothesis of decidability for the calculus. Upon the hypothesis that any list of 'A' predicates is an enumerable list, but with a last unit, the decidability assumption can be seen to hold.

Such demonstrations of a finite domain for deductive logic are linked to the definition of well formed sentences as concatenations of primitive signs where the signs that name classes or predicative conditions must obtain their specific meaning through an initial interpretation of the deductive system as a whole. In the case of logical schemata the commitment is merely to the logical domain *per se*; in the case of an interpreted mathematical calculus it must be to the assignation of particular values to particular constant-symbols of the calculus. It is for this reason that the practice of assigning values arbitrarily to the predicative variables or dummy constants of a typed deductive calculus suffices to blur the status of such a system with respect to vital questions of logical domain. It is a hallmark of inductive mathematical practice to assign meanings to functional variables upon the basis of recursive specification, as in the locution of 'giving a value to a variable'. But it is a requirement for the use of deductive systems that such an assignment of value be understood as the replacement of the variable with a constant that already has the value in question. When the assignment of value is not affected by a domain restriction inherent in the deductive system, the two locutions can be informally assimilated, as in the contemporary custom. But when the domain question is at issue, as in the case of the undecidability argument, it becomes necessary to make the notational discrimination.

Contemporary mathematical logic was developed in connection with an abrupt and largely unquestioned rejection of the nineteenth-century assumption of radical differences between the methodology of deductive logic and of mathematical practice. Once quantification over an infinite range of values was introduced into deductive logic, it became necessary to make a basic discrimination, in Russell's words [10], between statements about *all* such values and statements about *any* particular one—a distinction already used, he tells us, in mathematics. The *all-any* distinction is required in order to deduce directly from general to particular statements, as in the use of Hilbert's [4] ideal proposition '$(n)(n + 1 = 1 + n)$', to validate the inductive rule '$n + 1 = 1 + n$' in its application to any number. In this historical context the fact that a quantification statement transforms a free into a bound variable becomes its most definitive characteristic. Yet in *Principia* and early systems of mathematical logic, the implications of such a discrimination are not fully developed. For example, there is an imprecision in Russell's account of variables that deserves close attention. Strictly speaking, the deduction from a bound to a free variable is not a logical deduction at all unless the free variable is given a specific value, in which case it ceases to be a variable and functions in the manner of a constant or actual proper name. Russell's 'all-any' notion of the role of bound and free variables does not stand close examination, but it is closely related to his notational practice of supplying rules for inference by the use of uninterpreted schemata, where the only way to show the deduction of an 'Fx' statement from '$(x)Fx$' is to represent the former statement as 'Fx' itself, with 'x' taken to have an actual though unstated value. The result is an anomalous role for the free variable 'x' as having an indeterminate value and at the same time serving as the equivalent of a dummy constant.

In quantification systems subsequent to *Principia* the treatment of 'Fx' as a propositional function or open sentence was more rigidly affirmed, as was the deductive principle of supplying a definitive list of primitive symbols for the logical system and generating well formed sentences only as concatenations of such symbols or reducible to symbol concatenations. However, logicians have continued Russell's practice of treating free variables as the equivalent of constant-symbols, especially at higher type levels. The result is an incorporation into informal logical practice of the untested hypothesis of an unrestricted domain. For the question whether class variables can range over all natural classes can be submitted to critical investigation only if it is recognized that in deductive systems with quantifiable variables, only actual predicates or class names can designate specific sets or collections of individuals. The finitary restriction upon any axiomatic logic that contains names of properties or sets is a basic restriction upon what can be deduced from variable schemata by the replacement of free or bound variables by symbols with a definitive meaning. However, any use of such a system—any logical deduction carried out by its means—must result in what may be called a constructed sentence, or a sentence definable as a finite concatenation of particular symbols. If the rule is enforced that sentences expressing nonlogical truths cannot contain free variables, then it becomes evident that such sentences must contain actual predicates or class names, if they make definitive claims about set membership. Although it may be claimed that the variables of a deductive logic have an infinite range, their effective range in the generation of truth-functional sentences is limited, in this sense, to the finite lists of sentences that can in fact be constructed within the logic by a process of actual deduction.

The evidence of the class contradiction and Gödel's Proof tells us that the hypothesis of a domain both infinite and ontologically total for standard deductive logic, by contrast to basic number theory, is an hypothesis that must be rejected, and for reasons that were familiar to nineteenth-century logicians. By metalinguistic operations performed upon any finite list of the class names of an untyped logic we can conceptualize at least one class for which no such name can exist in the logic. And in the case of a typed logic subject to a system of Gödel numbering, our intuition of an unrepresentable set of numbers can be supported by inductive proof that such a set can in fact be recursively defined.

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