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ON UNSOLVABILITY IN SUBRECURSIVE CLASSES OF PREDICATES

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Introduction Decision problems for various classes of sets or languages have been studied extensively in the theory of automata and formal languages [7]. These results have been combined in [3] where certain closure properties important to formal language theory are used to show undecidability in classes meeting certain specifications, and in [6] where the "ability to count" provides the unsolvability. In this paper* closure properties prevalent in mathematical logic will be utilized to relate several decision problems to each other. First, several decision problems for bases of the r.e. sets will be located in the arithmetic hierarchy and then other decision problems will be related to them.

1 *Preliminaries* In the results presented below $\mathcal{R}_*$ will always be a class of recursive predicates [never the class of *all* recursive predicates] for which there is a recursive enumeration [denoted: $R_0, R_1, \ldots$]. The class of predicates $\mathcal{Q}_*$ will always be a subclass of $\mathcal{R}_*$. In addition, the class $\mathcal{R}_*$ will always possess an s_n^m Theorem, which can be stated:

Theorem (s_n^m) *There is a recursive function $s_n^m(i, x_1, \ldots, x_m)$ such that for any R_i in $\mathcal{R}_*$*:

$$R_{s_n^m(i,x_1,\ldots,x_m)}(y_1, \ldots, y_n) \equiv R_i(x_1, \ldots, x_m, y_1, \ldots, y_n).$$

This technical theorem will be useful in proofs since it provides a method to set several variables to specific values.

Several notational conventions should be given at this point. Script letters refer to classes of predicates while capital Roman letters represent predicates and small letters denote functions and variables. The empty set is denoted by $\emptyset$ and $\mathbf{N}$ represents the set of non-negative integers. The sequence $W_0, W_1, \ldots$ is the "standard enumeration" of the recursively

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enumerable sets. For completeness a few definitions from recursive function theory are necessary and are explained more fully in a text such as [13].

Definition The set A is *many-one reducible* [denoted $A \leqslant_m B$] to the set B iff there is a recursive function g such that for all x:

$$x \in A \Longleftrightarrow g(x) \in B.$$

Definition A predicate is in the class $\Sigma_n(\Pi_n)$ iff it can be expressed in prenex normal form with a prefix containing n alternating quantifiers beginning with an existential (universal) quantifier.

The collection of these classes [Σ_n and Π_n] is called the *arithmetic hierarchy*. A set in $\Sigma_n(\Pi_n)$ is called Σ_n-*complete* (Π_n-complete) if every other set in $\Sigma_n(\Pi_n)$ is reducible to it.

Since closure properties play an important role in the results which follow several common ones used with predicates are:

(a) The predicate P is formed from the predicate S by *explicit transformation* iff

$$\mathrm{P}(x_1, \ldots, x_n) \equiv \mathrm{S}(t_1, \ldots, t_m)$$

where each t_i is either an x_i or a constant.
(b) P is formed from S by *finite modification* iff $\mathrm{P}(x) \equiv \mathrm{S}(x)$ for all but a finite number of x.
(c) P is formed from S by *finite quantification* iff either:

$$\mathrm{P}(x, y) \equiv \exists z[z \leqslant x \wedge \mathrm{S}(z, y)],$$

or

$$\mathrm{P}(x, y) \equiv \forall z[z \leqslant x \Longrightarrow \mathrm{S}(z, y)].$$

The operations of *constructive definability* [14] are explicit transformation, finite quantification, and operations involving the propositional connectives [conjunction, disjunction, and negation].

For functions, several closure properties are

(a) f is formed from $h_1, \ldots, h_n$, and g by *composition* iff:

$$f(x_1, \ldots, x_k) = g[h_1(x_1, \ldots, x_k), \ldots, h_n(x_1, \ldots, x_k)].$$

(b) f is formed from g by *finite summation* iff:

$$f(x, y_1, \ldots, y_n) = \sum_{z=0}^{x} g(z, y_1, \ldots, y_n).$$

A class of functions is said to be closed under the operations of *substitution* if it is closed under composition and explicit transformation.

A class of predicates is said to be *effectively closed* under an operation if there is a recursive function which provides the index of the new predicate. That is, effective closure under conjunction requires a recursive g such that for all i, j, and x:

$$R_{g(i,j)}(x) \equiv R_i(x) \wedge R_j(x)$$

Predicates will usually be represented as having one argument and it is assumed that any class examined below possesses the *pairing predicates* which select members of a pair as follows:

$$P_1(n, x) \equiv \exists z \leqslant x[x = \langle n, z\rangle]$$
$$P_2(z, x) \equiv \exists n \leqslant x[x = \langle n, z\rangle].$$

Whenever predicates are written with several parameters [for emphasis on certain components] it is assumed that the "paired" form occurs in the standard enumeration of the class.

The *universal predicate* for the class of predicates $\mathcal{Q}_*$ is a predicate $Q(i, x)$ such that for each predicate in $\mathcal{Q}_*$ there is an integer n such that $Q(n, x)$ is exactly that predicate.

2 *Presenting classes of predicates* Often when a class of predicates is studied, results about its subclasses are proven. One of the first problems is to describe the subclass in terms of the original class of predicates. This may be done by defining a set which contains all of the members of the subclass. If a class of recursive predicates $\mathcal{R}_*$ can be enumerated by the predicates $R_0, R_1, \ldots$ then a subclass $\mathcal{Q}_*$ could be presented within $\mathcal{R}_*$ by a set which contains indices of the R_i which are members of $\mathcal{Q}_*$. Formally:

Definition The set A is a *presentation* of $\mathcal{Q}_*$ within $\mathcal{R}_*$ iff:

(a) For each predicate Q in $\mathcal{Q}_*$ there is an $i \in A$ such that for all x: $R_i(x) \equiv Q(x)$. [That is, A contains an index for each member of $\mathcal{Q}_*$.]

(b) For every $i \in A$, the predicate R_i is a member of the class $\mathcal{Q}_*$.

An interesting property of $\mathcal{Q}_*$ is just how it may be presented. If its presentation [the set A above] is recursively enumerable [r.e.] then the class is said to be *r.e.* also. If the presentation is recursive then the class is said to be *recursively presentable.* For example, the graphs of the recursive functions are not r.e. within the class of r.e. sets but the context sensitive predicates are easily recursively presented by indices of linear bounded automata. The r.e. classes of predicates can be thought of as "well behaved" due to the following straightforward result:

Theorem 1 *An* r.e. *class of predicates has a recursive universal predicate.*

Proof: Let the set A present $\mathcal{Q}_*$ within $\mathcal{R}_*$. If A is empty then the null predicate is universal for $\mathcal{Q}_*$. Otherwise let A be the range of the recursive function g. Then a recursive universal predicate for $\mathcal{Q}_*$ can be defined:

$$Q(i, x) \equiv R_{g(i)}(x).$$

Usually when a class is presented it is completely presented and this presentation is called an index set.

Definition $\Theta\mathscr{Q}_*$ is the *index set* for $\mathscr{Q}_*$ with respect to $\mathscr{R}_*$ and is defined:

$$\Theta\mathscr{Q}_* = \{i \mid \exists Q \in \mathscr{Q}_* \wedge \forall x[Q(x) \equiv R_i(x)]\}.$$

Index sets for many classes have been studied with respect to the class of r.e. sets and will be examined below in connection with subrecursive classes. Several immediate upper bounds in the arithmetic hierarchy for index sets are as follows:

Theorem 2 *Let $\mathscr{Q}_*$ be a subclass of $\mathscr{R}_*$. Then:*

(a) *If $\mathscr{Q}_*$ is trivial [either $\mathscr{Q}_*$ is empty or $\mathscr{Q}_* = \mathscr{R}_*$], then $\Theta\mathscr{Q}_*$ is recursive.*
(b) *If $\mathscr{Q}_*$ is finite, then $\Theta\mathscr{Q}_* \in \Pi_1$.*
(c) *If $\mathscr{Q}_*$ is* r.e., *then $\Theta\mathscr{Q}_* \in \Sigma_2$.*

Proof: (a) In this case $\Theta\mathscr{Q}_*$ is either $\emptyset$ or $\mathbf{N}$ and thus recursive.
(b) Let $R_{k_1}, \ldots, R_{k_n}$ be the members of $\mathscr{Q}_*$. Then membership in $\mathscr{Q}_*$ is provided by the Π_1 predicate:

$$i \in \Theta\mathscr{Q}_* \Longleftrightarrow \forall x[R_i(x) \equiv R_{k_1}(x)] \vee \ldots \vee \forall x[R_i(x) \equiv R_{k_n}(x)].$$

(c) If $\mathscr{Q}_*$ is r.e. then it has a recursive universal predicate [by the previous theorem]. Letting this predicate be Q, the Σ_2 predicate:

$$i \in \Theta\mathscr{Q}_* \Longleftrightarrow \exists n \forall x[R_i(x) \equiv Q(n, x)]$$

denotes membership in $\Theta\mathscr{Q}_*$.

3 *Bases and standard index sets* In this section several index sets will be located in the arithmetic hierarchy. Knowing where these sets are will help in locating other, more general sets. These "reference sets" [within $\mathscr{R}_* = \{R_0, R_1, \ldots\}$] are:

(a) $\Theta\emptyset = \{i \mid \forall x \neg R_i(x)\}$
(b) $\Theta\mathbf{N} = \{i \mid \forall x\, R_i(x)\}$
(c) $\Theta\,\text{Equal} = \{\langle i, j\rangle \mid \forall x[R_i(x) \equiv R_j(x)]\}$
(d) $\Theta\,\text{Finite} = \{i \mid R_i(x)$ for a finite number of $x\} = \{i \mid \exists z \forall x[x > z \Longrightarrow \neg R_i(x)]\}$
(e) $\Theta\,\text{Cofinite} = \{i \mid \neg R_i(x)$ for a finite number of $x\} = \{i \mid \exists z\, \forall x[x > z \Longrightarrow R_i(x)]\}$

It should be obvious that the upper bound in the arithmetic hierarchy for $\Theta\emptyset$, $\Theta\mathbf{N}$, and Θ Equal is Π_1 and the upper bound for Θ Finite and Θ Cofinite is Σ_2. If $\mathscr{R}_*$ possesses several specific closure properties then there are several immediate equivalences among these index sets [effective closure under complement insures that $\Theta\emptyset \equiv_m \Theta\mathbf{N}$ and Θ Finite $\equiv_m \Theta$ Cofinite] but the only immediate relationships are between $\Theta\mathbf{N}$, $\Theta\emptyset$, and Θ Equal. [It is assumed throughout this section that these index sets are non-trivial in $\mathscr{R}_*$.]

Lemma $\Theta\emptyset \leqslant_m \Theta$ Equal *and* $\Theta\mathbf{N} \leqslant_m \Theta$ Equal.

Proof: If $k \in \Theta\emptyset$ then $\Theta\emptyset \leqslant_m \Theta$ Equal via:

$$g(i) = \lambda i \langle i, k\rangle$$

Since: $i \in \Theta\emptyset \Longleftrightarrow \forall x \neg R_i(x) \Longleftrightarrow \forall x[R_i(x) \equiv R_k(x)] \Longleftrightarrow \langle i, k\rangle \in \Theta$ Equal.

The proof of $\Theta\mathbf{N} \leqslant_m \Theta$ Equal is quite similar.

Many classes of predicates can be used to characterize the r.e. sets and knowing this about a class provides exact placement of the above index sets in the arithmetic hierarchy. A class is said to be a basis for the r.e. sets if it is powerful enough to yield them when closed under existential quantification [14].

Definition The class of predicates $\mathcal{R}_*$ is a *basis for the r.e. sets* iff for each r.e. set W_i there is a predicate T_i in $\mathcal{R}_*$ such that for all x:

$$x \in W_i \Longleftrightarrow \exists x \ \mathrm{T}_i(x, z).$$

These T-predicates are usually known as Kleene T-predicates and can be intuitively described as:

$$\mathrm{T}_i(x, z) \equiv \text{Turing machine } \mathbf{M}_i \text{ halts in } z \text{ steps for the input } x.$$

The class $\mathcal{R}_*$ is known as an *effective basis for the r.e. sets* if there is a recursive function g such that:

$$\forall x, z\left[\mathrm{R}_{g(i)}(x, z) \equiv \mathrm{T}_i(x, z)\right].$$

[In other words, the T-predicates can be found effectively within $\mathcal{R}_*$.] Once a class of predicates is known to be a basis for the r.e. sets the positions of the index sets defined above are easily established.

Theorem 3 *If $\mathcal{R}_*$ is effectively closed under explicit transformations and is an effective basis for the* r.e. *sets then:*

(a) $\Theta\emptyset$, $\Theta\mathbf{N}$, *and* Θ Equal *are Π_1-complete;*
(b) Θ Finite *and* Θ Cofinite *are Σ_2-complete.*

Proof: Let the recursive function g provide the T-predicates, i.e., for all x and z:

$$\mathrm{R}_{g(i)}(x, z) \equiv \mathrm{T}_i(x, z).$$

(a) The Π_1-complete set

$$\overline{K} = \{i \mid \forall x \neg \mathrm{T}_i(i, x)\}$$

can be reduced to $\Theta\emptyset$ via f which is defined:

$$\mathrm{R}_{f(i)}(x) \equiv \mathrm{R}_{g(i)}(i, x).$$

Thus:

$$i \in \overline{K} \Longleftrightarrow \forall x \neg \mathrm{T}_i(i, x) \Longleftrightarrow \forall x \neg \mathrm{R}_{g(i)}(i, x) \Longleftrightarrow \forall x \neg \mathrm{R}_{f(i)}(x) \Longleftrightarrow f(i) \in \Theta\emptyset$$

and since $\overline{K} \leq_{\mathrm{m}} \Theta\emptyset$ and $\overline{K}$ is Π_1-complete then so is $\Theta\emptyset$.

Another Π_1-complete set is

$$A = \{i \mid \forall x \ \mathrm{T}_i(x, x)\}$$

which is just the indices of all Turing machines which halt in x steps for input x [9]. This set can be reduced to $\Theta\mathbf{N}$ via the recursive function f which is defined:

$$R_{f(i)}(x) \equiv R_{g(i)}(x, x).$$

Since:

$$i \in A \Longleftrightarrow \forall x\, T_i(x, x) \Longleftrightarrow \forall x\, R_{g(i)}(x, x) \Longleftrightarrow \forall x\, R_{f(i)}(x) \Longleftrightarrow f(i) \in \Theta \mathbf{N}$$

then $A \leqslant_m \Theta \mathbf{N}$ and thus $\Theta\mathbf{N}$ is Π_1-complete.

Θ Equal is also Π_1-complete because $\Theta\mathbf{N} \leqslant_m \Theta$ Equal and Θ Equal is a Π_1 set.

(b) A Σ_2-complete set which can be reduced to Θ Cofinite is

$$A = \{i \mid \exists n \forall x [x > n \Longrightarrow T_i(x, x)]\}$$

or the indices of all Turing machines which halt in x steps for all but a finite number of inputs [9]. The reduction is via f where:

$$R_{f(i)}(x) \equiv R_{g(i)}(x, x)$$

The placement of Θ Finite is quite similar.

Many well known classes of predicates are bases for the r.e. sets [such as the primitive recursive predicates, context sensitive languages, and all of the classes in the Grzegorczyk hierarchy] but some classes of predicates which are not have many subclasses with nonrecursive index sets. An example is the context free languages for which $\Theta\emptyset$ and Θ Finite are recursive but $\Theta\mathbf{N}$ and Θ Cofinite are not. This happens because they are a basis for the co-r.e. sets [5].

Definition The class of predicates $\mathcal{R}_*$ is an *effective basis for the co-r.e. sets* iff there is a recursive g such that for each r.e. set W_i and all x:

$$x \notin W_i \Longleftrightarrow \forall z\, R_{g(i)}(x, z).$$

[In this case the $R_{g(i)}$ predicates compute the negation of the T-predicates.]

Theorem 4 *If $\mathcal{R}_*$ is an effective basis for the* co-r.e. *sets and is effectively closed under explicit transformations then:*

(a) $\Theta\mathbf{N}$ *and* Θ Equal *are* Π_1*-complete;*
(b) Θ Cofinite *is* Σ_2*-complete.*

Proof: Let the recursive function g provide the negations of T-predicates, i.e., for all x and z:

$$R_{g(i)}(x, z) \equiv \neg T_i(x, z).$$

(a) Again $\overline{K}$ will be used in the reduction, this time to $\Theta\mathbf{N}$ via f which is defined:

$$R_{f(i)}(x) \equiv R_{g(i)}(i, x).$$

Since:

$$i \in \overline{K} \Longleftrightarrow \forall x \neg T_i(i, x) \Longleftrightarrow \forall x\, R_{g(i)}(i, x) \Longleftrightarrow \forall x\, R_{f(i)}(x) \Longleftrightarrow f(i) \in \Theta\mathbf{N}$$

then $\overline{K} \leqslant_m \Theta\mathbf{N}$ and $\Theta\mathbf{N}$ is Π_1-complete.

Θ Equal is Π_1-complete because $\Theta\mathbf{N} \leqslant_m \Theta$ Equal and Θ Equal is in Π_1.

(b) First, consider the set:

$$A = \{i \mid W_i \text{ is finite and } \forall x[x \in W_i \Rightarrow \mathrm{T}_i(x, x)]\}$$

The set $B = \{i \mid W_i$ is finite$\}$ can be reduced to A by means of the following procedure which defines the Turing machine $\mathbf{M}_{f(i)}$ on input x:

> Write down a description of the Turing machine $\mathbf{M}_i$—if this takes more than x steps then diverge, otherwise begin computing $\mathbf{M}_i(0)$, $\mathbf{M}_i(1)$, . . . in the following manner: one step of $\mathbf{M}_i(0)$, one step of $\mathbf{M}_i(1)$, two steps of $\mathbf{M}_i(0)$, one step of $\mathbf{M}_i(2)$, two steps of $\mathbf{M}_i(1)$, three steps of $\mathbf{M}_i(0)$, and so forth until $\mathbf{M}_{f(i)}$ has run for x steps. If $\mathbf{M}_i$ has accepted an input that was not detected during the computation of $\mathbf{M}_{f(i)}(0)$, . . ., $\mathbf{M}_{f(i)}(x - 1)$ then halt, otherwise diverge.

Note that $\mathbf{M}_{f(i)}$ halts for x in x steps or less whenever it halts and accepts a finite set iff $\mathbf{M}_i$ accepted one. Thus $B \leqslant_m A$ via f and so A is Σ_2-complete.

The reduction of A to Θ Cofinite is accomplished by means of the recursive f defined as follows:

$$\mathrm{R}_{f(i)}(x) \equiv \mathrm{R}_{g(i)}(x, x).$$

Recalling that $\mathrm{R}_{g(i)}(x, z) \equiv \neg \mathrm{T}_i(x, z)$ and:

$$z \leqslant x \wedge \mathrm{T}_i(x, z) \Rightarrow \mathrm{T}_i(x, x)$$

one arrives at the fact that:

$$\begin{aligned} i \in A &\Leftrightarrow \exists n \forall x, z[\mathrm{T}_i(x, z) \Rightarrow z \leqslant x \leqslant n] \\ &\Leftrightarrow \exists n \forall x[\mathrm{T}_i(x, x) \Rightarrow x \leqslant n] \\ &\Leftrightarrow \exists n \forall x[\neg \mathrm{R}_{g(i)}(x, x) \Rightarrow x \leqslant n] \\ &\Leftrightarrow \exists n \forall x[\neg \mathrm{R}_{f(i)}(x) \Rightarrow x \leqslant n] \\ &\Leftrightarrow f(i) \in \Theta \text{ Cofinite.} \end{aligned}$$

Since A is Σ_2-complete so is Θ Cofinite.

The context free languages form a basis for the co-r.e. sets [5] if one thinks of them as multi-parametered predicates under some simple string-theoretic pairing function, and therefore their Σ^* problem [i.e., $\Theta\mathbf{N}$] is Π_1-complete while their cofiniteness problem [Θ Cofinite] is Σ_2-complete. As mentioned above, it is quite well known that $\Theta\emptyset$ and Θ Finite are recursive for the context free languages [7].

4 *General index sets* Several index sets were classified in the last section so that arbitrary index sets could be located in the arithmetic hierarchy relative these "standard" sets. This can be done rather easily if some information is given about the class of predicates being examined and the index set which is to be classified. Again it is assumed that $\mathcal{R}_*$ is an enumerable class of recursive predicates [represented by R_0, R_1, . . .] and that $\mathcal{Q}_*$ is a non-trivial subclass of $\mathcal{R}_*$.

Theorem 5 *Let* $\mathcal{R}_*$ *be effectively closed under propositional connectives and finite quantification, and let either of the following conditions hold:*

(a) $\mathcal{Q}_*$ *contains no finite predicates;*
(b) $\mathcal{Q}_*$ *is closed under finite modification.*

Then for any non-trivial subclass $\mathcal{Q}_*$ *of* $\mathcal{R}_*$, $\Theta\emptyset \leq_m \Theta\mathcal{Q}_*$.

Proof: Since $\mathcal{Q}_*$ is a non-trivial subclass of $\mathcal{R}_*$ let k be a member of $\Theta\mathcal{Q}_*$ and let l be a member of $\overline{\Theta\mathcal{Q}_*}$. The theorem can now be proven by reduction.

(a) Given a predicate R_i a predicate $R_{g(i)}$ is to be constructed such that if R_i is the null predicate [i.e., $R_i(x)$ is always false] then $R_{g(i)}$ will be a predicate of $\mathcal{Q}_*$, namely R_k. $R_{g(i)}$ is defined as follows:

$$R_{g(i)}(x) \equiv R_k(x) \wedge \forall z \leq x \neg R_i(z)$$

If R_i is the null predicate then $R_{g(i)}$ will be true only where R_k is true. Thus:

$$i \in \Theta\emptyset \Rightarrow \forall x \neg R_i(x) \Rightarrow \forall x\,[R_{g(i)}(x) \equiv R_k(x)] \Rightarrow g(i) \in \Theta\mathcal{Q}_*.$$

But if there is some x for which $R_i(x)$ is true this forces $R_{g(i)}(z)$ to be false for all values of z greater than x, and so:

$$\begin{aligned} i \notin \Theta\emptyset &\Rightarrow \exists x\, R_i(x) \Rightarrow \exists x \forall z \leq x \neg R_{g(i)}(x) \\ &\Rightarrow R_{g(i)} \text{ is a finite predicate} \Rightarrow g(i) \notin \Theta\mathcal{Q}_*. \end{aligned}$$

Therefore $\Theta\emptyset$ is reducible to $\Theta\mathcal{Q}_*$.

(b) In this case if R_i is the null predicate $R_{g(i)}$ will be R_k as before but if for some x; $R_i(x)$ is true then $R_{g(i)}(z)$ will be the same as $R_l(z)$ for all z which are greater than x. This is accomplished by the definition of $R_{g(i)}$ as:

$$R_{g(i)}(x) \equiv [R_k(x) \wedge \forall z \leq x \neg R_i(z)] \vee [R_l(x) \wedge \exists z \leq x\, R_i(z)]$$

If R_i is the null predicate then $R_{g(i)}$ is the same as R_k [as in part (a)] but if $R_i(x)$ is true for some x then $R_{g(i)}$ is a finite modification of R_l and therefore $g(i)$ is a member of $\overline{\Theta\mathcal{Q}_*}$. Thus $\Theta\emptyset$ is reducible to $\Theta\mathcal{Q}_*$.

The method used in the previous proof was the selection of a member of $\mathcal{Q}_*$ if R_i was null and a member of $\mathcal{R}_* - \mathcal{Q}_*$ if R_i was true somewhere. This procedure required some knowledge of either the membership of $\mathcal{Q}_*$ or at least one of its closure properties. However, if $\mathcal{R}_*$ contains the machinery needed to diagonalize over $\mathcal{Q}_*$ then nothing need be known about $\mathcal{Q}_*$ other than the fact that it is non-trivial.

Theorem 6 *Let* $\mathcal{R}_*$ *be a class of recursive predicates which is effectively closed under propositional connectives and finite quantification and contains the pairing predicates. Then,* $\Theta\emptyset$ *is many-one reducible to the index set of any non-trivial subclass of* $\mathcal{R}_*$ *whose universal predicate is in* $\mathcal{R}_*$.

Proof: Let $\mathcal{Q}_*$ be a non-trivial subclass of $\mathcal{R}_*$, let $Q(i, x)$ be its universal predicate, and let $k \in \Theta\mathcal{Q}_*$. First a predicate will be defined [as in the previous theorem] which will be the same as R_k if the predicate R_i is null [for arbitrary i]. This predicate is:

$$S(i, x) \equiv R_k(x) \wedge \forall z \leq x \neg R_i(z),$$

and it should be obvious that:

$$i \in \Theta\emptyset \Rightarrow \forall x[S(i, x) \equiv R_k(x)] \Rightarrow S(i, x) \in \mathcal{Q}_*.$$

Next a predicate is defined which will diagonalize over $\mathcal{Q}_*$ whenever there is some z such that $R_i(z)$ is true. This is defined for $x = \langle n, z\rangle$ as:

$$D(i, \langle n, z\rangle) \equiv R_i(z) \wedge \neg Q(n, x),$$

or

$$D(i, x) \equiv \exists n, z \leq x[P_1(n, x) \wedge P_2(z, x) \wedge R_i(z) \wedge \neg Q(n, x)].$$

It is claimed that whenever $i \notin \Theta\emptyset$, the predicate $D(i, x)$ cannot be in $\mathcal{Q}_*$. Assuming that $i \notin \Theta\emptyset$ and that $D(i, x)$ is in $\mathcal{Q}_*$ then there must be:

(a) a z such that $R_i(z)$ is true,

and

(b) an n such that $\forall x[D(i, x) \equiv Q(n, x)]$.

Consider $D(i, x)$ for $x = \langle n, z\rangle$:

$$D(i, x) \equiv D(i, \langle n, z\rangle) \equiv R_i(z) \wedge \neg Q(n, x) \equiv \neg Q(n, x) \equiv \neg D(i, x).$$

This contradiction indicates that:

$$i \notin \Theta\emptyset \Rightarrow D(i, x) \notin \mathcal{Q}_*.$$

Since some form of the s_n^m Theorem is true for $\mathcal{R}_*$, there are recursive functions f and h such that for all i and x:

$$R_{f(i)}(x) \equiv S(i, x) \text{ and } R_{h(i)}(x) \equiv D(i, x).$$

Thus $\Theta\emptyset$ is many-one reducible to $\Theta\mathcal{Q}_*$ via the recursive function g which is defined as:

$$\begin{aligned} R_{g(i)}(x) &\equiv S(i, x) \vee D(i, x) \\ &\equiv R_{f(i)}(x) \vee R_{h(i)}(x). \end{aligned}$$

The previous theorems indicate that in order to classify $\Theta\mathcal{Q}_*$ [where $\mathcal{Q}_*$ is a subclass of $\mathcal{R}_*$] either a little information must be available about both classes or a lot of power must be present in $\mathcal{R}_*$. The next result shows the other side of this trade-off: that when many things are known about $\mathcal{Q}_*$, then only a minimal amount of machinery need be contained in $\mathcal{R}_*$.

Theorem 7 *If $\mathcal{Q}_*$ is a non-trivial subclass of $\mathcal{R}_*$ containing the null predicate, $\mathcal{R}_*$ is effectively closed under finite existential quantification, and either of the following holds:*

(a) $\mathcal{Q}_*$ *is closed under finite modification and* $\mathcal{R}_*$ *is effectively closed under conjunction;*
(b) $\mathcal{Q}_*$ *contains no cofinite predicates;*

then $\Theta\emptyset \leqslant_{\mathrm{m}} \Theta\mathcal{Q}_*$.

Proof: Again the required reductions are accomplished by means of predicate constructions. Assuming that $l \notin \Theta\mathcal{Q}_*$, these are:

(a) $\mathrm{R}_{g(i)}(x) \equiv \mathrm{R}_l(x) \wedge \exists z \leqslant x \mathrm{R}_i(z)$
(b) $\mathrm{R}_{g(i)}(x) \equiv \exists z \leqslant x \mathrm{R}_i(z)$.

The details of the arguments are similar to those used in the previous theorems.

The same kind of classification may be accomplished with the other "standard" Π_1-complete index set: $\Theta\mathrm{N}$. A result which expresses resource trade-offs like the previous ones is stated in the following theorem.

Theorem 8 *If* $\mathcal{Q}_*$ *is a non-trivial subclass of* $\mathcal{R}_*$ *containing no finite predicates,* $\mathcal{R}_*$ *is effectively closed under finite universal quantification, and either of the following holds:*

(a) $\mathcal{R}_*$ *is effectively closed under conjunction;*
(b) $\mathcal{Q}_*$ *contains the "always true" predicate;*

then $\Theta\mathrm{N} \leqslant_{\mathrm{m}} \Theta\mathcal{Q}_*$.

Proof: By the reductions:

(a) $\mathrm{R}_{g(i)}(x) \equiv \mathrm{R}_k(x) \wedge \forall z \leqslant x \mathrm{R}_i(z)$ for some $k \in \Theta\mathcal{Q}_*$,
(b) $\mathrm{R}_{g(i)}(x) \equiv \forall z \leqslant x \mathrm{R}_i(z)$.

5 *Using characteristic functions* In this section, Θ Finite and Θ Cofinite will be reduced to index sets of classes of predicates. The major technique used will be diagonalization, but these results differ from Theorem 6 in that counting is used as a major tool in the diagonalization. To accomplish this with ease much of the work will be done within the class of functions related to the class of predicates being examined. If $\mathcal{R}$ is a class of recursive functions then $\mathcal{R}_*$ is the class of predicates whose characteristic functions are in $\mathcal{R}$. It is assumed that the class $\mathcal{R}$ is recursively enumerable and that its standard enumeration is represented by the sequence of functions $r_0, r_1, \ldots$. An enumeration of $\mathcal{R}_*$ can then be specified as:

$$\mathrm{R}_i(x) \equiv r_i(x) > 0.$$

A universal function for a subclass $\mathcal{Q}_*$ of $\mathcal{R}_*$ is of course the characteristic function for a universal predicate of $\mathcal{Q}_*$. Two special functions used in the next proof are the "sign functions" which are defined:

$$\mathrm{sgn}(0) = 0$$
$$\mathrm{sgn}(x + 1) = 1,$$

and

$$\overline{\mathrm{sgn}}(0) = 1$$
$$\overline{\mathrm{sgn}}(x+1) = 0.$$

Since reductions from Θ Finite and Θ Cofinite to an index set can be quite similar only the result involving Θ Finite is given here.

Theorem 9 *Let $\mathcal{R}$ be a class of recursive functions which contains the sign functions, is effectively closed under composition and finite summation, and whose class of predicates $\mathcal{R}_*$ is effectively closed under propositional connectives. Then Θ Finite is many-one reducible to the index set of any non-trivial subclass of $\mathcal{R}_*$ which is closed under finite modification and whose universal function is in $\mathcal{R}$.*

Proof: Let $\mathcal{Q}_*$ be a non-trivial subclass of $\mathcal{R}_*$ whose universal function is $q(i, x)$, and let $k \in \Theta\mathcal{Q}_*$. First a predicate is defined which will be a finite modification of R_k if R_i is true for only a finite number of x. This predicate is:

$$\mathrm{S}(i, x) \equiv \mathrm{R}_k(x) \wedge \neg \mathrm{R}_i(x)$$

and obviously:

$$\begin{aligned} i \in \Theta \text{ Finite} &\Rightarrow \exists x \forall z [z > x \Rightarrow \neg \mathrm{R}_i(z)] \\ &\Rightarrow \exists x \forall z [z > x \Rightarrow \mathrm{S}(i, z) \equiv \mathrm{R}_k(z)] \\ &\Rightarrow \mathrm{S}(i, x) \in \mathcal{Q}_*. \end{aligned}$$

Next a function is defined which counts the number of times R_i has been true on values less than or equal to x. This is

$$n(i, x) = \sum_{z=0}^{x} \mathrm{sgn}[r_i(z)]$$

and it is combined with the universal function to yield:

$$d(i, x) = \overline{\mathrm{sgn}}[q(n(i, x), x)].$$

Let $d(i, x)$ be the characteristic function for the predicate $\mathrm{D}(i, x)$, which of course is in $\mathcal{R}_*$ since $d(i, x)$ is a member of $\mathcal{R}$. It should be noted that if R_i is true for an infinite number of values then every integer occurs in the range of $n(i, x)$ and thus the predicate $\mathrm{D}(i, x)$ diagonalizes over $\mathcal{Q}_*$. Therefore:

$$i \notin \Theta \text{ Finite} \Rightarrow \mathrm{D}(i, x) \notin \mathcal{Q}_*.$$

The reduction is accomplished via the recursive function g which is defined:

$$\mathrm{R}_{g(i)}(x) \equiv \mathrm{S}(i, x) \vee [\mathrm{D}(i, x) \wedge \mathrm{R}_i(x)].$$

This theorem can be applied to the problem of deciding whether a deterministic linear bounded automaton accepts a regular set. The deterministic lba form is a basis for the r.e. sets [8], [10], and meet all of the conditions of the theorem [12]. Therefore Θ Finite is Σ_2-complete by Theorem 3. Since the regular sets are closed under finite modification and have a universal function which can be computed by a linear bounded automaton, Θ Finite $\leqslant_m \Theta$ Regular. The regular sets are an r.e. class; thus Θ Regular is in Σ_2 and is therefore Σ_2-complete.

6 *Discussion and open problems* One thing which should be noted is that in several of the above theorems the restrictions placed on classes of predicates were often too severe. When classes were required to be effectively closed under explicit transformation usually only the substitution of a constant was used in the proof. Also, instead of finite quantification, some sort of bounded [by an increasing recursive function] quantification would have sufficed. Thus any computational complexity class which is a basis for the r.e. sets (such as the class of polynomial time computable functions [1]) will satisfy the above theorems.

Several open problems related to this material are:

(a) When the universal predicate for a class of functions is contained in that class, what does this imply about its index set?

(b) The basic unsolvability result concerning sets of indices of r.e. sets is due to Rice [11]. It states that the only recursive index sets are $\emptyset$ and $\mathbf{N}$. A similar result for index sets in subrecursive classes would be quite interesting. An approximation to the desired result might be:

Theorem 10 *If $\mathcal{R}_*$ is an effective basis for the* r.e. *sets which is effectively closed under the operations of constructive definability, then its only recursive index sets which are closed under finite modification are $\emptyset$ and $\mathbf{N}$.*

Proof: The argument is identical to that of Rice. Let A be a non-trivial index set of $\mathcal{R}_*$. Then if $\Theta\emptyset \subseteq A$, let l be an index not in A and define:

$$R_{g(i)}(x) \equiv \exists z \leqslant x\, T_i(i, z) \wedge R_l(x).$$

Upon examination one discovers that:

$$i \in \overline{K} \Rightarrow g(i) \in \Theta\emptyset.$$

while

$$i \in K \Rightarrow R_{g(i)} \text{ is a finite modification of } R_l.$$

Therefore A cannot be recursive.

(c) A study of class properties such as recursively enumerable, recursively presentable, and completely recursively enumerable might be carried out along the lines established by Dekker and Myhill [2].

(d) A notion of subrecursively inseparable could easily be defined for these classes and possibly used in computational complexity. [e.g., if A and B are index sets which are subrecursively inseparable within $\mathcal{R}$ then they must contain indices of predicates which are not in $\mathcal{R}$.]

(e) Can subpart quantification [$\exists x$ such that x is a substring of y] be used in place of finite quantification in the above results?

(f) What interesting observations emerge when Turing reducibility rather than many-one reducibility is used? An easy result is that if Θ Equal is in Π_1 (as it is when one considers a basis for the r.e. sets) then no r.e. class has its index set above Σ_2. And obviously no class which is both r.e. and co-r.e. has an index set above Π_1.

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