Notre Dame Journal of Formal Logic
Volume 26, Number 4, October 1985

Definable Partitions and Reflection Properties for Regular Cardinals

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The purpose of the present paper is to study the relation between definable partitions and reflection properties of regular cardinals. It turns out that in contrast to Σ_1^1 reflection, which does not lead to a large cardinal axiom (see Section 2), Π_1^1 reflection, which is studied in association with definable stationary subsets of κ (see Section 3) and definable partition properties (see Section 4), leads to a large cardinal axiom. In particular it follows (see Section 4) that the least regular uncountable cardinal which satisfies a certain partition relation lies strictly between the first uncountable inaccessible and the first uncountable Mahlo cardinal (assuming the axiom of constructibility $V = L$).

1 Introduction and preliminaries The Jensen hierarchy $(J_\alpha : \alpha \in \mathrm{Ord})$ of constructible sets is defined in [2]. L is the universe of constructible sets. Only structures of the form $\mathbf{M} = (M, \in, R_1, \ldots, R_r)$ will be considered, where M is a nonempty set and $R_1, \ldots, R_r$ are relations on M. The Levy hierarchies Σ_n, Π_n of formulas in the language with predicate symbols $\in, S_1, \ldots, S_n$ (the arity of each S_i is the same as the arity of R_i), and the corresponding sets of $\Sigma_n(\mathbf{M})$, $\Pi_n(\mathbf{M})$, $\Delta_n(\mathbf{M})$ of relations on the set M, are defined as usual (see [2]). A formula ϕ is a first-order formula if it is in Σ_n, for some $n \geq 0$. The set of first-order formulas is denoted by Σ_ω. Any formula of the form $\exists V_1 \ldots \exists V_m \phi$, $\forall V_1 \ldots \forall V_m \phi$, where the formula $\phi = \phi(V_1, \ldots, V_m, x_1, \ldots, x_k)$ is first order, $V_1, \ldots, V_m$ are second-order variables, $x_1, \ldots, x_k$ are first-order variables, is respectively called Σ_1^1, Π_1^1.

*The present research was carried out at the Universität Heidelberg. During the preparation of this paper the author was supported by the Minna James Heinaman Stiftung Hannover. I would like to thank I. Phillips for pointing out numerous errors on earlier drafts of the paper.

Received September 14, 1983; revised October 8, 1984

The structure $\mathbf{M} = (M, \in, R_1, \ldots, R_m)$ will usually be abbreviated by $\mathbf{M} = (M, R_1, \ldots, R_m)$. The symbol $\mathbf{M} \prec_n \mathbf{N}$ means that $M \subseteq N$ and the structures $\mathbf{M}$, $\mathbf{N}$ satisfy exactly the same Σ_n formulas with parameters in M. If $J\alpha \prec_n J_\beta$ then α is called $\Sigma_n - \beta$-stable; the set of all $\Sigma_n - \beta$-stables $< \alpha$ is denoted by S_β^n. It is well known (e.g., see [4]) that for each $n \geq 1$ there exists a Π_n formula $\phi_n(v)$, without any parameters, such that for all $\alpha < \beta$, $J_\beta \models \phi_n(\alpha) \Leftrightarrow J_\alpha \prec_n J_\beta$. The symbol $\mathbf{M} \prec \mathbf{N}$ means that $\mathbf{M} \prec_n \mathbf{N}$, for all $n \geq 0$.

The concept of reflection was introduced in [5] in order to characterize the closure ordinals of certain inductive definitions. If Φ is a class of formulas (in the given $\in$-language) and X is a nonempty subset of α, then α is called Φ reflecting on X if and only if for any formula $\phi(v_1, \ldots, v_k)$ and any parameters $a_1, \ldots, a_k \in J_\alpha$, $J_\alpha \models \phi(a_1, \ldots, a_k) \Rightarrow (\exists \beta \in X \cap \alpha)\, J_\beta \models \phi(a_1, \ldots, a_k)$. If in the above definition $X = \alpha$ then α is called Φ reflecting.

For any nonempty set X let $[X]^2$ be the set of all unordered pairs of elements from X. The partition symbol $\alpha \xrightarrow{\Sigma_n} (\alpha)_2^2$ means that every $\Sigma_n(J_\alpha)$ function $h : [\alpha]^2 \to 2$ has a homogeneous set $H \subseteq \alpha$ (i.e., the set $h''[H]^2$ is a singleton) of order type α. The partition symbol $\alpha \xrightarrow{\Sigma_n} (\alpha - \Sigma_\omega)_2^2$ means that every $\Sigma_n(J_\alpha)$ function $h : [\alpha]^2 \to 2$ has a $\Sigma_\omega(J_\alpha)$ definable homogeneous set $H \subseteq \alpha$ (i.e., the set $h''[H]^2$ is a singleton) of order type α.

Throughout the present paper κ will always denote a regular uncountable cardinal. The concepts of Inaccessible, Mahlo, as well as stationary subset of κ can be found in any standard book on set theory (e.g., [3]). Knowledge of the fine structure of L will be essential (see [2]).

2 Σ_1^1 reflection This section clarifies the differences between Π_1^1 and Σ_1^1 reflection.

Theorem 2.1 *Every uncountable cardinal is Σ_1^1 reflecting.*

Proof: Let α be an uncountable cardinal and $\phi(S_1, \ldots, S_m)$ a first-order formula with parameters in J_α such that $J_\alpha \models (\exists S_1) \ldots (\exists S_m)\phi(S_1, \ldots, S_m)$. Also, let $R_1, \ldots, R_m \subseteq J_\alpha$ such that $(J_\alpha, R_1, \ldots, R_m) \models \phi(R_1, \ldots, R_m)$. By the Löwenheim-Skolem theorem, there exists a structure

$$\mathbf{M} = (M, P_1, \ldots, P_m) \prec (J_\alpha, R_1, \ldots, R_m)$$

of cardinality less than α such that M contains the transitive closure of the set which contains all the parameters occurring in ϕ. Using Jensen's condensation lemma, one can find an ordinal $\beta < \alpha$ and $T_1, \ldots, T_m \subseteq J_\beta$ such that the structures $\mathbf{M}$ and $(J_\beta, T_1, \ldots, T_m)$ are isomorphic. It follows that J_β satisfies the formula $(\exists S_1) \ldots (\exists S_m)\phi(S_1, \ldots, S_m)$, and the proof is complete.

Theorem 2.2 considers a Σ_1^1 property of κ, assuming that κ is Mahlo (the proof arose after a discussion with P. Welch). IN_κ denotes the set of inaccessibles below κ.

Theorem 2.2 *$(V = L)$ If κ is Mahlo then κ is Σ_1^1 reflecting on IN_κ.*

Proof: As in [2] one constructs a sequence of elementary submodels of J_{κ^+} by induction as follows:

$$N_0 = \text{smallest } N \prec J_{\kappa^+} \text{ such that } N \cap \kappa \in \text{Ord}, \; N_\lambda = \bigcup_{\nu<\lambda} N_\nu, \text{ for } \lambda \text{ limit },$$

$$N_{\nu+1} = \text{smallest } N \prec J_{\kappa^+} \text{ such that } N \cap \kappa \in \text{Ord and } N_\nu \cup \{N_\nu\} \subseteq N .$$

The sequence $(\alpha_\nu = N_\nu \cap \kappa : \nu < \kappa)$ is normal; hence there exists an innaccessible cardinal $\nu < \kappa$ such that $\nu = \alpha_\nu$. If π is the transitive collapse $\pi : N_\nu \cong J_\gamma$ then $\pi(\kappa) = \nu$. To show that κ is Σ^1_1 reflecting on IN_κ, let ϕ be a first-order formula such that $J_\kappa \models (\exists S)\phi$. Construct an elementary chain $(N_\nu : \nu < \kappa)$, as above, where N_0 contains the transitive closure of the set which contains all the parameters in ϕ. Let ν, π be as above. Since

$$J_{\kappa^+} \models (\exists S \subseteq \kappa)\phi^{(J_\kappa)} \text{ and } N_\nu \prec J_{\kappa^+}$$

it follows that

$$J_\gamma \models (\exists S \subseteq \nu)\phi^{(J_\nu)}, \text{ and hence, } J_\nu \models (\exists S)\phi .$$

It follows easily that

Theorem 2.3 $(V = L)$ *[the least κ such that $\kappa \xrightarrow{\Sigma_n} (\kappa)^2_2$] $\neq$ [the least Mahlo].*

Proof: It is enough to note that there exists a Σ^1_1 formula Φ without any parameters such that for all ordinals ν, $J_\nu \models \Phi \Leftrightarrow \nu \xrightarrow{\Sigma_n} (\nu)^2_2$, and then use Theorem 2.2.

3 Π^1_1 reflection and stationary sets In this section Π^1_1 reflection is studied. The first two results contrast the difference between Π^1_1 reflection for countable and uncountable ordinals.

Theorem 3.1 *For all $n \geq 0$ and all admissible $\lambda \leq \omega_1$,*
1. $(\forall \alpha \in S^{n+1}_\lambda)$ *(α is Π^1_1 reflecting on S^n_α).*
2. *If S^{n+1}_λ is cofinal in λ, so is $\{\alpha \in S^{n+1}_\lambda : \alpha$ is Π^1_1 reflecting on $S^n_\alpha\}$.*

Proof: It is clear that (2) follows from (1). To prove (1) let $\alpha \in S^{n+1}_\lambda$. Let Φ be a Π^1_1 sentence true in J_α. By a result of [5] there exists a Σ_1 sentence Φ^+ with the same parameters as those of Φ such that for all countable ordinals $\gamma \geq \alpha$ and all admissible ordinals $\delta > \gamma$, $J_\gamma \models \Phi \Leftrightarrow J_\delta \models \Phi^+(\gamma)$. Since α is countable and λ is admissible, $J_\lambda \models \Phi^+(\alpha)$. Hence, $J_\lambda \models (\exists x)(\phi_n(x)$ and $\Phi^+(x))$. Thus, the above Σ_{n+1} sentence must also be true in J_α.

Theorem 3.2 *For all cardinals $\lambda \geq \omega_1$,*
1. *λ is Π^1_1 reflecting on $S^1_\lambda \Leftrightarrow \lambda$ is Π^1_1 reflecting.*
2. *λ is Π^1_1 reflecting $\Rightarrow \lambda$ is a limit cardinal.*

Proof: This is easy. Notice the notion of regular uncountable is expressible via a Π^1_1 sentence and then use Levy's absoluteness principle (see [1]).

Theorem 3.3 $(V = L)$ *For any $0 \leq n \leq \omega$, and any κ the following are equivalent*

1. *κ is Π^1_1 reflecting on S^n_κ.*
2. *For any $\Sigma_{n+1}(J_\kappa)$ stationary set E there exists an ordinal $\alpha \in S^n_\kappa$ such that $\alpha \cap E$ is stationary in α.*

3. *For any $\Delta_2(J_\kappa)$ stationary set E there exists an ordinal $\alpha \in S^n_\kappa$ such that $\alpha \cap E$ is stationary in α.*

Proof: The proof of (2) ⇒ (3) is trivial. To prove (1) ⇒ (2), let ϕ be a Σ_{n+1} formula defining the $\Sigma_{n+1}(J_\kappa)$ set E with parameters in J_κ. But the following Π^1_1 formula is true in J_κ,

$$(\forall C)(C \text{ closed and unbounded} \Rightarrow (\exists\alpha)(C(\alpha) \text{ and } \phi(\alpha))) \ .$$

Now the proof of (1) ⇒ (2) follows easily using the reflection property satisfied by κ. Finally, to prove (3) ⇒ (1) assume by way of contradiction that κ is not Π^1_1 reflecting on S^n_κ. Let $\phi(S)$ be a first-order formula such that

$$J_\kappa \models \forall S\phi(S) \text{ and } (\forall\alpha \in S^n_\kappa)J_\alpha \not\models \forall S\phi(S) \ .$$

As in [2] (Theorem 11.1), one defines the $\Delta_2(J_\kappa)$ definable set E of all limit cardinals α such that there exists $\beta > \alpha$ for which the following hold

1. α is regular at β,
2. β is α-minimal, and
3. $(\forall R \in J_\kappa)(R \subseteq J_\alpha \Rightarrow J_\alpha \models \phi(R))$.

A contradiction can be obtained as in [2] by showing that $(\forall\alpha \in S^n_\kappa)$ $\alpha \cap E$ is not stationary in α.

Theorem 3.4 $(V = L)$ *($\kappa \xrightarrow{\Sigma_n} (\kappa)^2_2 \Rightarrow \kappa$ is Π^1_1 reflecting on S^n_κ), $n \geq 1$.*

Proof: Assume the hypothesis but that the conclusion fails. By Theorem 3.3 there exists $\Delta_{n+1}(J_\kappa)$ stationary set E such that $(\forall\alpha \in S^n_\kappa)$ $\alpha \cap E$ is not stationary in α. Without loss of generality it can be assumed that $E \subseteq S^n_\kappa$. As in Theorem 9.1 of [2], construct a $\Delta_{n+1}(J_\kappa)$ diamond sequence $\Diamond_\kappa(E)$, $(S_\alpha : \alpha \in E)$ by $\Sigma_{n+1}(J_\kappa)$ induction on E. As in the construction of a Souslin tree in L, one can now construct a $\Delta_{n+1}(J_\kappa)$ tree of height κ which has no branch of height κ. This contradicts the partition hypothesis and completes the proof of the theorem.

4 The sizes of definable partition cardinals This section is concerned with determining the size of the least regular cardinal which satisfies the partition property $\kappa \xrightarrow{\Sigma_n} (\kappa)^2_2$. For each ordinal α let $S^\omega_\alpha = \{\gamma < \alpha : J_\gamma \prec J_\alpha\}$. The proof of the following theorem can be found in [3].

Theorem 4.1 *If κ is Mahlo then the set of uncountable regular cardinals α such that $J_\alpha < J_\kappa$ and α is Π^1_1 reflecting on S^ω_κ is cofinal in κ.*

Theorem 4.2 *If κ is Π^1_1 reflecting on S^{n+1}_κ then $\kappa \xrightarrow{\Sigma_n} (\kappa)^2_2$, $n \geq 1$.*

Proof: Let $\mathbf{T} = (T, <_T)$ be a $\Sigma_n(J_\kappa)$ tree of height κ. Consider a $\Sigma_{n+1}(J_\kappa)$ definable function $f : \kappa \to \kappa$ such that for all $\alpha < \kappa$, $|T_\alpha| \leq |f(\alpha)| = |J_{f(\alpha)}|$, where T_α is the α'th level of the tree $\mathbf{T}$. Define a new $\Sigma_{n+1}(J_\kappa)$ tree $\mathbf{S}$ of height κ, by defining for each $\alpha < \kappa$, the α'th level S_α of the tree $\mathbf{S}$. This is done by embedding the level T_α into the set $J_{f(\alpha)+1} - J_{f(\alpha)}$, and then taking an appropriate subset of the previous set theoretic difference to be the new level S_α of the tree $\mathbf{S}$. It will be shown that $\mathbf{S}$ has a branch of length κ. Indeed, assume on the contrary that S has no such branch. This means that $J_\kappa \models \forall X$ (X is a branch of $\Rightarrow \exists z(X \subseteq z)$). Let $\alpha \in S_\kappa^{n+1}$ such that $f''(\alpha) \subseteq \alpha$, J_α reflects the above Π_1^1 formula. Consider an element $t \in S$ of tree rank α and let $X = \{s \in S : s <_S t\}$. Since for each $\gamma < \alpha$, $S_\gamma \subseteq J_{f(\gamma)+1}$, it follows that $X \subseteq J_\alpha$; in addition X is unbounded in tree rank below α. However, it follows from the above Π_1^1 formula which is true in J_α, that $J_\alpha \models \exists z(X \subseteq z)$. But this is a contradiction. It is now easy to see that for regular uncountable cardinals, the above proved property on $\Sigma_n(J_\kappa)$ definable trees on κ implies that $\kappa \xrightarrow{\Sigma_n} (\kappa)_2^2$.

For each $n \geq 1$ let

$$\kappa_n = \text{least } \kappa \text{ such that } \kappa \xrightarrow{\Sigma_n} (\kappa)_2^2 \ ,$$

and

$$\kappa_I = \text{least inaccessible } \kappa, \ \kappa_M = \text{least Mahlo } \kappa \ .$$

As an immediate consequence of the above results one obtains

Theorem 4.3 *($V = L$) For all $n \geq 1$, $\kappa_I \neq \kappa_2$ and $\kappa_I \leq \kappa_1 \leq \ldots \leq \kappa_n \leq \ldots \leq \kappa_\omega \leq \kappa_M$.*

The above ideas can also be used to obtain

Theorem 4.4 *($V = L$) For all $n \geq 2$, the partition relation $\kappa_n \xrightarrow{\Sigma_n} (\kappa - \Sigma_\omega)_2^2$ is false.*

Proof: Notice that the partition relation $\alpha \xrightarrow{\Sigma_n} (\alpha - \Sigma_\omega)_2^2$ can be defined by a Π_1^1 formula Φ such that for all α, $J_\alpha \models \Phi$ if and only if $\alpha \xrightarrow{\Sigma_n} (\alpha - \Sigma_\omega)_2^2$.

It is still an open question whether $\kappa_1 = \kappa_I$ or $\kappa_n < \kappa_{n+1} < \kappa_\omega$. In addition, it would be useful to study the above partition properties for exponents higher than 2.

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