

Two λ -Extensions of the Theory of Homogeneous Simple Types as a Second-Order Logic

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In the theory of simple logical types as originally conceived, it is meaningless for one predicate expression to occur in one of the subject or argument positions of another unless the latter is assigned a higher logical type than the former *within the grammar of the object language*; and therefore it is meaningless in particular for any predicate expression to apply to itself, i.e., to occur in one of its own subject positions. Russell's paradox of predication is thereby avoided, of course, but the price is high, for the resulting theory is not an accurate representation of the role of predicates in natural language where predicate expressions can apply not only to the nominalizations of other predicates but to their own nominalizations as well—and without regard at all for the notion of a logical type.

In the theory of logical types as a second-order logic, on the other hand, predicate expressions are typed within the grammar of the object language only in the way they are typed in standard second-order logic, i.e., only with respect to their degree or number of subject positions (adicity), and they are allowed otherwise to meaningfully occur in the subject or argument positions of other predicates, and of themselves as well, without regard to the notion of a logical type. Russell's paradox of predication can be avoided, it turns out, not by resorting to the notion of a logical type as a part of the grammar of the object language but rather only as a part of the metalinguistic description of the conditions under which properties and relations are to be posited by means of the grammar of the object language. The difference is crucial, needless to say, since it allows for a more accurate representation of the role of predicates and predication in natural language. The resulting theory is not, to be sure, a second-order logic in the "standard" sense used today (though it does contain the latter), but it is a second-order logic in the traditional or pre-type-theoretical sense in which quantifier expressions are allowed to reach into both subject and

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predicate positions without obliterating the logical and conceptually important distinctions between the two.

We return in the present essay to the theory of homogeneous simple types (HST) as a second-order logic, and in particular to the theory HST^* described in [2]. We shall be concerned only with homogeneous simple types since, as was noted in [2], the full relational theory ST^* of heterogeneous simple types is inconsistent as a second-order logic. We shall extend the grammar of HST^* , however, by introducing the λ -operator as a new logical primitive, thereby allowing for the formation of λ -abstracts as complex predicate expressions. Our first goal is to formulate a system λHST^* which together with a principle of individuation for properties and relations is a conservative extension of HST^* (with that same principle of individuation); and in fact the system we formulate is not only a conservative extension but an inessential one since λ -abstraction is in principle eliminable therein. Unfortunately, however, the realization of this goal requires that we constrain the grammar of the object language and allow as meaningful only those λ -abstracts which are homogeneously stratified (in the metalinguistic sense); i.e., we must impose on the grammar of the object language for λHST^* just those conditions which in HST^* applied only to the conditions for positing properties and relations. A secondary goal, accordingly, is to formulate a system HST^*_λ which does not impose these grammatical constraints on λ -abstracts but which posits *as individuals* only those properties and relations posited in HST^* —and of course which is consistent if HST^* is, even though it is not a conservative extension of HST^* . This is achieved in part by rendering the first-order logic contained in HST^*_λ free of existential presuppositions and allowing some λ -abstracts when occurring in subject positions to denote a value of the bound individual variables while admitting that others, of necessity, will fail to do so.

As a means towards realizing these goals, we reformulate the extensional Fregean semantics given for HST^* in [2], and we show that both λHST^* and HST^*_λ , together with an extensional principle of individuation for properties and relations, are complete with respect to their extensional Fregean semantics. (A Fregean semantics, as we understand it here, is one in which nominalized predicates are interpreted as denoting certain individuals called concept-correlates rather than the concepts designated by these predicates in their predicate position occurrences.) We also intensionalize this semantics in a concluding section and show that two modal extensions of λHST^* and HST^*_λ , together with an intensional principle of individuation for properties and relations, are complete with respect to this intensionalized Fregean semantics.

Finally, it should be noted that as a counterpart to the original theory of simple logical types much of the motivation for our development of λHST^* and HST^*_λ parallels Quine's motivation for the construction of his two first-order set theories, *viz.*, the system NF of [5] and the later system ML of [6]. Moreover, in this regard, even our proof of the consistency of HST^*_λ relative to λHST^* parallels Wang's proof in [7] of the consistency of ML relative to NF . Nevertheless, notwithstanding this similarity and parallel motivation, the differences in deductive powers between NF and ML , on the one hand, and λHST^* and HST^*_λ , on the other, are not insignificant. For example, as shown in [2], HST^* , and therefore λHST^* as well, is equiconsistent with the original theory of simple

logical types, *ST*, at least when both systems are supplemented with an extensional principle of individuation for properties and relations. *NF*, on the other hand, not only exceeds *ST* in deductive powers but does so in inappropriate ways, e.g., by disproving the axiom of choice and thereby proving an axiom of infinity. Neither of these axioms, however, is either provable or disprovable in *ST*, λHST^* , or HST_λ^* . In fact, unlike *NF* and *ML*, both λHST^* and HST_λ^* (with and without axioms of choice and infinity) as well as *ST* can be shown to be consistent if weak Zermelo set theory (with and without axioms of choice and infinity) is consistent. In other words, as theories of predication, λHST^* and HST_λ^* are better type-free representations of the original theory of simple logical types than is either *NF* or *ML* as a theory of membership. Moreover, many, if not all, of the controversial theses of *NF* and *ML* as set theories independent of any superseding theory of predication (e.g., their admission of a universal set and of the set-existence of the complement of every set, and their countenance therefore of a symmetry between the small and the large) are neither dubious nor controversial in λHST^* and HST_λ^* when viewed as theses about properties and relations in a theory of predication. Finally, even the problematic distinction between proper classes and sets in *ML* has a natural explanation when rephrased in terms of the distinction in HST_λ^* between those λ -abstracts which denote a value of the bound individual variables and those which do not (even though all λ -abstracts designate or express values of bound predicate variables). In particular, whereas nominalized predicates may be taken as always purporting to denote an independently real property or relation (though some of necessity must fail to denote anything at all), the use of these same predicates in their role as predicates, i.e., their occurrences in predicate positions, may be assumed always to express only unsaturated predicable concepts in the sense of (mind dependent) cognitive capacities (or cognitive structures otherwise based upon such capacities). That is, HST_λ^* can be construed as representing a form of conceptual realism where the use of a predicate *as such* amounts to the application of an (unsaturated) predicable concept in an act of thought (wherein the concept becomes saturated), but also where the denotative use of the same predicate, i.e., the use of its nominalization, amounts instead to an attempt *to refer* to a real property or relation which is purportedly represented by the concept expressed by that predicate. That not every predicable concept which we can form and articulate can succeed in denoting a real property or relation when applied denotatively will in that case be an interesting but hardly problematic fact. What would be problematic is the assumption that concepts are themselves the independently real properties and relations they purport to represent when applied denotatively while admitting that only some of them are individuals while others are not. Given that assumption, in other words, it would be better to stick with λHST^* where (at least on its intended interpretation) all values of bound predicate variables are also values of the bound individual variables.

1 *HST revisited** We begin by reviewing the syntax of HST^* . This consists essentially of the wffs of standard second-order logic extended so as to allow predicate variables and constants to occupy the subject or argument positions that go with predicates. Predicates occupying such positions are said to be

nominalized (at such positions) and are understood to represent the predicate nominalizations that occur in natural language. (Adding suffixes such as ‘-ity’, ‘-ness’, or ‘-hood’ to nominalized occurrences of predicates would be superfluous here, incidentally, since such occurrences are already formally identified as subject position occurrences.)

We shall use ‘ x ’, ‘ y ’, ‘ z ’, with or without numerical subscripts, to refer (in the metalanguage) to individual variables, and we use ‘ F^n ’ and ‘ G^n ’ similarly to refer to n -place predicate variables. (We shall usually delete the superscript when the context makes clear the number of subject or argument positions that go with a predicate variable or constant.) We shall also use ‘ a ’ and ‘ b ’, with or without numerical subscripts, to refer ambiguously (in the metalanguage) to both predicate and individual variables and constants. Predicate and individual constants we assume to be provided by different (formal) languages; i.e., a *language* is a set of predicate and individual constants.

As primitive logical constants, we take \rightarrow , the (material) conditional sign, \sim , the negation sign, and \forall , the universal quantifier. (The other logical constants, *viz.*, $\&$, \vee , \leftrightarrow , and \exists , are assumed to be defined as abbreviatory devices of the metalanguage.) Where L is a language, we understand the *terms* of L to be the members of L together with all individual and predicate variables. The atomic wffs of L are all the expressions of the form: $P^n(a_1, \dots, a_n)$, where $n \in \omega$, P^n is either an n -place predicate constant in L or an n -place predicate variable, and a_1, \dots, a_n are terms of L . (When $n = 0$, we take $P^n(a_1, \dots, a_n)$ to be just P^n itself.) An expression is a *wff* of L iff it belongs to every set K containing the atomic wffs of L and such that $\sim\phi$, $(\phi \rightarrow \psi)$, $\forall a\phi$ are in K whenever ϕ , ψ are in K and a is an individual or predicate variable. We use ‘ ϕ ’, ‘ ψ ’, and ‘ χ ’ to refer (in the metalanguage) to wffs.

We assume the notions of bondage and freedom of occurrences of variables in wffs to be defined in the usual way, and similarly we assume what it means for a term a to be free for a term b in a wff ϕ . We take $\phi(a/b)$ to be the result of replacing all free occurrences of b in ϕ by free occurrences of a , if such a wff exists; and otherwise we take $\phi(a/b)$ to be just ϕ itself. For convenience, we use \equiv to abbreviate (in the metalanguage) *indiscernibility* as follows (where F is the first one-place predicate variable different from both a and b):

$$a \equiv b =_{df} \forall F[F(a) \leftrightarrow F(b)] \ .$$

Finally, we say that a wff ϕ is *homogeneously stratified* if there is an assignment t of natural numbers to the terms that occur in ϕ such that for every subwff of ϕ of the form $P^n(a_1, \dots, a_n)$: (1) $t(a_i) = t(a_j)$, for all positive integers $i, j \leq n$, and (2) $t(P^n) = 1 + t(a_1)$.

2 An improved axiom set for HST^* * The axiomatization of HST^* given in [2] can be improved, it turns out, in several ways that are conducive to the formulation of the systems λHST^* and HST^*_λ . Briefly, the original axioms of HST^* consisted of all tautologous wffs and all wffs of one of the following forms:

- (A1) $\forall a(\phi \rightarrow \psi) \rightarrow (\forall a\phi \rightarrow \forall a\psi)$, where a is any variable
 (A2) $\phi \rightarrow \forall a\phi$, where a is a variable not occurring free in ϕ

- (A3) $\exists x(a \equiv x)$, where a is any term in which the individual variable x does not occur free
- (A4) $\exists F^n(P \equiv F \ \& \ \forall x \dots \forall x_n[F(x_1, \dots, x_n) \leftrightarrow P(x_1, \dots, x_n)])$, where x_1, \dots, x_n are pairwise distinct individual variables, P^n is an n -place predicate variable or constant, and F^n is an n -place predicate variable different from P^n
- (A5) $a \equiv b \rightarrow (\phi \rightarrow \psi)$, where ϕ is an atomic wff and ψ is obtained from ϕ by replacing a *subject position occurrence* of b by a subject position occurrence of a ,

together with all *homogeneously stratified* wffs that are instances of the following *comprehension principle*:

$$(HSCP^*) \quad \exists F^n \forall a_1 \dots \forall a_n [F(a_1, \dots, a_n) \leftrightarrow \phi],$$

where F^n is an n -place predicate variable not occurring free in ϕ and a_1, \dots, a_n are pairwise distinct *individual* or *predicate variables* occurring free in ϕ . (Note: the entire biconditional must be homogeneously stratified in this principle, not just the comprehending wff ϕ alone. Also, in general, we use a *-label for any thesis which explicitly involves predicate nominalizations.)

The only inference rules of HST^* (and later of λHST^* and HST^*_λ as well) are modus ponens (MP) and universal generalization (UG) of either a predicate or individual variable. Theoremhood is assumed defined in the usual way, and $\Gamma \frac{}{HST^*} \phi$ is taken to mean that $\frac{}{HST^*}(\psi_1 \ \& \ \dots \ \& \ \psi_n \rightarrow \phi)$, for some $\psi_1, \dots, \psi_n \in \Gamma$. (For $n = 0$, we take the above “conditional” to be just ϕ itself.) As observed in [2], the following theses are provable without utilizing ($HSCP^*$) at all:

(1) *Leibniz’ law for subject position occurrences of terms:*

$$(LL_1^*) \quad a \equiv b \rightarrow (\phi \leftrightarrow \psi),$$

where ψ is obtained from ϕ by replacing one or more free subject position occurrences of a by free subject position occurrences of b .

(2) *The principle of universal instantiation of any term a for a generalized individual variable x :*

$$(UI_1^*) \quad \forall x \phi \rightarrow \phi(a/x).$$

(3) *The principle of universal instantiation of an n -place predicate variable or constant P^n for a generalized n -place predicate variable:*

$$(UI_n^*) \quad \forall F^n \phi \rightarrow \phi(P^n/F).$$

Now because

$$\exists R \forall F \forall G (R(F, G) \leftrightarrow \forall x_1 \dots \forall x_n [F(x_1, \dots, x_n) \leftrightarrow G(x_1, \dots, x_n)])$$

is an instance of ($HSCP^*$), the following thesis,

$$(Ind^*) \quad \forall F^n \forall G^n (F \equiv G \rightarrow \forall x_1 \dots \forall x_n [F(x_1, \dots, x_n) \leftrightarrow G(x_1, \dots, x_n)])$$

is provable in HST^* ; and therefore, by a simple induction on the subwffs of ϕ , the above restricted version of Leibniz' law can be strengthened into the full version:

$$(LL^*) \quad \frac{}{HST^*} a \equiv b \rightarrow (\phi \leftrightarrow \psi),$$

where ψ is obtained from ϕ by replacing one or more free occurrences of a by free occurrences of b (if such a wff exists, and otherwise ψ is just ϕ itself¹). Indiscernibility in HST^* , in other words, suffices as a characterization of identity.

(LL^*) is also derivable of course if we simply drop the restriction in (A5) that only subject position occurrences of a term are to be replaced under the hypothesis of indiscernibility. (Note: although unimportant in HST^* , the restriction is crucial in certain other theories, specifically the system $T^*(^*)$ of Section 2 in [2], where (Ind_n^*) is disprovable; i.e., the distinction is crucial in theories in which indiscernibility necessarily fails as a characterization of identity.) Strengthening (A5) in this way, needless to say, allows us to derive (Ind^*) without resorting to $(HSCP^*)$ at all. (We take (Ind^*) to be the set of wffs (Ind_n^*) , for all $n \in \omega$). Now this is significant because given (Ind^*) , we can weaken or rather restrict $(HSCP^*)$ to just those homogeneously stratified wffs of the form:

$$\exists F^n \forall x_1 \dots \forall x_n [F(x_1, \dots, x_n) \leftrightarrow \phi] ,$$

where F_n is not free in ϕ and x_1, \dots, x_n are pairwise distinct *individual variables* occurring free in ϕ . That is, the original quantifier prefix $\forall a, \dots, \forall a_n$ of $(HSCP^*)$ can be restricted to those cases where none of a_1, \dots, a_n is a predicate variable.

Proof: Suppose

$$F(a_1, \dots, a_n) \leftrightarrow \phi$$

is a homogeneously stratified wff as in the original version of $(HSCP^*)$. Let x_1, \dots, x_n be distinct individual variables new to ϕ , and let b_i be a variable new to ϕ but of the same type as a_i . Then, on the basis of (LL^*) and the remaining axioms of HST^* other than $(HSCP^*)$,

$$\phi \leftrightarrow \exists b, \dots, \exists b_n [a_1 \equiv b_1 \ \& \dots \ \& \ a_n \equiv b_n \ \& \ \phi(b_1/a_1, \dots, b_n/a_n)]$$

is provable; and since

$$\begin{aligned} & \forall x_1 \dots \forall x_n [F(x_1, \dots, x_n) \leftrightarrow \exists b_1 \dots \exists b_n [x_1 \equiv b_1 \ \& \dots \ \& \ x_n \equiv b_n \ \& \\ & \quad \phi(b_1/a_1, \dots, b_n/a_n)]] \rightarrow \\ (F(a_1, \dots, a_n) \leftrightarrow \exists b_1 \dots \exists b_n [a_1 \equiv b_1 \ \& \dots \ \& \ a_n \equiv b_n \ \& \ \phi(b_1/a_1, \dots, b_n/a_n)]) \end{aligned}$$

is provable on the basis of (UI_1^*) (which does not itself depend on $(HSCP^*)$), then by the interchange law for wffs provably equivalent (which also does not depend on $(HSCP^*)$), universal generalization on a_1, \dots, a_n, F_n , and tautologous transformations utilizing (A1) and (A2),

$$\begin{aligned} & \exists F \forall x_1 \dots \forall x_n (F(x_1, \dots, x_n) \leftrightarrow \exists b_1 \dots \exists b_n [x_1 \equiv b_1 \ \& \dots \ \& \ x_n \equiv b_n \ \& \\ & \quad \phi(b_1/a_1, \dots, b_n/a_n)]] \rightarrow \\ & \exists F \forall a_1 \dots \forall a_n [F(a_1, \dots, a_n) \leftrightarrow \phi] \end{aligned}$$

is also provable without utilizing (*HSCP**). Of course since by hypothesis the consequent of this last conditional is an instance of the original version of (*HSCP**), it follows that the antecedent is also an instance of (*HSCP**), and therefore of the restricted version in question as well. That is, on the basis of (*LL**) and the remaining axioms of *HST**, the original version of (*HSCP**) is derivable from the restricted version.

Hereafter, in referring to *HST** we shall take this to be the system described above with (*HSCP**) replaced by its restricted version and (*A5*) replaced by (*LL**) (or by (*A5*) strengthened in the way indicated). This is an improvement over the original version, incidentally, if only because it explains why definitionally extending an applied form of *HST** by means of an explicit definition for a predicate constant must have the form of a homogeneously stratified biconditional:

$$\forall x_1 \dots \forall x_n [P^n(x_1, \dots, x_n) \leftrightarrow \phi] ,$$

where ϕ is a wff in which the n -place predicate constant P^n does not occur, x_1, \dots, x_n are distinct individual variables occurring free in ϕ , and no variables other than x_1, \dots, x_n occur free in ϕ . That is, as is appropriate for any second-order logic with nominalized predicates, the defining conditions for a new predicate must always be such as to be meaningful for all individuals, whether these be concrete or abstract, and not just for individuals of a certain type (as might have been thought on the basis of the original form of (*HSCP**)). Definitions of this form, needless to say, can easily be shown to be noncreative, and defined predicates are provably eliminable on the basis of the restricted version of (*HSCP**).

3 The grammar of *HST with λ -abstracts** Given the λ -operator and the fact that we are dealing with a logic with nominalized predicates, explicit definitions of predicate constants can be given the even more natural form of an identity sentence:

$$P^n \equiv [\lambda x_1, \dots, x_n \phi] ,$$

where $[\lambda x_1 \dots x_n \phi]$ is a homogeneously stratified λ -abstract. Note that only individual variables are allowed to be bound by the λ -operator, since to do otherwise might suggest that we have predicate expressions which on grammatical grounds alone cannot be meaningfully predicated of all individuals, thereby defeating the whole point of reconstructing type theory as a second-order logic. Individual variables, in other words, are subject position indicators *par excellence*.

Now in taking the λ -operator as a new primitive logical constant, we shall describe the syntax of λHST^* in terms of the somewhat broader syntax of HST^*_λ , i.e., in terms of a syntax which does not require all λ -abstracts to be homogeneously stratified. For convenience, we identify the (*category*) *types* of the syntax with the set of natural numbers, where semantically 0 is understood to be the type of all individuals, 1 is the type of all propositions, 2 is the type of all properties, and $n + 1$, for $n > 1$, the type of all n -ary relations. Accordingly, individual variables are said to be of type 0, and n -place predicate

variables are said to be of type $n + 1$. Where L is a language and $n \in \omega$, then *the meaningful expressions of L of type n* , in symbols $ME_n(L)$, are recursively defined as follows:

- (1) if a is an individual variable or constant in L , then $a \in ME_0(L)$; and if a is an n -place predicate variable or constant in L , then $a \in ME_{n+1}(L)$;
- (2) if $\pi \in ME_{n+1}(L)$ and $a_1, \dots, a_n \in ME_0(L)$, then $\pi(a_1, \dots, a_n) \in ME_1(L)$;
- (3) if $\phi \in ME_1(L)$ and x_1, \dots, x_n are distinct individual variables, then $[\lambda x_1 \dots x_n \phi] \in ME_{n+1}(L)$;
- (4) if $\phi \in ME_1(L)$, then $\sim \phi \in ME_1(L)$;
- (5) if $\phi, \psi \in ME_1(L)$, then $(\phi \rightarrow \psi) \in ME_1(L)$;
- (6) if $\phi \in ME_1(L)$ and a is an individual or predicate variable, then $\forall a \phi \in ME_1(L)$;
- (7) if $\phi \in ME_1(L)$, then $[\lambda \phi] \in ME_0(L)$; and
- (8) if $n \in \omega - \{1\}$, then $ME_n(L) \subseteq ME_0(L)$.

The *terms* of a language L , needless to say, are now all of the members of $ME_0(L)$; and for $n > 0$, the members of $ME_{n+1}(L)$ are the n -place *predicate expressions* of L . Naturally, the *wffs* of L are now the members of $ME_1(L)$. Note that whereas by clause (8) every predicate expression is a term, not every wff is also a term; but then, by clause (3), if ϕ is a wff, so is $[\lambda \phi]$, and, by clause (7), $[\lambda \phi]$ is a term. Wffs are terms, in other words, when prefixed by the λ -operator. (We read $[\lambda \phi]$ as ‘that ϕ ’ when it occurs in a subject position. The *subject positions* of a wff ϕ or of a complex predicate expression containing ϕ are of course any one of the subject or argument positions of the predicate expressions occurring in ϕ .) For convenience, we shall also speak of wffs that are terms as 0-place predicate expressions. We assume the obvious definitions of *bondage* and *freedom* of terms in wffs and λ -abstracts, as well as the notion of one term being *free for* another in a wff or λ -abstract.

Now where ξ is a meaningful expression, i.e., $\xi \in ME_n(L)$, for some language L and $n \in \omega$, we say that ξ is *homogeneously stratified* iff there is a function t with the set of terms of L occurring in ξ as domain such that: (1) for all $k \in \omega$, all k -place predicate expressions π of L and all terms a_1, \dots, a_k of L , if $\pi(a_1, \dots, a_k)$ is a wff occurring in ξ , then (i) $t(a_i) = t(a_j)$, for $1 \leq i \leq j \leq k$, and (ii) $t(\pi) = t(a_1) + 1$; and (2) for all $k \in \omega$, all individual variables x_1, \dots, x_k and all wffs ϕ of L , if $[\lambda x_1 \dots x_k \phi]$ occurs in ξ , then $t(x_i) = t(x_j)$, for $1 \leq i \leq j \leq k$, and $t([\lambda x_1 \dots x_k \phi]) = t(x_1) + 1$. In terms of this notion we say that ϕ is a *wff of λHST^** iff ϕ is a wff in which every predicate expression of the form $[\lambda x_1 \dots x_n \psi]$ is homogeneously stratified. It is clear of course that not every wff is a wff of λHST^* .

4 The system λHST^* In turning now to an axiomatization of λHST^* we again take all tautologous wffs (only now of λHST^* as well) and all wffs of λHST^* of the form of (A1), (A2), (A3), and (LL*) together with all those of the following forms:

- (Id $_{\lambda}^*$) $[\lambda x_1 \dots x_n P(x_1, \dots, x_n)] \equiv P$, where P^n is an n -place predicate variable or constant.

(λ -Conv*) $[\lambda x_1 \dots x_n \phi](a_1, \dots, a_n) \leftrightarrow \phi(a_1/x_1, \dots, a_n/x_n)$, where each a_i is free for x_i in ϕ .

(HSCP $^*_\lambda$) $\exists F^n([\lambda x_1 \dots x_n \phi] \equiv F)$, where F^n does not occur free in ϕ .

Note that since only wffs of λHST^* are involved here, the λ -abstracts which occur in instances of (λ -Conv*) and (HSCP $^*_\lambda$) must be homogeneously stratified. As *inference rules* we again take modus ponens and universal generalization (as applied to any variable of any type) and we assume that theoremhood and derivability from premises are defined as in HST^* .

By (LL^*) and ($A3$), the principle (UI^*_1) of universal instantiation of any term for a generalized individual variable is also provable in λHST^* . Similarly, by (LL^*) and (HSCP $^*_\lambda$), the principle of universal instantiation of an n -place λ -abstract of λHST^* for a generalized n -place predicate variable is also provable in λHST^* :

($UI^*_{2\lambda}$) $\forall F^n \psi \rightarrow \psi([\lambda x_1 \dots x_n \phi]/F^n)$,

and therefore, by (Id^*_λ) and (LL^*), so is the principle (UI^*_n). Of course, also provable in λHST^* , by (LL^*), universal generalization, and elementary transformations, is the thesis (Ind^*).

The converse of (Ind^*), viz., the principle of extensionality:

(Ext *_n) $\forall F^n \forall G^n (\forall x_1 \dots \forall x_n [F(x_1, \dots, x_n) \leftrightarrow G(x_1, \dots, x_n)] \rightarrow F \equiv G)$,

is not provable in λHST^* , however, which is as it should be except for those theories which explicitly assume otherwise. (Frege would assume (Ext *), though on his interpretation nominalized predicates do not denote concepts but concept-correlates.) We would clearly want to reject (Ext *), for example, if we were to introduce the modal operator \Box into the grammar of λHST^* along with, say, the axioms of $S5$ modal propositional logic, the Carnap-Barcan wff for quantifiers and \Box , and the rule of modal generalization into λHST^* itself—calling the result $\Box \lambda HST^*$. In that case, however, it is not clear that we would also want to reject the corresponding *principle of intensionality*²:

(\Box Ext *_n) $\forall F^n \forall G^n (\Box \forall x_1 \dots \forall x_n [F(x_1, \dots, x_n) \leftrightarrow G(x_1, \dots, x_n)] \rightarrow F \equiv G)$.

One of the benefits of assuming either (Ext *) or (\Box Ext *) is that we are able on that basis to eliminate all occurrences of the λ -operator; that is, the wffs of λHST^* are in that case reducible to those of HST^*). Thus, using (LL^*), ($UI^*_{2\lambda}$), (λ -Conv*) and elementary transformations, the following lemma is easily provable. (We use ' \Box ') to indicate the corresponding lemma in case (\Box Ext *) and $\Box \lambda HST^*$ are used instead of (Ext *) and λHST^* .)

Lemma³ *If ϕ, ψ are wffs of λHST^* and F^n does not occur in ϕ or ψ , then:*

$$\begin{aligned} ((\Box)Ext^*) \quad & \frac{}{(\Box)\lambda HST^*} \phi \leftrightarrow \exists F^n [(\Box)\forall x_1 \dots \forall x_n (F(x_1, \dots, x_n) \leftrightarrow \psi) \\ & \& \phi(F^n/[\lambda x_1 \dots x_n \psi])] . \end{aligned}$$

Accordingly, since the λ -abstracts occurring in any wff ϕ of λHST^* always occur in certain subwffs of ϕ relative to which those occurrences are free (even

if they are not free in the broader context of ϕ itself), then by systematically interchanging these subwffs with λ -free subwffs on the basis of the above lemma, we are able to establish the following λ -elimination theorem.⁴

λ -Elimination Theorem *For every wff ϕ of λHST^* there is a λ -free wff ψ of λHST^* such that ψ is homogeneously stratified if ϕ is, and*

$$((\Box)Ext^*) \vdash_{(\Box)\lambda HST^*} (\phi \leftrightarrow \psi) .$$

It follows, accordingly, that if $\lambda HST^* + (Ext^*)$ is a conservative extension of $HST^* + (Ext^*)$ and $HST^* + (Ext^*)$ is consistent, then so is $(\Box)\lambda HST^* + ((\Box)Ext^*)$. (Note that $(\Box)\lambda HST^* + ((\Box)Ext^*)$ is easily shown to be consistent if $\lambda HST^* + (Ext^*)$ is consistent, by simply interpreting \Box as double negation.) We assume in the statement of the following theorem what we show in Section 6; viz., that $\lambda HST^* + (Ext^*)$ is a conservative extension of $HST^* + (Ext^*)$.

Theorem *If $HST^* + (Ext^*)$ is consistent, then so is $(\Box)\lambda HST^* + ((\Box)Ext^*)$.*

In [2], incidentally, we showed that HST^* is consistent if *monadic HST^** is; and, furthermore, by using a result from [4], that *monadic $HST^* + (Ext^*)$* is consistent if weak Zermelo set theory is consistent. (Weak Zermelo set theory is the restriction of Zermelo set theory to those instances of the *Aussonderungssaxiom* in which all quantifiers are limited or restricted, i.e., have the forms $(\forall x \in y)\phi$ or $(\exists x \in y)\phi$.) Accordingly, it follows that if weak Zermelo set theory is consistent, then so is $(\Box)\lambda HST^* + ((\Box)Ext^*)$.

Theorem *If weak Zermelo set theory is consistent, then so is $(\Box)\lambda HST^* + ((\Box)Ext^*)$.*

Finally, where $n \neq k$, it might be noted that despite the apparent plausibility of

$$(Inf^*) \quad \forall F^n \forall G^k (F \neq G)$$

as a thesis of HST^* or λHST^* , it is not a theorem of either. We have avoided taking it as a basic axiom schema here only because the set of its instances amounts to requiring that there be at least a potential infinity of individuals. The thesis is plausible, moreover, primarily only for the Platonist who identifies the individual denoted by a nominalized predicate with the property or relation designated by that predicate when it occurs in a predicate position. It need not be accepted by a Fregean, for example, who denies that concepts are individuals and who interprets nominalized predicates as standing not for concepts but for concept-correlates. In either case, however, it can be shown that if weak Zermelo set theory together with an axiom of infinity is consistent then so is $\lambda HST^* + (Inf^*)$.

5 The system HST^*_λ We turn now to our second goal, viz., the formulation of a system HST^*_λ which does not restrict itself grammatically to homogeneously stratified λ -abstracts but which takes only those properties and relations to be individuals that are already posited as such in HST^* . The fulfillment of

this goal, however, at least insofar as every λ -abstract (homogeneously stratified or not) will express a concept (as a value of the bound predicate variables), requires that we drop the (axiomatic) assumption (A3) that every nominalized predicate expression actually denotes an individual (as a value of the bound individual variables). That is, the quantificational logic contained in HST_λ^* must be free of existential presuppositions regarding singular terms. We do this initially by replacing (A3) with

$$(A3') \quad \forall y \exists x (x \equiv y),$$

and allow for the possible truth of $\sim \exists x (a \equiv x)$.

This is not the only change that will be required, however; e.g., (Ind^*) will no longer be validated except in the qualified form:

$$(\exists/Ind_n^*) \quad \forall F^n \forall G^n (\exists y (F \equiv y) \ \& \ (F \equiv G) \rightarrow \forall x_1 \dots \forall x_n [F(x_1, \dots, x_n) \leftrightarrow G(x_1, \dots, x_n)]),$$

and therefore (LL^*) will no longer hold in general either, though (LL_1^*), i.e., Leibniz' law for *subject position* occurrences, remains unproblematic. (LL_1^*) and (\exists/Ind^*) will of course validate the following qualified form of (LL^*):

$$(\exists/LL^*) \quad \exists x (a \equiv x) \rightarrow [a \equiv b \rightarrow (\phi \leftrightarrow \psi)],$$

where x is not free in a , and ψ is obtained from ϕ by replacing one or more free occurrences of a by free occurrences of b .

Note that since every λ -abstract will express a concept (as a value of the bound predicate variables), the following *comprehension principle*:

$$(CP_\lambda^*) \quad \exists F^n ([\lambda x_1 \dots x_n \phi] \equiv F)$$

will be valid even in those cases where

$$\sim \exists y ([\lambda x_1 \dots x_n \phi] \equiv y)$$

is provable (such as will hold for the λ -abstract $[\lambda x \exists G (x \equiv G \ \& \ \sim G(x))]$). However, since (LL^*), and (Ind^*) in particular, fails in general, (CP_λ^*) will not suffice to validate ($UI_{2\lambda}^*$) (as applied now to arbitrary λ -abstracts). ($UI_{2\lambda}^*$) is valid, however, and it trivially implies (CP_λ^*) as well as

$$\exists F^n ([\lambda x_1 \dots x_n \phi] \equiv F \ \& \ \forall x_1 \dots \forall x_n [F(x_1, \dots, x_n) \leftrightarrow \phi]) ,$$

and for this reason we will take ($UI_{2\lambda}^*$) rather than (CP_λ^*) as an axiom schema of HST_λ^* . Similarly, insofar as (\exists/Ind^*) does not suffice to validate

$$(\exists/Id_\lambda^*) \quad [\lambda x_1 \dots x_n P(x_1, \dots, x_n)] \equiv P \ \& \ [P(a_1, \dots, a_n) \leftrightarrow [\lambda x_1 \dots x_n P(x_1, \dots, x_n)](a_1, \dots, a_n)],$$

we shall replace (Id_λ^*) by (\exists/Id_λ^*).

Finally, although

$$[\lambda x_1 \dots x_n \phi](a_1, \dots, a_n) \rightarrow \phi(a_1/x_1, \dots, a_n/x_n)$$

remains valid, where each a_i is free for x_i in ϕ , nevertheless its converse fails to also be valid. (A λ -abstract binds individual variables, after all, and therefore can be true only of values of the *bound* individual variables.) However, where

no x_i is free in any a_j , (λ -Conv*) can be validated in the following qualified form:

$$(\exists/\lambda\text{-Conv}^*) \quad [\lambda x_1 \dots x_n \phi](a_1, \dots, a_n) \leftrightarrow \exists x_1 \dots \exists x_n (a_1 \equiv x_1 \ \& \dots \ \& \ a_n \equiv x_n \ \& \ \phi).$$

Because arbitrary wffs with arbitrary λ -abstracts are now allowed, we need the following notion in order to stipulate that the properties and relations of HST^* are individuals in HST_λ^* . Accordingly, we say that a meaningful expression ξ is *bound to individuals* iff for all $n \in \omega$, all n -place predicate variables F^n , and all wffs ψ , if $\forall F^n \psi$ is a subwff of ξ , then for some individual variable x and some wff χ , ψ is $[\exists x(F^n \equiv x) \rightarrow \chi]$. (Semantically, this amounts to stipulating that the bound predicate variables occurring in ξ refer only to properties and relations which are individuals. Such a notion is pointless in λHST^* , needless to say, since by (A3) every wff of λHST^* is provable in that system to a wff which is bound to individuals.) The relevant axiom, accordingly, is the following *nominalized version of (HSCP*)*:

$$(\exists/\text{HSCP}_\lambda^*) \quad \exists y(a_1 \equiv y) \ \& \dots \ \& \ \exists y(a_k \equiv y) \rightarrow \exists y([\lambda x_1 \dots x_n \phi] \equiv y),$$

where $[\lambda x_1 \dots x_n \phi]$ is homogeneously stratified, ϕ is bound to individuals, y is an individual variable not occurring free in ϕ , and a_1, \dots, a_k are all the variables or nonlogical constants occurring free in $[\lambda x_1 \dots x_n \phi]$.

Finally, by an *axiom of HST_λ^** we understand any wff which is tautologous or of one of the forms (A1), (A2), (A3'), (LL_1^*), (\exists/Ind^*), (\exists/Id_λ^*), ($\exists/\lambda\text{-Conv}^*$), ($UI_{2\lambda}^*$), or ($\exists/\text{HSCP}_\lambda^*$). (Again, it should be emphasized that we are dealing now with all wffs, whether they are also wffs of λHST^* or not, i.e., whether the λ -abstracts they contain are homogeneously stratified or not. This is important since *when restricted to the wffs of λHST^** every axiom of HST_λ^* is a theorem of λHST^* . The instances of (A3) which are refutable in HST_λ^* , in other words, all involve λ -abstracts which are not homogeneously stratified.) We take modus ponens and universal generalization to be the only inference rules of HST_λ^* , and we assume that theoremhood and derivability from premises is defined for HST_λ^* in the way it is defined for λHST^* .

Note that since (A3) is not an axiom of HST_λ^* , the principle (UI_1^*) of universal instantiation of any term for a generalized individual variable is not provable in HST_λ^* except in the following qualified form:

$$(\exists/UI_1^*) \quad \exists y(a \equiv y) \rightarrow [\forall x \phi \rightarrow \phi(a/x)],$$

where a is free for x in ϕ and y is an individual variable which does not occur free in a . By ($UI_{2\lambda}^*$) and (\exists/Id_λ^*), however, (UI_n^*) is provable in HST_λ^* without qualification.

The second-order universal instantiation thesis ($UI_{2\lambda}^*$), incidentally, should not be confused with the different principle:

$$(UI_2^*) \quad \forall F^n \phi \rightarrow \check{S}_\psi^{F(x_1, \dots, x_n)} \phi,$$

which is not provable in HST_λ^* and in fact is refutable for certain instances. E.g., by ($\exists/\lambda\text{-Conv}^*$),

$$\frac{}{HST_\lambda^*} [\lambda x F(x)](a) \rightarrow \exists x(a \equiv x) ,$$

where x is not free in a ; and therefore by (\exists/Id_λ^*) and (UG) ,

$$\frac{}{HST_\lambda^*} \forall F[F(a) \rightarrow \exists x(a \equiv x)] .$$

That is, according to HST_λ^* , only real individuals have properties (which is not the same as to say that wffs containing denotationless singular terms cannot be true). Thus, if *per contra* the above universal instantiation principle (UI_2^*) of wffs for predicate variables were provable in HST_λ^* , then by substituting $\sim \exists x(a \equiv x)$ for $F(a)$ in the above theorem, it would follow that $(A3)$ is provable in HST_λ^* after all; i.e., every term would in that case denote a real individual according to HST_λ^* .

With arbitrary λ -abstracts, however, there are terms for which it is provable that they must fail to denote a real individual. In particular, it is provable in HST_λ^* that there is no such individual as *the Russell property*:

$$\frac{}{HST_\lambda^*} \sim \exists y([\lambda x \exists G(x \equiv G \ \& \ \sim G(x))] \equiv y) ,$$

and similarly that there is no such individual as *the Russell (binary) relation*:

$$\frac{}{HST_\lambda^*} \sim \exists y([\lambda xz \exists R(x \equiv R \ \& \ \sim R(x, z))] \equiv y) ;$$

and therefore the above instantiation principle (UI_2^*) is actually disprovable in HST_λ^* when ψ is $\sim \exists x(a \equiv x)$, where a is either of these λ -abstracts, and ϕ is the above theorem that only real individuals have properties in HST_λ^* .

One interesting consequence of the theorem that only real individuals can have properties, incidentally, is the further thesis that nonreal “individuals” are indiscernible from one another:

$$\frac{}{HST_\lambda^*} \sim \exists x(a \equiv x) \ \& \ \sim \exists x(b \equiv x) \rightarrow a \equiv b ,$$

where x is not free in either a or b . Thus, in particular, every property which the Russell property has the Russell relation also has, and conversely; that is,

$$\frac{}{HST_\lambda^*} [\lambda x \exists G(x \equiv G \ \& \ \sim G(x))] \equiv [\lambda xz \exists R(x \equiv R \ \& \ \sim R(x, z))] ,$$

and therefore by $(EG_{2\lambda}^*)$, i.e., by the contrapositive of $(UI_{2\lambda}^*)$,

$$\frac{}{HST_\lambda^*} \exists F^1 \exists G^2 (F \equiv G) .$$

In other words, the thesis (Inf^*) is refutable in HST_λ^* . The qualified version of this thesis, however, i.e., where $n \neq k$,

$$(\exists/Inf^*) \quad \forall F^n \forall G^k [\exists x(F \equiv x) \vee \exists x(G \equiv x) \rightarrow F \equiv G] ,$$

is not refutable in HST_λ^* and in fact can be shown to be consistent in HST_λ^* if $HST^* + (Inf^*)$ is consistent.

Finally, although (Ind^*) holds in HST_λ^* only in its qualified form, there is no problem about its converse, (Ext^*) . That is, although (Ext^*) is not provable in HST_λ^* (which is as it should be), nevertheless it can be shown that $HST_\lambda^* + (Ext^*)$ is consistent if $HST^* + (Ext^*)$ is. Similarly, where $HST_{\lambda\Box}^*$ stands to HST_λ^* the way $\Box \lambda HST^*$ stands to λHST^* , i.e., where $HST_{\lambda\Box}^*$ is HST_λ^* supplemented with the $S5$ modal propositional logic, the Carnap-Barcan

wffs for quantifiers and \Box , and the rule of modal generalization, then it can be shown that $HST_{\lambda\Box}^* + (\Box Ext^*)$ is consistent if $HST^* + (Ext^*)$ is.

Given either (Ext^*) or $(\Box Ext^*)$, needless to say, we can prove a λ -elimination theorem for HST_{λ}^* just the way we did for λHST^* . However, because (UI_1^*) does not hold in HST_{λ}^* except in its qualified form, the leading lemma for this result must be stated in the following somewhat more complex form.

Lemma *If ϕ, ψ are wffs, F^n and y_1, \dots, y_n do not occur in either ϕ or ψ , and $[\lambda x_1 \dots x_n \psi](a_1, \dots, a_n), \dots, [\lambda x_1 \dots x_n \psi](a_{k_1}, \dots, a_{k_n})$ are all the subwffs of ϕ in which $[\lambda x_1 \dots x_n \psi]$ occurs free in ϕ in a predicate position, then where*

$$\begin{aligned} \chi_i =_{df} & (\Box)(F(a_1, \dots, a_n) \leftrightarrow \exists y_1 \dots \exists y_n [a_{i_1} \equiv y_1 \\ & \& \dots \& a_{i_n} \equiv y_n \& \psi(y_1/x_1, \dots, y_n/x_n)]) , \\ ((\Box)Ext^*) & \frac{}{HST_{\lambda(\Box)}^*} \phi \leftrightarrow \exists F^n [(\Box)\forall x_1 \dots \forall x_n (F(x_1, \dots, x_n) \leftrightarrow \psi) \\ & \& \chi_1 \& \dots \& \chi_k \& \phi(F^n/[\lambda x_1 \dots x_n \psi])] . \end{aligned}$$

λ -Elimination Theorem *For each wff ϕ there is a λ -free wff ψ such that ψ is homogeneously stratified if ϕ is, and $((\Box)Ext^*) \frac{}{HST_{\lambda(\Box)}^*} (\phi \leftrightarrow \psi)$.*

6 An extensional Fregean semantics for nominalized predicates A Fregean interpretation of nominalized predicates, we have already noted, interprets such occurrences as standing not for the concepts these predicates stand for when occurring in predicate positions but for certain individuals called concept-correlates. Now whether or not we agree with Frege that what predicates stand for in predicate positions cannot themselves be individuals, the idea of a correlation between such entities and individuals can nevertheless be easily incorporated into the well-known semantics or model theory of standard second-order logic in such a way as to provide a simple and direct semantics for nominalized predicates as well. Of course, because this sort of semantics is extensional to begin with, the one drawback in proceeding in this way is that we validate (Ext^*) . (Later, in intensionalizing this semantics, we validate $(\Box Ext^*)$ instead.)

Now we do not claim that the extensional (and later intensional) frames and models described below provide the intended interpretation of either λHST^* or HST_{λ}^* (unless of course one adopts either an extensional or intensional Fregean view of properties and relations as unsaturated entities). What we do claim, however, is that these frames and models serve as a guide into the structure, if not the specific content, of whatever one might take to be the intended interpretation of either λHST^* or HST_{λ}^* ; and in that regard they provide an important and indispensable service which should not be shunned. In particular, because they contain the general frames and models of standard second-order logic, the Fregean frames and models for λHST^* and HST_{λ}^* allow us to exploit and utilize the well-known techniques and results of standard model theory. Thus, because of the completeness of $\lambda HST^* + (Ext^*)$ and $HST_{\lambda}^* + (Ext^*)$ with respect to the appropriate classes of Fregean frames and models, we are able to prove the consistency of $HST_{\lambda}^* + (Ext^*)$ (and therefore of HST_{λ}^* alone) relative to the already proved consistency of $HST^* + (Ext^*)$ relative to weak Zermelo

set theory. Of course, once we are assured of the consistency of HST_λ^* (to the extent that we are assured of the consistency of weak Zermelo set theory), we can, as it were, kick away the Fregean semantical ladder we have climbed and adopt HST_λ^* as a semantical framework in its own right from which to view the world (especially from the perspective of natural language). For those who do not believe in the reality of this view of the world, on the other hand, the Fregean semantics we provide below can help them better understand, or at least to model, what they take to be the ravings of those who do.

Accordingly, by an *extensional Fregean frame* let us understand an indexed triple $\mathfrak{A} = \langle D, X_n, f \rangle_{n \in \omega}$ such that (1) D is a nonempty domain of discourse; (2) for $n \in \omega$, X_n is a nonempty subset of $\mathcal{P}(D^n)$; and (3) there is a set D' such that $D \subseteq D'$ and f is a function with $D' \cup \bigcup_{n \in \omega} X_n$ as domain such that for all $d \in D'$, $f(d) = d$, and for $n \in \omega$, $K \in X_n$, $f(K) \in D'$. We take the function f in this definition to represent the Fregean correlation of objects with concepts. We add to this correlation the identity function on the superset D' only for the convenience of the semantic definitions that are to follow. Note that by definition $D' = \text{rng}(f)$; i.e., D' consists of all the Fregean correlates of the objects and concepts represented in the frame. It is this set, it should be noted, which we shall take as the range of values of the *free* individual variables. Of course, D , the domain of discourse, is taken as the range of values of the *bound* individual variables, and in frames of models of λHST^* it will turn out that $D = D' = \text{rng}(f)$. Note that a frame is *standard*, incidentally, if $X_n = \mathcal{P}(D^n)$, for all $n \in \omega$.⁵

Now by an *assignment (of values to variables)* in an extensional Fregean frame defined as above we understand a function A with the set of variables (of all types) as domain such that: (1) for all $n \in \omega$ and all F^n , $A(F^n) \in X_n$, and (2) $A(x) \in \text{rng}(f)$, for all individual variables x . In addition, where L is a language, \mathfrak{A} is an extensional Fregean frame, and g is a function with L as domain such that: (1) $g(P^n) \in X_n$, for all n -place predicate constants in L and (2) $g(c) \in \text{rng}(f)$, for all individual constants in L , then we say that $I = \langle g, \mathfrak{A} \rangle$ is an (*extensional*) *model for L*. Also, for convenience, where $n \in \omega$, we identify *the set of possible extensions of type n in \mathfrak{A}* as follows:

- (1) $Ext_{0, \mathfrak{A}} = D$
- (2) $Ext_{n+1, \mathfrak{A}} = X_n$.

If L is a language and $I = \langle g, \mathfrak{A} \rangle$ is an extensional model for L , then we shall say that I is a *general model for L* if there is a semantic function ext_I defined for each assignment A in \mathfrak{A} in such a way that $ext_{I,A}$ is a function with $\bigcup_{n \in \omega} ME_n(L)$ as domain such that for $n \in \omega$ and $\xi \in ME_{n+1}(L)$, $ext_{I,A}(\xi) \in Ext_{n+1, \mathfrak{A}}$ and also satisfying the following conditions:

- (1) if ξ is a variable, then $ext_{I,A}(\xi) = A(\xi)$; and if ξ is a descriptive constant in L , then $ext_{I,A}(\xi) = g(\xi)$;
- (2) if ξ is $\pi(a_1, \dots, a_k)$, where $\pi \in ME_{k+1}(L)$ and $a_1, \dots, a_k \in ME_0(L)$, then $ext_{I,A}(\xi) = 1$ iff $\langle f(ext_{I,A}(a_1)), \dots, f(ext_{I,A}(a_k)) \rangle \in ext_{I,A}(\pi)$;
- (3) if ξ is $[\lambda x_1 \dots x_k \phi]$, where $\phi \in ME_1(L)$, then $ext_{I,A}(\xi) = \{ \langle d_1, \dots, d_k \rangle \in D^k : ext_{I,A}(d_1/x_1, \dots, d_k/x_k)(\phi) = 1 \}$;

- (4) if ξ is $\sim\phi$, where $\phi \in ME_1(L)$, then $ext_{I,A}(\xi) = 1$ iff $ext_{I,A}(\phi) = 0$;
 (5) if ξ is $(\phi \rightarrow \psi)$, where $\phi, \psi \in ME_1(L)$, then $ext_{I,A}(\xi) = 1$ iff either $ext_{I,A}(\phi) = 0$ or $ext_{I,A}(\psi) = 1$;
 (6) if ξ is $\forall a\phi$, where $\phi \in ME_1(L)$ and a is a variable of type k , then $ext_{I,A}(\xi) = 1$ iff for all $J \in Ext_{k,\mathfrak{A}}$, $ext_{I,A(J/a)}(\phi) = 1$; and
 (7) if ξ is $[\lambda\phi]$, where $\phi \in ME_1(L)$, then $ext_{I,A}(\xi) = ext_{I,A}(\phi)$.

If \mathfrak{A} is a standard Fregean frame, needless to say, then such a semantic function as ext_I is easily seen to exist. But whether \mathfrak{A} is standard or not, given such a function we can define the satisfaction and truth of a wff ϕ of L as follows:

- (1) A satisfies ϕ in I iff $ext_{I,A}(\phi) = 1$; and
 (2) ϕ is true in I iff every assignment in \mathfrak{A} satisfies ϕ in I .

The semantic function ext_I , it should be noted, is defined for all meaningful expressions of L and not just for those which are well-formed in λHST^* , i.e., those in which every λ -abstract is homogeneously stratified. For convenience, let us say that a model I is a *general model for L* (λHST^*) if there is a semantic function ext_I defined as above for all assignments A in \mathfrak{A} and all meaningful expressions of L which are well-formed in λHST^* regardless whether $ext_{I,A}$ is also defined for expressions containing λ -abstracts which are not homogeneously stratified. Note that although every general model for L is a general model for $L(\lambda HST^*)$, not every general model for $L(\lambda HST^*)$ need also be a general model for L .

Whether I is a general model for L or only for $L(\lambda HST^*)$, note that every instance of (Ext^*) , $(A1)$, $(A2)$, $(A3')$, (LL_1^*) , $(\exists/\lambda-Conv^*)$, (\exists/Id_λ^*) , and therefore (Id_λ^*) as well, for which ext_I is defined is true in I . Every instance of (LL^*) for which ext_I is defined would be true in I if every instance of (Ind^*) were true in I ; but note that there are some general models in which instances of (Ind^*) are false (such as those for the system $T^*(*)$ of [1]). Also if every instance of (Ind^*) were true in I , then where $\mathfrak{A} = \langle D, X_n, f \rangle_{n \in \omega}$ is the Fregean frame upon which I is based, it would follow that $f \upharpoonright X_n$ is one-to-one, for all $n \in \omega$. If every instance of $(A3)$ for which ext_I is defined is also true in I , then every instance of $(\lambda-Conv^*)$ for which ext_I is defined is also true in I , and, moreover, $D = rng(f)$.

Now we shall say that I is a *model of $\lambda HST^* + (Ext^*)$* iff for some language L , I is a general model for $L(\lambda HST^*)$ in which every instance of $(A3)$, $(HSCP_\lambda^*)$ and (Ind^*) which is a wff of L and of λHST^* is true. It follows, accordingly, that if I is a model of $\lambda HST^* + (Ext^*)$ then every theorem of $\lambda HST^* + (Ext^*)$ which is a wff of L is true in I .

Similarly, we shall say that I is a *model of $HST^* + (Ext^*)$* iff for some language L , I is a general model for L in which every instance of $(UI_{2\lambda}^*)$, $(\exists/HSCP_\lambda^*)$ and (\exists/Ind^*) which is a wff of L is true. Accordingly, if I is a model of $HST^* + (Ext^*)$, then every theorem of $HST^* + (Ext^*)$ which is a wff of L is true in I . Note, incidentally, that since $\forall F \exists x (F \equiv x)$ is true in any model of $\lambda HST^* + (Ext^*)$ and false in any model of $HST^* + (Ext^*)$, then no model of the one system is a model of the other.

Finally, if $\Gamma \cup \{\phi\}$ is a set of wffs, then we define *logical consequence* in $\lambda HST^* + (Ext^*)$ and $HST_\lambda^* + (Ext^*)$, respectively, as follows:

- (1) $\Gamma \frac{}{\lambda HST^* + (Ext^*)} \phi$ iff $\Gamma \cup \{\phi\}$ is a set of wffs of λHST^* and for all languages L , all general models $I = \langle g, \mathfrak{A} \rangle$ for $L(\lambda HST^*)$ and all assignments A in \mathfrak{A} , if I is a model of $\lambda HST^* + (Ext^*)$, $\Gamma \cup \{\phi\}$ is a set of wffs of L , and A satisfies every member of Γ in I , then A satisfies ϕ in I ; and
- (2) $\Gamma \frac{}{HST_\lambda^* + (Ext^*)} \phi$ iff for all languages L , all general models $I = \langle g, \mathfrak{A} \rangle$ for L , and all assignments A in \mathfrak{A} , if I is a model of $HST_\lambda^* + (Ext^*)$, $\Gamma \cup \{\phi\}$ is a set of wffs of L , and A satisfies every member of Γ in I , then A satisfies ϕ in I .

Because the proof of the following completeness theorems is similar to that for the completeness theorems of the intensional semantics which is to follow we avoid giving the details of the proof here.

Completeness theorem for $\lambda HST^* + (Ext^*)$ *If $\Gamma \cup \{\phi\}$ is a set of wffs of λHST^* , then:*

$$\Gamma \frac{}{\lambda HST^* + (Ext^*)} \phi \text{ iff } \Gamma \cup (Ext^*) \frac{}{\lambda HST^*} \phi .$$

Completeness theorem for $HST_\lambda^* + (Ext^*)$ *If $\Gamma \cup \{\phi\}$ is a set of wffs, then:*

$$\Gamma \frac{}{HST_\lambda^* + (Ext^*)} \phi \text{ iff } \Gamma \cup (Ext^*) \frac{}{HST_\lambda^*} \phi .$$

Finally, since a model of $\lambda HST^* + (Ext^*)$ is based on what in [2] we called a homogeneously stratified Fregean frame, then it follows that I is a model of $\lambda HST^* + (Ext^*)$ iff it is also a model of $HST^* + (Ext^*)$ as defined in [2]. Accordingly, by the above completeness theorem for $\lambda HST^* + (Ext^*)$ and the completeness theorem for $HST^* + (Ext^*)$ in [2], it follows that a λ -free wff is a theorem of $\lambda HST^* + (Ext^*)$ iff it is already a theorem of $HST^* + (Ext^*)$; i.e., $\lambda HST^* + (Ext^*)$ is a conservative extension of $HST^* + (Ext^*)$.

Theorem $\lambda HST^* + (Ext^*)$ is a conservative extension of $HST^* + (Ext^*)$.

7 The relative consistency of $HST_\lambda^* + (Ext^*)$ to $\lambda HST^* + (Ext^*)$ A completeness theorem, it might be objected, does not show that a system is consistent, and in particular it does not show that there are any models of $\lambda HST^* + (Ext^*)$ or $HST_\lambda^* + (Ext^*)$. However, since we have already shown that $\lambda HST^* + (Ext^*)$ is consistent relative to monadic $HST^* + (Ext^*)$, and that monadic $HST^* + (Ext^*) (+Inf^*)$ is consistent relative to weak Zermelo set theory $(+inf)$, then it follows that there is a model of $\lambda HST^* + (Ext^*) (+Inf^*)$ if weak Zermelo set theory $(+Inf)$ is consistent. We now show that any model of $\lambda HST^* + (Ext^*) (+Inf^*)$ can be converted into a model of $HST_\lambda^* + (Ext^*) (+\exists/Inf^*)$.

Theorem *If $\lambda HST^* + (Ext^*) (+Inf^*)$ is consistent, then so is $HST_\lambda^* + (Ext^*) (+\exists/Inf^*)$.*

Proof: Suppose $\lambda HST^* + (Ext^*) + (Inf^*)$ is consistent and that L is any language. Then, by the completeness theorem there is a model $I = \langle g, \mathfrak{A} \rangle$ of $\lambda HST^* + (Ext^*)$, i.e., a general model I for $L(\lambda HST^*)$ in which every wff of L which is a theorem of $\lambda HST^* + (Ext^*)$ is true. Where $\mathfrak{A} = \langle D, X_n, f \rangle_{n \in \omega}$, let $Y_n = \mathcal{P}(D^n)$, for $n \in \omega$, and let f_λ be that function with $D \cup \bigcup_{n \in \omega} Y_n$ as domain such that $f_\lambda(d) = f(d) = d$, for all $d \in D$, and for $n \in \omega$, $K \in Y_n$, $f_\lambda(K) = f(K)$ if $K \in X_n$, and otherwise $f_\lambda(K) = \langle K, D \rangle$; and finally let $\mathfrak{A}_\lambda = \langle D, Y_n, f_\lambda \rangle_{n \in \omega}$. It is clear of course that \mathfrak{A}_λ is an extensional Fregean frame and that $I_\lambda = \langle g, \mathfrak{A}_\lambda \rangle$ is a general model for L (since \mathfrak{A}_λ is standard). It remains only to show, accordingly, that every wff of L which is an instance of either $(UI_{2\lambda}^*)$, $(\exists/HSCP_\lambda^*)$ or (\exists/Ind^*) (or of (\exists/Inf^*) if (Inf^*) is true in I) is true in I_λ .

(1) We note first that by construction, for all assignments A in \mathfrak{A}_λ and all $n \in \omega$: A satisfies $\exists x(y \equiv x)$ in I_λ iff $A(y) \in D$; and A satisfies $\exists x(F^n \equiv x)$ in I_λ iff $A(F^n) \in X_n$. (Observe that $D^1 \in X_1$ since $\frac{}{\lambda HST^*} \forall y[\lambda x(x \equiv x)](y)$, and therefore for all $d \in D$, $\langle d \rangle \in D^1 \in X_1$. Thus, if $A(d/x)$ satisfies $(y \equiv x)$ in I_λ , for some $d \in D$, then $\langle f_\lambda(A(y)) \rangle \in D^1$; and since $f_\lambda(A(y)) = A(y)$, by definition, then $A(y) \in D$. Similarly, if $A(d/x)$ satisfies $(F \equiv x)$ in I_λ , then $f_\lambda(A(d/x)(F^n)) \in D$, and since $A(d/x)(F^n) = A(F^n)$, then $f_\lambda(A(F^n)) \in D$. But if $A(F^n) \notin X_n$, then $f_\lambda(A(F^n)) = \langle A(F^n), D \rangle$, which is impossible if $f_\lambda(A(F^n)) \in D$. That is, if A satisfies $\exists x(F \equiv x)$ in I_λ , then $A(F^n) \in X_n$. The converse direction in both cases is similar but even more trivial.)

(2) By a simple induction on $ME_n(L)$, for $n \in \omega$, we note that for all $\xi \in ME_n(L)$, if ξ is bound to individuals, every λ -abstract occurring in ξ is homogeneously stratified, and A, B are assignments in \mathfrak{A} and \mathfrak{A}_λ , respectively, such that for all variables a free in ξ , $A(a) = B(a)$, then $ext_{I,A}(\xi) = ext_{I_\lambda,B}(\xi)$. (We consider only the quantifier case where ξ is of the form $\forall F^n \phi$, for some wff ϕ of L . Note that since ξ is bound to individuals, then ϕ must be of the form $(\exists x(F \equiv x) \rightarrow \psi)$; and therefore, by (1) above and the inductive hypothesis, $ext_{I_\lambda,B}(\forall F^n \phi) = 1$ iff for all $K \in X_n$, $ext_{I,A(K/F^n)}(\psi) = 1$; i.e., iff $ext_{I,A}(\forall F^n \phi) = 1$.)

(3) Every wff of L which is an instance of $(UI_{2\lambda}^*)$ is true in I_λ . (This is an immediate consequence of the fact that each Y_n of \mathfrak{A}_λ is "full", i.e., $Y_n = \mathcal{P}(D^n)$.)

(4) Every wff of L which is an instance of $(\exists/HSCP_\lambda^*)$ is true in I_λ . (To show this, assume A is an assignment in \mathfrak{A}_λ which satisfies $\exists y(a_i \equiv y)$ in I_λ , for each variable or nonlogical constant a_i free in $[\lambda x_1 \dots x_n \phi]$, where $[\lambda x_1 \dots x_n \phi]$ is homogeneously stratified, ϕ is bound to individuals, and y is new to ϕ . Note that by (1) above, $A(a_i) \in D$ if a_i is an individual variable, and $A(a_i) \in X_k$ if a_i is a k -place predicate variable, for all $k \in \omega$. Now let B be any assignment in \mathfrak{A} such that $B(a_i) = A(a_i)$ (and of course, by the preceding note, there must be at least one such assignment). Note that by (A3),

$$\frac{}{\lambda HST^*} \exists y([\lambda x_1 \dots x_n \phi] \equiv y) ,$$

and therefore B satisfies $\exists y([\lambda x_1 \dots x_n \phi] \equiv y)$ in I , from which it follows, by (2) above, that A satisfies $\exists y([\lambda x_1 \dots x_n \phi] \equiv y)$ in I_λ .)

(5) Every instance of (\exists/Ind^*) is true in I_λ . This follows from the fact that

$$\frac{}{HST_\lambda^*} \exists y(F^n \equiv y) \ \& \ (F^n \equiv G^n) \rightarrow \exists y(G^n \equiv y) \ ,$$

and therefore

$$\frac{}{HST_\lambda^*} (\exists/Ind_n^*) \leftrightarrow \forall F^n [\exists y(F \equiv y) \rightarrow \forall G^n (\exists y(G \equiv y) \rightarrow [F \equiv G \rightarrow \forall x_1 \dots \forall x_n [F(x_1, \dots, x_n) \leftrightarrow G(x_1, \dots, x_n)])]] \ .$$

But the right-hand side of this last theorem is a wff bound to individuals and, by (A3), is equivalent in λHST^* to (Ind^*) , which, by hypothesis, is true in I . Therefore, by (2) above, (\exists/Ind_n^*) is true in I_λ .

(6) Every instance of (\exists/Inf^*) is true in I_λ if every instance of (Inf^*) is true in I . (Note that where $n \neq k$,

$$\frac{}{HST_\lambda^*} \exists x(F^n \equiv x) \ \& \ \sim \exists x(G^k \equiv x) \rightarrow F^n \neq G^k \ ,$$

$$\frac{}{HST_\lambda^*} \sim \exists x(F^n \equiv x) \ \& \ \exists x(G^k \equiv x) \rightarrow F^n \neq G^k \ ,$$

and therefore by tautologous transformations,

$$\frac{}{HST_\lambda^*} [\exists x(F^n \equiv x) \vee \exists x(G^k \equiv x) \rightarrow F^n \neq G^k] \leftrightarrow [\exists x(F^n \equiv x) \ \& \ \exists x(G^k \equiv x) \rightarrow F^n \neq G^k] \ .$$

Thus, by universal generalization and distribution:

$$\frac{}{HST_\lambda^*} (\exists/Inf^*) \leftrightarrow \forall F^n \forall G^k [\exists x(F \equiv x) \ \& \ \exists x(G \equiv y) \rightarrow F^n \neq G^k] \ .$$

But the right-hand side here is bound to individuals and, by (A3), is equivalent to (Inf^*) in λHST^* . That is, the right-hand side is true in I if (Inf^*) is true in I ; and therefore by (2) above, the right-hand side is true in I_λ if (Inf^*) is true in I . But by (3)–(5), I_λ is a model of $HST_\lambda^* + (Ext^*)$, and therefore (\exists/Inf^*) is true in I_λ if (Inf^*) is true in I .

8 An intensional Fregean semantics for nominalized predicates Let us now add the necessity operator \Box to our list of primitive logical constants (and take \Diamond as an abbreviation of $\sim\Box\sim$). Where L is a language, let us add the following clause to the recursive definition of a meaningful expression of L :

(9) if $\phi \in ME_1(L)$, then $\Box\phi \in ME_1(L)$;

and let us understand all our previous definitions about the wffs of L , the wffs of λHST^* , etc., to now include expressions with \Box as well.

We intensionalize the extensional Fregean semantics given in Section 6 as follows. A doubly indexed triple $\langle D, X_n, f_i \rangle_{n \in \omega, i \in W}$ is said to be an *intensional Fregean frame* if: (1) D and W are nonempty sets, (2) $X_n \subseteq \mathcal{P}(D^n)^W$, for $n \in \omega$, and (3) there is a set D' such that $D \subseteq D'$ and for $i \in W$, f_i is a function with $D' \cup \bigcup_{n \in \omega} X_n$ as domain such that $f_i(d) = d$, for all $d \in D'$, and for $n \in \omega$, $K \in X_n$, $f_i(K) \in D'$. We take W to be the set of possible worlds (or possible contexts of use) of the frame. Properties and relations (or concepts) are

then represented by functions from possible worlds to extensions of the relevant type. The frame is said to be *standard* if $X_n = \mathcal{P}(D^n)^W$, for all $n \in \omega$.

As in the extensional frames we take the set D of an intensional Fregean frame to be the domain of discourse of “real” individuals, i.e., D is the range of values of bound individual variables. Free individual variables, as in the extensional semantics, are allowed to range over the superset D' which by definition is identical with $\bigcup_{i \in W} \text{rng}(f_i)$. Note that we allow nominalized predicates to denote different individuals in different possible worlds. This is allowed in order to accommodate those Fregeans who interpret nominalized predicates as denoting the extensions of the concepts otherwise expressed by these predicates in their role as predicates. (A concept’s extension in one possible world, needless to say, need not be the same as its extension in a different possible world.) Having made this allowance, however, we shall not in fact pursue such an interpretation of nominalized predicates here—or rather we shall not when dealing with the intensional models of $\Box\lambda HST^* + (\Box Ext^*)$ and $HST^*_{\lambda\Box} + (\Box Ext^*)$. For that purpose we say that an intensional Fregean frame as defined above is *rigid* if for all $i, j \in W$, $f_i = f_j$. It is only on the assumption of rigidity, in other words, that we will be able to validate $\forall F^n \forall G^n [F \equiv G \rightarrow \Box(F \equiv G)]$ and reject $\exists F^n \exists G^n [F \equiv G \ \& \ \Diamond(F \neq G)]$.

Note that the domain of discourse of an intensional Fregean frame is common to all the possible worlds (or contexts of use) in the frame. We assume in this regard that being-the-value-of-a-bound-individual-variable is a semantic property which corresponds only to the *being* of an individual and not to its *concrete existence*, where the latter is a property which an individual can have in some possible worlds and not have in others. Thus if properties and relations are individuals, then their *being as individuals* should not be identified with their existence in a given possible world; and, indeed, if by existence we mean (world bound) concrete existence, then properties and relations are individuals which *cannot exist*. (Cf. [3] for a development of such a view.) The individuals denoted by nominalized predicates, in other words, all have *being* (as values of the bound individual variables), but it might be that none of these individuals ever *exist* if existence can be a property only of concrete individuals. We shall not concern ourselves here, however, with the representation of (concrete) existence, and we leave its analysis to more specialized applications of $\Box\lambda HST^*$ or $HST^*_{\lambda\Box}$.

Now where $\mathfrak{A} = \langle D, X_n, f_i \rangle_{n \in \omega, i \in W}$ is an intensional Fregean frame, we say that A is an *assignment in* \mathfrak{A} if A is a function with the set of variables as domain such that: (1) for all $n \in \omega$, all n -place predicate variables F^n , $A(F^n) \in X_n$ and (2) for all individual variables x , $A(x) \in \bigcup_{i \in W} \text{rng}(f_i)$. In addition, if L is a language and g is a function with L as domain such that for $n \in \omega$ and $P^n \in L$, $g(P^n) \in X_n$, and for each individual constant $c \in L$, $g(c) \in \bigcup_{i \in W} \text{rng}(f_i)$, then we say that $I = \langle g, \mathfrak{A} \rangle$ is an *intensional model for* L .

Individual constants are interpreted as “rigid designators”, incidentally, since we will be concerned below only with rigid intensional Fregean frames. We now identify *the set of possible intensions of type n in* \mathfrak{A} as follows:

- (1) $Int_{0, \mathfrak{A}} = D$,
- (2) $Int_{n+1, \mathfrak{A}} = X_n$.

If L is a language and $I = \langle g, \mathfrak{A} \rangle$ is an intensional model for L , then we shall say that I is a *general intensional model for L* if there is a semantic function int_I defined for each assignment A in \mathfrak{A} in such a way that $int_{I,A}$ is a function with $\bigcup_{n \in \omega} ME_n(L)$ as domain such that for $n \in \omega$ and $\xi \in ME_{n+1}(L)$, $int_{I,A}(\xi) \in Int_{n+1, \mathfrak{A}}$ and also satisfying the following conditions for all $i \in W$:

- (1) if ξ is a variable, then $int_{I,A}(\xi) = A(\xi)$; and if $\xi \in L$, then $int_{I,A}(\xi) = g(\xi)$;
- (2) if ξ is $\pi(a_1, \dots, a_k)$, where $\pi \in ME_{k+1}(L)$ and $a_1, \dots, a_k \in ME_0(L)$, then $int_{I,A}(\xi)(i) = 1$ iff $\langle f_i(int_{I,A}(a_1)), \dots, f_i(int_{I,A}(a_k)) \rangle \in int_{I,A}(\pi)(i)$;
- (3) if ξ is $[\lambda x_1 \dots x_k \phi]$, where $\phi \in ME_1(L)$, then $int_{I,A}(\xi)(i) = \{ \langle d_1, \dots, d_k \rangle \in D^k : int_{I,A(d_1/x_1, \dots, d_k/x_k)}(\phi)(i) = 1 \}$;
- (4) if ξ is $\sim \phi$, where $\phi \in ME_1(L)$, then $int_{I,A}(\xi)(i) = 1$ iff $int_{I,A}(\phi)(i) = 0$;
- (5) if ξ is $\Box \phi$, where $\phi \in ME_1(L)$, then $int_{I,A}(\xi)(i) = 1$ iff for all $j \in W$, $int_{I,A}(\phi)(j) = 1$;
- (6) if ξ is $(\phi \rightarrow \psi)$, where $\phi, \psi \in ME_1(L)$, then $int_{I,A}(\xi)(i) = 1$ iff either $int_{I,A}(\phi)(i) = 0$ or $int_{I,A}(\psi)(i) = 1$;
- (7) if ξ is $\forall a \phi$, where $\phi \in ME_1(L)$ and a is a variable of type k , then $int_{I,A}(\xi)(i) = 1$ iff for all $J \in Int_{k, \mathfrak{A}}$, $int_{I,A(J/a)}(\phi)(i) = 1$; and
- (8) if ξ is $[\lambda \phi]$, where $\phi \in ME_1(L)$, then $int_{I,A}(\xi) = int_{I,A}(\phi)$.

The existence of the semantic function int_I is assured, needless to say, if \mathfrak{A} is a standard intensional Fregean frame. But, again, whether \mathfrak{A} is standard or not, given such a function we can define satisfaction, truth, and validity in I of a wff ϕ of L as follows, where A is an assignment in \mathfrak{A} and $i \in W$:

- (1) A satisfies ϕ in I at i iff $int_{I,A}(\phi)(i) = 1$;
- (2) ϕ is true in I at i iff every assignment in \mathfrak{A} satisfies ϕ in I at i ; and
- (3) ϕ is valid in I iff for all $j \in W$, ϕ is true in I at j .

As in Section 6, we observe that int_I is defined for all meaningful expressions of L and not just for those that are well-formed in $\Box \lambda HST^*$. For the restricted class of meaningful expressions let us say that I is a *general intensional model for $L(\Box \lambda HST^*)$* if there is a semantic function int_I defined as above for all assignments A in \mathfrak{A} and all meaningful expressions of L which are well formed in $\Box \lambda HST^*$ regardless whether int_I is also defined for meaningful expressions of L containing λ -abstracts which are not homogeneously stratified. Again, we observe that although every general intensional model for L is a general intensional model for $L(\Box \lambda HST^*)$, the converse does not also hold.

Whether I is a general intensional model for L or only for $L(\Box \lambda HST^*)$, however, note that every instance of $(\Box Ext^*)$, $(A1)$, $(A2)$, $(A3')$, $(\exists/\lambda-Conv^*)$, (\exists/Id_λ^*) , and (Id_λ^*) for which int_I is defined is valid in I . If I is based on a rigid intensional Fregean frame, then every instance of (LL_1^*) for which int_I is defined is valid in I ; and, as in Section 6, if every instance of (Ind^*) is also valid in I , then every instance of (LL^*) for which int_I is defined is valid in I . Note also, however, that any instance of any one of the following modal schemas for which int_I is defined is also valid in I :

- (M1) $\Box\phi \rightarrow \phi$
(M2) $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$
(M3) $\Diamond\phi \rightarrow \Box\Diamond\phi$
(M4) $\forall a\Box\phi \leftrightarrow \Box\forall a\phi$ (where a is any variable).

Accordingly, where $\Box\lambda HST^*$ is the result of adding all wffs of λHST^* which are instances of the above modal theses to the axioms of λHST^* (now understood to apply to wffs with \Box as well) along with the rule of *modal generalization* as a new primitive inference rule, we shall say that a general intensional model for $L(\Box\lambda HST^*)$ is an *intensional model of $\Box\lambda HST^* + (\Box Ext^*)$* iff I is based on a rigid intensional Fregean frame and every wff of L which is a wff of $\Box\lambda HST^*$ and an instance of (A3), $(HSCP_\lambda^*)$, or (Ind^*) is valid in I . It follows, of course, that if I is an intensional model of $\Box\lambda HST^* + (\Box Ext^*)$, then every theorem of $\Box\lambda HST^* + (\Box Ext^*)$ which is a wff of L is valid in I .

Similarly, where $HST_{\lambda\Box}^*$ is HST_λ^* supplemented with the same modal theses (but for all wffs, not only just those with homogeneously stratified λ -abstracts) and the new rule of inference, we say that a general intensional model for a language L is an *intensional model of $HST_{\lambda\Box}^* + (\Box Ext^*)$* iff I is based on a rigid intensional Fregean frame and every wff of L which is an instance of $(UI_{2\lambda}^*)$, $(\exists/HSCP_\lambda^*)$ or (\exists/Ind^*) is valid in I . Accordingly, if I is an intensional model of $HST_{\lambda\Box}^* + (\Box Ext^*)$, then every theorem of $HST_{\lambda\Box}^* + (\Box Ext^*)$ which is a wff of L is valid in I .

Before moving on to the completeness theorems for these modal logics it is perhaps worth noting that whatever metaphysical assumption justifies accepting $(\Box Ext^*)$ in $HST_{\lambda\Box}^*$ also justifies defining necessity as follows:

$$\Box\phi =_{df} [\lambda\phi] \equiv [\lambda\forall x(x \equiv x)] .$$

These two wffs are provable equivalent in $HST_{\lambda\Box}^* + (\Box Ext^*)$, in other words, but not in $HST_{\lambda\Box}^*$ alone. Thus while adding \Box as a new logical primitive is pointless in $HST_{\lambda\Box}^* + (\Box Ext^*)$, it is not pointless in $HST_{\lambda\Box}^*$ alone.

It is not pointless to introduce \Box as a new logical primitive of $\Box\lambda HST^* + (\Box Ext^*)$, incidentally, since in that context the above definition would not suffice to explain the use of \Box with wffs that are not homogeneously stratified. That is, as a wff of λHST^* the above definiens would restrict the application of \Box to wffs that are homogeneously stratified (since only these wffs can occur within the λ -abstracts of λHST^*); and since there are wffs of λHST^* , including tautologous wffs, which are not homogeneously stratified, the above definition would in that case fail to validate the rule of modal generalization when applied to any axiom of λHST^* .

We assume that the semantic relations of *logical consequence* in $\Box\lambda HST^* + (\Box Ext^*)$ and $HST_{\lambda\Box}^* + (\Box Ext^*)$, respectively, have been defined as in the extensional semantics of Section 6 except for using satisfaction in I at any given world i of I in the definiens in place of the extensional notion of satisfaction in I simpliciter.

Completeness theorem for $\Box\lambda HST^* + (\Box Ext^*)$ *If Γ is a set of wffs of $\Box\lambda HST^*$ which is consistent in $\Box\lambda HST^* + (\Box Ext^*)$, then there are a language*

L , a general intensional model $I = \langle g, \mathfrak{A} \rangle$ for $L(\Box\lambda HST^*)$ which is an intensional model of $\Box\lambda HST^* + (\Box Ext^*)$, a possible world i of \mathfrak{A} , and an assignment A in \mathfrak{A} such that A satisfies every member of Γ in I at i .

Completeness theorem for $HST_{\lambda\Box}^* + (\Box Ext^*)$ *If Γ is a set of wffs which is consistent in $HST_{\lambda\Box}^* + (\Box Ext^*)$, then there are a language L , a general intensional model $I = \langle g, \mathfrak{A} \rangle$ for L which is an intensional model of $HST_{\lambda\Box}^* + (\Box Ext^*)$, a possible world i of \mathfrak{A} , and an assignment A in \mathfrak{A} such that A satisfies every member of Γ in I at i .*

Proof: We shall give a proof here only for $HST_{\lambda\Box}^* + (\Box Ext^*)$, since the proof for $\Box\lambda HST^* + (\Box Ext^*)$ is similar (and somewhat simpler since in that case free logic considerations can be ignored). We shall also assume, but only for convenience of presentation, that Γ is a set of wffs of a countable language. Maximal consistency (in $HST_{\lambda\Box}^*$) is defined as usual, and where L is a language we say that K is an ω -complete set of wffs of L if for all variables a and wffs ψ of L , $\exists a\psi \in K$ only if $\psi(b/a) \in K$ and $\exists a(b \equiv a) \in K$, for some term b of L of the same type as a which is free for a in ψ .

We now assume the hypothesis and take $c_0, \dots, c_n, \dots (n \in \omega)$ to be pairwise distinct individual constants new to all the wffs in Γ . Also, for $n \in \omega$, let $P_0^k, \dots, P_n^k, \dots (n \in \omega)$ be pairwise distinct k -place predicate constants new to Γ , and let L be the set of all these new descriptive constants together with those which occur in the wffs in Γ . We assume an ordering $\delta_1, \dots, \delta_n, \dots (n \in \omega)$ of all the wffs of L of the form $\exists a\phi$, where a is any variable. We define the chain $\Gamma_0, \dots, \Gamma_n, \dots (n \in \omega)$ by recursion as follows:

- (1) $\Gamma_0 = \Gamma$;
- (2) if δ_{n+1} is $\exists a\phi$ and ϕ is not an identity wff ($\xi \equiv a$), then

$$\Gamma_{n+1} = \Gamma_n \cup \{ \exists a\phi \rightarrow \phi(b/a) \ \& \ \exists a(b \equiv a) \} ,$$

where b is the first constant in L of the same type as a which is new to $\Gamma_n \cup \{ \delta_{n+1} \}$;

- (3) if δ_{n+1} is $\exists x(\xi \equiv x)$, then

$$\Gamma_{n+1} = \Gamma_n \cup \{ [\exists x(\xi \equiv x) \rightarrow \xi \equiv b], [\sim \exists x(\xi \equiv x) \rightarrow \xi \equiv b] \} ,$$

where b is the first individual constant in L which is new to $\Gamma_n \cup \{ \delta_{n+1} \}$.

We observe that, by hypothesis, Γ_0 is consistent in $HST_{\lambda\Box}^* + (\Box Ext^*)$. Also, if Γ_{n+1} is constructed as in clause (2) then by the usual argument Γ_{n+1} is consistent in $HST_{\lambda\Box}^* + (\Box Ext^*)$ if Γ_n is. If Γ_{n+1} is constructed as in clause (3), then it is not consistent only if $(\xi \neq b)$ is derivable from Γ_n in $HST_{\lambda\Box}^* + (\Box Ext^*)$; but since b is new to ξ and to Γ_n then it can be shown that $(\xi \neq \xi)$ is also derivable from Γ_n , which is impossible if Γ_n is consistent in $HST_{\lambda\Box}^* + (\Box Ext^*)$. We conclude, accordingly, that $\bigcup_{n \in \omega} \Gamma_n$ is consistent and that there is a maximally consistent set K of wffs of L such that $\Gamma_n \subseteq K$, for all $n \in \omega$. Note that by construction K is ω -complete. Also, for each $\xi \in ME_k(L)$, for $k \in \omega$, there is a descriptive constant b in L also of type k such that $(\xi \equiv b) \in K$.

Now let W be the set of ω -complete sets J of wffs of L which are maximally consistent in $HST_{\lambda\Box}^* + (\Box Ext^*)$ and such that for all wffs ψ of L , if $\Box\psi \in K$, then $\Box\psi \in J$. Of course, by definition, $K \in W$. We also claim that: (I) for all wffs ψ of L and all $J, J' \in W$, if $\Box\psi \in J$, then $\Box\psi \in J'$. For if $\Box\psi \notin J'$, then $\Box\psi \notin K$, in which case $\sim\Box\psi \in K$ and therefore by (M3) $\Box\sim\Box\psi \in K$, from which it follows that $\sim\Box\psi \in J$, which is impossible by hypothesis. In addition, we claim that: (II) for all $J \in W$ and all wffs ψ of L , if $\psi \in J$ and $\sim\Box\psi \in J$, then for some $J' \in W$, $\sim\psi \in J'$. To prove (II) we return to the above ordering of wffs of L of the form $\exists a\phi$ and define a new recursion as follows:

- (1) $\theta_0 = \sim\psi$
- (2) if $\Diamond(\theta_0 \& \dots \& \theta_n \& \delta_{n+1}) \notin K$, then $\theta_{n+1} = \theta_n$
- (3) if $\Diamond(\theta_0 \& \dots \& \theta_n \& \delta_{n+1}) \in K$ and $\delta_{n+1} = \exists a\phi$, then $\theta_{n+1} = \phi(b/a)$, where b is the first constant in L of the same type as a which is new to $(\theta_0 \& \dots \& \theta_n \& \delta_{n+1})$ and such that $\exists a(b \equiv a) \& \Diamond[\theta_0 \& \dots \& \theta_n \& \phi(b/a)] \in K$. (Note: by the modal thesis (M4) and the way K was constructed there must be at least one, and therefore a first, such constant b .)

Now it is easily seen by induction that for all $n \in \omega$, $\Diamond(\theta_0 \& \dots \& \theta_n) \in K$, and therefore that $\{\theta_0, \dots, \theta_n\}$ is consistent in $HST_{\lambda\Box}^* + (\Box Ext^*)$. Thus if $\Diamond\theta_0 \notin K$, then $\sim\Diamond\theta_0 \in K$, i.e., then $\Box\psi \in K$, and therefore $\Box\psi \in J$, which is impossible by assumption. Similarly, if $\Diamond(\theta_0 \& \dots \& \theta_n) \in K$, then by definition of θ_{n+1} it follows that $\Diamond(\theta_0 \& \dots \& \theta_{n+1}) \in K$. Accordingly where $M = \{\theta_n: n \in \omega\} \cup \{\Box\chi: \Box\chi \in K\}$, note that M is consistent in $HST_{\lambda\Box}^* + (\Box Ext^*)$. For by the S5 modal axioms $[\Diamond\phi_1 \& \Box\phi_2 \rightarrow \Diamond(\phi_1 \& \Box\phi_2)]$ is a provable schema, and therefore for all $n, k \in \omega$, $\Diamond(\theta_0 \& \dots \& \theta_n \& \Box\chi_1 \& \dots \& \Box\chi_k) \in K$, where $\Box\chi_i \in K$, from which it follows that M is consistent as claimed. Consequently, by Lindenbaum's lemma, there is a set M' of wffs L which is maximally consistent in $HST_{\lambda\Box}^* + (\Box Ext^*)$ and such that $M \subseteq M'$. But since $\sim\psi$ is θ_0 , then $\sim\psi \in M'$. Of course by construction M' is ω -complete in L , and therefore $M' \in W$, which concludes our argument for (II).

Now where a is a term of L , let $[a] = \{b: b \text{ is a term of } L \text{ and } (a \equiv b) \in K\}$, and let $D' = \{[a]: a \text{ is a term of } L\}$ and $D = \{[a] \in D': \text{for some individual variable } x \text{ not free in } a, \exists x(a \equiv x) \in K\}$. We observe that by (A3'), modal generalization and (M4), $\forall y\Box\exists x(x \equiv y)$ is provable in $HST_{\lambda\Box}^*$, and therefore by (\exists/UI_1^*) , $\exists x(a \equiv x) \rightarrow \Box\exists x(a \equiv x)$ is provable in $HST_{\lambda\Box}^*$. Thus, if $[a] \in D$, then $\exists x(a \equiv x) \in J$, for all $J \in W$.

Where $J \in W$, let \parallel_J be that function with the set of predicate expressions of L as domain such that for all $n \in \omega$ and all n -place predicate expressions π of L , $|\pi|_J = \{\langle [a_1], \dots, [a_n] \rangle \in D^n: \pi(a_1, \dots, a_n) \in J\}$. Now for $n \in \omega$ let $X_n = \{|\pi|: \pi \text{ is an } n\text{-place predicate expression of } L\}$; and for $J \in W$, let f_J be that function with $D' \cup \bigcup_{n \in \omega} X_n$ as domain such that for $[a] \in D'$, $f_J([a]) = [a]$, and for $n \in \omega$ and all n -place predicate expressions π of L , $f_J(|\pi|) = [\pi]$. Finally, let $\mathfrak{A} = \langle D, X_n, f_J \rangle_{n \in \omega, J \in W}$; and where g is that function with L as domain such that $g(c) = [c]$, for each individual constant c in L , and $g(P) = |P|$, for each predicate constant P in L , let $I = \langle g, \mathfrak{A} \rangle$.

We observe that by construction \mathfrak{A} is a rigid intensional Fregean frame and that I is an intensional model for L . We also claim (III) that I is a general intensional model for L .

To show (III) note that if $k \in \omega$ and ξ is a term of L of type k , then there is a constant b in L also of type k such that $(\xi \equiv b) \in K$ —and therefore $(\xi \equiv b) \in J$, for all $J \in \mathcal{W}$ since $\Box(\xi \equiv b) \in K$ by (LL^*) . For if $k > 0$, then the existence of such a constant follows by (CP_λ^*) and the fact that K is ω -complete in L ; and if $k = 0$, then its existence follows from the definition of Γ_{n+1} which is contained in K . Accordingly, if A is an assignment in \mathfrak{A} , $\xi \in ME_n(L)$, for some $n \in \omega$, and a_1, \dots, a_k are all the distinct variables occurring free in ξ , then there are constants b_1, \dots, b_k in L such that, for $1 \leq i \leq k$, b_i is of the same type as a_i and $A(a_i) = [b_i]$ if a_i is an individual variable and $A(a_i) = |b_i|$ if a_i is a predicate variable. Where b_1, \dots, b_k are the first such constants we set:

$$\xi_A =_{df} \xi(b_1/a_1, \dots, b_k/a_k) .$$

On the basis of the above identification we now recursively define the semantic function val_A with the set of meaningful expressions of L as domain and such that for $\xi \in ME_{n+1}(L)$, $val_A(\xi) \in X_n$. We will then show that $val_A(\xi)$ fulfills all of the conditions required for the function $int_{I,A}(\xi)$ in the definition of a general intensional model for L .

- (1) if ξ is a variable, then $val_A(\xi) = A(\xi)$; and if $\xi \in L$, then $val_A(\xi) = g(\xi)$;
- (2) if ξ is $\pi(a_1, \dots, a_k)$, then $val_A(\xi) = |[\lambda\pi(a_1, \dots, a_k)]_A|$;
- (3) if ξ is $[\lambda x_1 \dots x_n \phi]$, then $val_A(\xi) = |\xi_A|$;
- (4) if ξ is $\sim\phi$, then $val_A(\xi) = |[\lambda \sim\phi]_A|$;
- (5) if ξ is $(\phi \rightarrow \psi)$, then $val_A(\xi) = |[\lambda(\phi \rightarrow \psi)]_A|$;
- (6) if ξ is $\Box\phi$, then $val_A(\xi) = |[\lambda\Box\phi]_A|$;
- (7) if ξ is $\forall a\phi$, then $val_A(\xi) = |[\lambda\forall a\phi]_A|$;
- (8) if ξ is $[\lambda\phi]$, then $val_A(\xi) = |\xi_A|$.

By induction over the meaningful expressions of L it can be shown that val_A fulfills all the conditions for $int_{I,A}$. This is immediate of course in the case where ξ is a variable or constant in L . Thus, if ξ is $\pi(a_1, \dots, a_k)$, then by definition of $\|_J$, $val_A(\xi)(J) = 1 = \{0\}$ iff $[\lambda\pi(a_1, \dots, a_k)]_A \in J$, and therefore iff $\pi(a_1, \dots, a_k) \in J$ (since $(\phi \leftrightarrow [\lambda\phi])$ is a provable schema of $HST_{\lambda\Box}^*$). But since $[\pi(a_1, \dots, a_k) \rightarrow \exists x(a_i \equiv x)]$ is provable in $HST_{\lambda\Box}^*$, then $\pi(a_1, \dots, a_k) \in J$ iff $\langle [a_{1A}], \dots, [a_{kA}] \rangle \in |\pi_A|_J$; and also since $f_J(val_A(a_i)) = [a_{iA}]$, then $val_A(\pi(a_1, \dots, a_k))(J) = 1$ iff $\langle f_J(val_A(a_1)), \dots, f_J(val_A(a_k)) \rangle \in val_A(\pi)(J)$.

If ξ is $[\lambda x_1 \dots x_n \phi]$, then by definition $val_A(\xi)(J) = \{ \langle [a_1], \dots, [a_n] \rangle \in D^n : [\lambda x_1 \dots x_n \phi]_A(a_1, \dots, a_n) \in J \}$. But by $(\exists/\lambda\text{-Conv}^*)$, $[\lambda x_1 \dots x_n \phi]_A(a_1, \dots, a_n) \in J$ iff $\langle [a_1], \dots, [a_n] \rangle \in D^n$ and $\phi_{A([a_1]/x_1, \dots, [a_n]/x_n)} \in J$, and therefore iff $\langle [a_1], \dots, [a_n] \rangle \in D^n$ and $[\lambda\phi]_{A([a_1]/x_1, \dots, [a_n]/x_n)} \in J$. But by definition $val_{A([a_1]/x_1, \dots, [a_n]/x_n)}(\phi) = |[\lambda\phi]_{A([a_1]/x_1, \dots, [a_n]/x_n)}|$, and therefore $\{ \langle [a_1], \dots, [a_n] \rangle \in D^n : [\lambda x_1 \dots x_n \phi]_A(a_1, \dots, a_n) \in J \} = \{ \langle [a_1], \dots, [a_n] \rangle \in D^n : val_{A([a_1]/x_1, \dots, [a_n]/x_n)}(\phi)(J) = 1 \}$.

Now if ξ is $\sim\phi$, $(\phi \rightarrow \psi)$, $\Box\phi$ or $[\lambda\phi]$, then it is easily seen by the inductive hypothesis that $val_A(\xi)$ fulfills the corresponding defining clause for $int_{I,A}(\xi)$. Suppose finally that ξ is $\forall a\phi$, where a is a variable of type k . Then by definition $val_A(\xi)(J) = 1$ iff $[\lambda\forall a\phi]_A \in J$, and therefore iff $\forall a\phi_A \in J$. But since every term of L of type k is represented (in an indiscernibility wff) by a constant of type k , then by (\exists/UI_1^*) and $(UI_{2\lambda}^*)$, $\forall a\phi_A \in J$ iff for all constants b of type k , if $\exists a(b \equiv a) \in J$, then $\phi_A(b/a) \in J$. Note, however, that $\exists a(b \equiv a) \in J$ iff $\exists a(b \equiv a) \in K$, and therefore in general, iff $b \in Int_{k,\mathfrak{A}}$. Also, since $\phi_A(b/a) \in J$ iff $[\lambda\phi]_{A(g(b)/a)} \in J$, then $\phi_A(b/a) \in J$ iff $val_{A(g(b)/a)}(\phi)(J) = 1$. It follows, accordingly, that $val_A(\forall a\phi)(J) = 1$ iff for all $u \in Int_{k,\mathfrak{A}}$, $val_{A(u/a)}(\phi)(J) = 1$. We conclude then that $val_A(\xi)$ does indeed fulfill all the defining conditions for $int_{I,A}(\xi)$, from which our claim (III) above follows; i.e., I is a general intensional model for L .

Now where A is any assignment in \mathfrak{A} , it is clear by the definition of $val_A(\xi)$ that if ξ is a wff of L , then for all $J \in \mathcal{W}$, $val_A(\xi)(J) = 1$ iff $\xi_A \in J$. Accordingly, since every axiom of $HST_{\lambda\Box}^* + (\Box Ext^*)$ is a member of every $J \in \mathcal{W}$, then ϕ is an axiom of $HST_{\lambda\Box}^* + (\Box Ext^*)$ and a wff of L only if $val_A(\phi)(J) = 1$, for all $J \in \mathcal{W}$. That is, every axiom of $HST_{\lambda\Box}^* + (\Box Ext^*)$ is valid in I . It follows, accordingly, that I is an intensional model of $HST_{\lambda\Box}^* + (\Box Ext^*)$.

Finally, where A is that assignment in \mathfrak{A} such that $A(x) = [x]$, for each individual variable x , and $A(F) = |F|$, for each predicate variable F , then if ϕ is a wff of L and $J \in \mathcal{W}$, $val_A(\phi)(J) = 1$ iff $\phi \in J$. Accordingly, since $\Gamma \subseteq K$, then for all $\phi \in \Gamma$, $val_A(\phi)(K) = 1$, i.e., A satisfies every member of Γ in I at K .

Theorem If $\Gamma \cup \{\phi\}$ is a set of wffs of $\Box\lambda HST^*$, then

$$\Gamma \frac{}{\Box\lambda HST^* + \Box Ext^*} \phi \text{ iff } \Gamma \cup (\Box Ext^*) \frac{}{\Box\lambda HST^*} \phi .$$

Theorem If $\Gamma \cup \{\phi\}$ is a set of wffs, then

$$\Gamma \frac{}{HST_{\lambda\Box}^* + (\Box Ext^*)} \phi \text{ iff } \Gamma \cup (\Box Ext^*) \frac{}{HST_{\lambda\Box}^*} \phi .$$

9 Identity versus indiscernibility in HST_{λ}^* Before concluding this paper, perhaps it should be noted that we can obtain the full form of Leibniz' law in HST_{λ}^* if we are willing to introduce the identity sign into our logical grammar as a purely syncategorematic sign and not as a 2-place predicate constant. As a two-place predicate constant, say I , the identity sign, by (\exists/Id_{λ}^*) , would be a legitimate substituent for the bound predicate variable in the thesis that only real individuals have properties or stand in relations. That is, as a two-place predicate constant,

$$I(a, b) \rightarrow (\exists x)(a \equiv x) \ \& \ (\exists x)(b \equiv x)$$

would be provable in HST_{λ}^* ; and therefore since the Russell property is identical with itself, i.e., since

$$I(a, a)$$

would be a basic thesis of HST_{λ}^* with identity, for all terms a , it would follow that the Russell property is a real individual after all, i.e.,

$$(\exists x)(a \equiv x)$$

would be provable for all terms a .

If we introduce the identity sign, $=$, as a purely syncategorematic sign, however, we can avoid this problem and obtain Leibniz' law in its unqualified form. It would follow in particular from (LL_1^*) (now formulated in terms of $=$ instead of \equiv) and the identity counterpart of (Ind^*) :

$$(Id^*/Ext^*) \quad \forall F^n \forall G^n (F = G \rightarrow \forall x_1 \dots \forall x_n [F(x_1, \dots, x_n) \leftrightarrow G(x_1, \dots, x_n)]) .$$

Conversely, we can simply assume (LL^*) (now formulated with $=$ instead of \equiv) and derive (LL_1^*) and (Id^*/Ext^*) instead. Note, however, that although real individuals are indiscernible in HST_{λ}^* with identity if, and only if, they are identical, i.e.,

$$\frac{}{HST_{\lambda(=)}^*} \forall x \forall y (x \equiv y \leftrightarrow x = y) ,$$

nevertheless it will be provable in HST_{λ}^* with identity that there are concepts for any denotative applications of which identity and indiscernibility must diverge; i.e.,

$$\frac{}{HST_{\lambda(=)}^*} \exists F^n \exists G^k (F \equiv G \ \& \ F \neq G) .$$

Thus, e.g., the Russell property ($n = 1$) and the Russell relation ($k = 2$) are (vacuously) indiscernible but are still not identical. Similarly where $n = k$. E.g., since there is no such individual as the Russell relation,

$$\frac{}{HST_{\lambda(=)}^*} \sim \exists y ([\lambda x z \exists R (x = R \ \& \ \sim R(x, z))] = y) ,$$

nor such an individual as the predication relation,

$$\frac{}{HST_{\lambda(=)}^*} \sim \exists y ([\lambda x z \exists F (x = F \ \& \ F(z))] = y) ,$$

then the one has all and only the same properties as the other (viz., none):

$$\frac{}{HST_{\lambda(=)}^*} [\lambda x z \exists R (x = R \ \& \ \sim R(x, z))] \equiv [\lambda x z \exists F (x = F \ \& \ F(z))] .$$

But, notwithstanding their (vacuous) indiscernibility, the Russell relation is not identical with the predication relation:

$$\frac{}{HST_{\lambda(=)}^*} [\lambda x z \exists R (x = R \ \& \ \sim R(x, z))] \neq [\lambda x z \exists F (x = F \ \& \ F(z))] .$$

This is because, not being coextensive, as relational concepts the Russell relation and the predication relation cannot even purport to denote the same individual. That is, by (Id^*/Ext^*) it is provable that noncoextensive concepts cannot even purport to denote the same individual. Such a conclusion may be philosophically suspect, however; for whereas the (vacuous) truth of an indiscernibility wff ($a \equiv b$) is completely unproblematic when neither a nor b denote, one can only wonder as to the real significance of the truth of a nonidentity wff

($a \neq b$) when it is logically necessary that neither a nor b can denote (as singular terms). This problem of course does not arise in λHST^* with identity since in that framework all values of bound predicate variables are also values of bound individual variables and identity is provably equivalent with indiscernibility for all terms whatsoever.

With identity as a new logical constant we must of course add the following clause to the recursive definition of a meaningful expression of a language L :

(10) if $a, b \in ME_0(L)$, then $(a = b) \in ME_1(L)$.

Also, in the definitions of the *extension* and *intension* of a meaningful expression of L relative to an extensional and intensional (free) model or interpretation I for L and assignment A , we need to add the following clauses:

(7) if ξ is $(a = b)$, where $a, b \in ME_0(L)$, then $ext_{I,A}(\xi) = 1$ iff $f^+(ext_{I,A}(a)) = f^+(ext_{I,A}(b))$;

(8) if ξ is $(a = b)$, where $a, b \in ME_0(L)$, then for all $i \in W$, $int_{I,A}(\xi)(i) = 1$ iff $f_i^+(int_{I,A}(a)) = f_i^+(int_{I,A}(b))$.

With regard to an axiomatization of HST_λ^* with identity, we shall say that an *axiom* of $HST_{\lambda(=)}^*$ is any wff which is either tautologous or of one of the following forms:

(A1) $\forall a(\phi \rightarrow \psi) \rightarrow (\forall a\phi \rightarrow \forall a\psi)$, where a is any variable

(A2) $\phi \rightarrow \forall a\phi$, where a is any variable not occurring free in ϕ

(A3') $\forall x\exists y(x = y)$

(A4) $(a = a)$, where a is any term

(LL*) $a = b \rightarrow (\phi \leftrightarrow \psi)$, where ψ is obtained from ϕ by replacing one or more free occurrences of a by free occurrences of b

(Id $_\lambda^*$) $[\lambda x_1 \dots x_n P(x_1, \dots, x_n)] = P$, where P is an n -place predicate variable or constant,

(\exists/λ -Conv*) $[\lambda x_1 \dots x_n \phi](a_1, \dots, a_n) \leftrightarrow \exists x_1 \dots \exists x_n (a_1 = x_1 \ \& \dots \ \& \ a_n = x_n \ \& \ \phi)$, where no x_i is free in any a_j ; for all i, j such that $1 \leq i, j \leq n$,

(CP $_\lambda^*$) $\exists F^n([\lambda x_1 \dots x_n \phi] = F)$, where F^n does not occur free in ϕ ,

($\exists/HSCP_\lambda^*$) $\exists y(a_1 = y) \ \& \dots \ \& \ \exists y(a_k = y \rightarrow \exists y([\lambda x_1 \dots x_n \phi] = y))$, where $[\lambda x_1 \dots x_n \phi]$ is homogeneously stratified, ϕ is bound to individuals, y is an individual variable not occurring free in ϕ , and a_1, \dots, a_k are all the variables or nonlogical constants occurring free in $[\lambda x_1 \dots x_n \phi]$.

Using modus ponens and universal generalization as the only inference rules, note that by (LL*),

(Id*/Ext*) $\frac{}{HST_{\lambda(=)}^*} \forall F^n \forall G^n (F = G \rightarrow \forall x_1 \dots \forall x_n [F(x_1, \dots, x_n) \leftrightarrow G(x_1, \dots, x_n)])$.

Also, by (LL*),

$$\frac{}{HST_{\lambda(=)}^*} [\lambda x_1 \dots x_n \phi] = F \rightarrow [\psi \leftrightarrow \psi([\lambda x_1 \dots x_n \phi]/F^n)]$$

and therefore by universal generalization, (A1), and (CP $_\lambda^*$),

(UI $_{2\lambda}^*$) $\frac{}{HST_{\lambda(=)}^*} \forall F \psi \rightarrow \psi([\lambda x_1 \dots x_n \phi]/F^n)$.

Accordingly, since by (LL^*)

$$\frac{}{HST_{\lambda(=)}^*} a = b \rightarrow a \equiv b ,$$

then every identity-free wff which is a theorem of HST_{λ}^* is a theorem of $HST_{\lambda(=)}^*$. Note also that unlike the refutability of (Inf^*) in HST_{λ}^* , where $n \neq k$,

$$(Inf_{=}^*) \quad \forall F^n \forall G^k (F \neq G)$$

is not refutable in $HST_{\lambda(=)}^*$.

Where L is a language, let us now say that a general free model I for L is a *model* of $HST_{\lambda(=)}^* + (Ext^*)$ if every wff of L which is an instance of (CP_{λ}^*) , $(\exists/HSCP_{\lambda}^*)$ or (Id^*/Ext^*) is true in I . The extensional notion of *logical consequence* is then understood to be redefined accordingly.

Note now that the general free model $I_{\lambda} = \langle g, \mathfrak{A}_{\lambda} \rangle$ which is constructed in the relative consistency proof of Section 7 is not only a model of $HST_{\lambda}^* + (Ext^*)$ but is also a model of $HST_{\lambda(=)}^* + (Ext^*)$. This is because the Fregean correlation function f_{λ} constructed in that proof is defined in such a way as to be one-to-one even in the outer domain; i.e., $f_{\lambda}(K) = \langle K, D \rangle$, for all $K \in Y_n - X_n$, where D, Y_n, X_n are as constructed in that proof. Using this fact and part (1) of that proof it can then be easily seen that (Id^*/Ext^*) is true in I_{λ} (and so is $(Inf_{=}^*)$ if (Inf^*) is true in I). The rest of the proof is otherwise exactly as in Section 7 except of course for replacing all considerations of wffs with \equiv by the corresponding wffs with $=$ instead.

Theorem *If $\lambda HST^* + (Ext^*)(+ (Inf^*))$ is consistent, then so is $HST_{\lambda(=)}^* + (Ext^*)(+ (Inf_{=}^*))$.*

Finally, by the modal or intensional version of HST_{λ}^* with identity, let us understand $HST_{\lambda(=)\square}^*$ to be obtained from $(HST_{\lambda(=)}^*)$ just the way that $HST_{\lambda\square}^*$ was obtained from HST_{λ}^* . Then, where L is a language, let us say that a general (intensional) free interpretation I for L is an *intensional free model* of $HST_{\lambda(=)\square}^* + (\square Ext^*)$ iff I is based on a rigid free Fregean frame and every wff of L which is an instance of (CP_{λ}^*) , $(E/HSCP_{\lambda}^*)$ or (Id^*/Ext^*) is always true in I . Again, we understand the intensional notion of *logical consequence* to be redefined accordingly.

Needless to say the proof of the completeness theorem for $HST_{\lambda(=)\square}^* + (\square Ext^*)$ proceeds in exactly the same way as the proof for $HST_{\lambda\square}^* + (\square Ext^*)$ except that wherever wffs with \equiv were used in that proof we now are to use the corresponding wff with $=$ instead. In part(3) of that proof we also need to show that $int_{I,A}(a = b)(J) = 1$ iff $(a = b) \in J$; but the proof for this quite straightforward.

Completeness theorem for $HST_{\lambda(=)\square}^* + (\square Ext^*)$ *If $\Gamma \cup \{\phi\}$ is a set of wffs, then*

$$\Gamma \frac{}{HST_{\lambda(=)\square}^* + (\square Ext^*)} \phi \text{ iff } \Gamma \cup (\square Ext^*) \frac{}{HST_{\lambda(=)\square}^*} \phi .$$

We conclude by noting that from a purely philosophical standpoint we remain unsure whether to recommend HST_{λ}^* with or without identity. Without

identity, the treatment of denotationless nominalized predicates is unproblematic, and with identity there is the question of the real significance or content of the provable and therefore true nonidentity wffs containing such denotationless terms. On the other hand, with identity, albeit as a purely syncategorematic sign, we do have Leibniz' law in its full unqualified form; and perhaps that alone is enough to recommend HST_{λ}^* with identity after all. Of course, there is also always λHST^* where indiscernibility suffices as a full characterization of identity and where whatever is the value of a bound predicate variable is also (on its intended interpretation) a value of the bound individual variables. But then the choice between λHST^* and HST_{λ}^* with or without identity will depend on which stand one takes in the realism/conceptualism debate (regarding the nature of predication) whereas the choice between HST_{λ}^* with and HST_{λ}^* without identity seems entirely to be an "in-house" problem for conceptualism on how to interpret true nonidentity wffs with denotationless nominalized predicates. Space precludes our saying more on any of these issues here.

NOTES

1. Note that where $n \neq k$, (LL^*) does not validate $F^n \equiv G^k \rightarrow [F^n(a_1, \dots, a_n) \leftrightarrow G^k(a_1, \dots, a_n)]$, since $G^k(a_1, \dots, a_n)$ is not well formed, i.e., it is not a wff. Thus, where (LL^*) involves nominalized predicates of different degrees (number of argument positions), then either a occurs free in ϕ only in subject positions (and ψ is ϕ with these free subject position occurrences of a replaced by free subject position occurrences of b) or (LL^*) reduces to the tautologous wff: $a \equiv b \rightarrow (\phi \leftrightarrow \phi)$.
2. In stating theses in this manner, we always assume that x_1, \dots, x_n are pairwise distinct individual variables. Also, by (Ext^*) and $(\Box Ext^*)$ we mean the sets of wffs (Ext_n^*) and $(\Box Ext_n^*)$, respectively, for all $n \in \omega$.
3. Although we are only interested in those applications of this lemma in which $[\lambda x_1 \dots x_n \psi]$ occurs free in ϕ , we note that this is not essential to the proof of the lemma since $\phi(F^n/[\lambda x_1 \dots x_n \psi])$ is otherwise just ϕ itself. Note that the right-hand side of this provable wff is homogeneously stratified if ϕ is.
4. The interchange law for λHST^* is easily proved by an inductive argument for all wffs of λHST^* —so long as the wffs interchanged do not occur within the scope of the λ -operator; and if they do so occur, then the interchange law holds only on the condition that the wffs interchanged are homogeneously stratified. Note that in proving the λ -elimination theorem, the subwffs of ϕ which both contain and occur within λ -abstracts must be homogeneously stratified (since ϕ otherwise would not be a wff of λHST^*); and therefore such subwffs can be interchanged with any provably equivalent homogeneously stratified wff. The interchange to be effected on the basis of the above lemma, needless to say, concerns only homogeneously stratified wffs if the subwff in question is itself homogeneously stratified.
5. We assume throughout that each natural number is the set of natural numbers less than it, and therefore take 0 to be the empty set and $1 = \{0\}$. We take $\mathcal{P}(D^n)$ to be the power-set of D^n , the set of n -tuples drawn from D ; and therefore $\mathcal{P}(D^0) = \{0, 1\}$.

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