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# A Note on the Hanf Number of Second-Order Logic

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The Hanf number of a logic $L$ is the least cardinal $\kappa$ such that every sentence of $L$ that has a model of power at least $\kappa$ has arbitrarily large models. The Hanf number $\kappa^{\Pi}$ of second-order logic is very large; for example, it is readily seen to exceed the first measurable cardinal (if there is one). In fact, Barwise [1] showed that one cannot prove that $\kappa^{\Pi}$ exists within the theory $ZF_1$, which is $ZFC$ with the full subset schema but with collection only for $\Sigma_1(\mathcal{P})$ formulas. (Here $\mathcal{P}$ is a unary function symbol, and the power set axiom reads: $\forall x \forall y (y \subseteq x \leftrightarrow y \in \mathcal{P}(x))$.) Moreover, within $ZF_1$ he proved that $R_{\kappa^{\Pi}} \models ZF_1$, and in fact $\kappa^{\Pi}$ is the $(\kappa^{\Pi})^{\text{th}}$ cardinal with this property. Friedman [3] improved this result by showing that even in the weaker theory $T = ZF_0 + (\beta)$, where $ZF_0 = KP(\mathcal{P}) +$ [Power set axiom], if $\kappa^{\Pi}$ exists then $R_{\kappa^{\Pi}} \prec_{\Sigma_1(\mathcal{P})} V$.[1]

In this short note we use Friedman's result to give a new characterization of $\kappa^{\Pi}$ (Theorem 1 below). A related characterization is given in Väänänen [5] (Corollary 5.7):

(1) $\kappa^{\Pi} = sup\{\alpha\colon \alpha \textit{ is } \Sigma_2\textit{-definable}\}$,

where a set $S$ is $\Sigma_2$-definable if the predicate "$x \in S$" is a $\Sigma_2$-definable predicate of set theory.[2] (Väänänen's result is actually more general.) Here is an outline of a proof of (1). For $\geq$, if $\phi(x)$ is a $\Sigma_2$ (or $\Sigma_1(\mathcal{P})$) definition of "$x \in \alpha$" then consider the following sentence $\psi$ of second-order logic, which holds in $(R_\kappa, \in)$ if $\kappa$ is least such that $\phi(x)$ defines "$x \in \alpha$" in $(R_\kappa, \in)$:

$$\psi \equiv \text{"The universe is of the form } (R_\delta, \in)\text{"} \wedge (\exists! \beta)(\forall x)(\phi(x) \leftrightarrow x \in \beta) \wedge \forall \beta [\forall x (\phi(x) \leftrightarrow x \in \beta) \rightarrow \forall \gamma \exists x \in \beta (R_\gamma \models \neg \phi(x))] \ .$$

Then $\psi$ has a model of power at least $|\alpha|$, but it's easy to see that $\psi$ does not have arbitrarily large models. For $\leq$, observe that if $\phi$ is a sentence of second-

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order logic that has a model of power $\lambda$ but does not have arbitrarily large models, then the following $\Sigma_1(\mathcal{P})$-formula defines an ordinal $\alpha > \lambda$: $\psi(\beta) \equiv$ "there is a model of $\phi$ of power at least $\beta$". It seems that none of the equivalences of Theorem 1 below follow readily from Väänänen's result (1), but rather they hinge on the theorem of Friedman cited above. In fact, the reader can check that the equivalence of (a) and (b) in Theorem 1 is itself equivalent to the following characterization:

(2) $\kappa^{\Pi} = sup\{\alpha\colon \{\alpha\}$ *is* $\Sigma_2(\mathcal{P})$*-definable*$\}$.

(The inequality $\leq$ follows easily from (1), but the other direction is less obvious.)
The following theorem can be proved in the theory $T$ defined above.

**Theorem 1** *If any of the following cardinals exist, then they all exist and are equal.*

(a) $\kappa = \kappa^{\Pi}$.
(b) $\lambda$ *is least such that whenever* $V \models \exists x \forall y \phi$ *with* $\phi \in \Delta_0(\mathcal{P})$, *then* $V \models (\exists x \in R_\lambda)\forall y \phi$.
(c) $\mu$ *is least such that whenever* $\psi \in \Sigma_2(\mathcal{P})$ *and* $V \models \psi$, *then* $R_\mu \models \psi$; *and whenever* $\theta(x) \in \Sigma_1(\mathcal{P})$, $a \in R_\mu$, *and* $V \models \theta(a)$, *then* $R_\mu \models \theta(a)$.

*Proof:* Note that the predicate "$R_\alpha \models \phi(\vec{a})$" is a $\Delta_1(\mathcal{P})$ predicate (of $\alpha$ and $\vec{a}$). To show $\kappa \leq \lambda$, suppose $\psi$ is a sentence of second-order logic which does not have arbitrarily large models. Then $V \models \exists \nu \forall \mathfrak{A}(\nu \leq \|\mathfrak{A}\| \rightarrow \mathfrak{A} \not\models \psi)$. By definition of $\lambda$, $V \models \exists \nu < \lambda \forall \mathfrak{A}(\nu \leq \|\mathfrak{A}\| \rightarrow \mathfrak{A} \not\models \psi)$. Hence $\psi$ has no model of power $\geq\lambda$. By (the contraposition of) the definition of $\kappa$, $\kappa \leq \lambda$.

Next, observe that $\lambda \leq \mu$ since $\mu$ clearly satisfies the property in (b) for which $\lambda$ is least.

It remains to show $\mu \leq \kappa$. For this it suffices to show that $\kappa$ has the properties in (c) for which $\mu$ is least. The first of these two properties is easy, for if $R_\kappa \models \forall x \exists y \phi$ with $\phi \in \Delta_0(P)$, then $R_\alpha \models \forall x \exists y \phi$ for arbitrarily large $\alpha$ by [1], 1.6. (That is, apply the definition of $\kappa$ to the sentence $\forall x \exists y \phi \wedge \psi$, where $\psi$ is true in exactly those structures of the form $(R_\alpha, \in)$.) It follows that $V \models \forall x \exists y \phi$. The second property is proved in Friedman [3] (Theorem 2).

Remark 1: Theorem 1 has a straightforward generalization to infinitary second-order logic. Let $L^{\Pi}_{\infty\omega}$ be the closure of first-order logic under second-order quantification $\exists R$ and infinite disjunctions. For any admissible set $A$, let $L^{\Pi}_A$ be $L^{\Pi}_{\infty\omega} \cap A$. Then Theorem 1 remains true when $\kappa$ is the Hanf number of $L^{\Pi}_A$ in (a), and in (b) and (c) one allows parameters from $A$ in the formulas $\phi$ and $\psi$. The proof contains no surprises relative to the proof of Theorem 1, so we omit it. Notice that one could define $L^{\Pi}_{\infty\infty}$ in the obvious way, but in fact $L^{\Pi}_A$ and $L^{\Pi}_{\infty\infty} \cap A$ are essentially equivalent (for example, they have the same $\Delta$-closure), and their Hanf numbers are the same. However, the Hanf number of $L^{\Pi}_{HC}$ ($HC$ = hereditarily countable sets) exceeds that of (finitary) second-order logic, for the following sentence has a model of power $\geq\kappa^{\Pi}$ but does not have arbitrarily large models: $\bigwedge ZF_1 \wedge \forall \alpha \exists \mathfrak{A}(\|\mathfrak{A}\| \geq \alpha \wedge \bigvee\{$"$\mathfrak{A} \models \phi$": $\phi \in L^{\Pi}_{\omega\omega}$ does not have arbitrarily large models$\})$. On the other hand, the Hanf number of

$L^{\Pi}_{HYP}$ (where $HYP$ is the least admissible set to which $\omega$ belongs) does in fact equal $\kappa^{\Pi}$, because $L^{\Pi}_{HYP}$ has the same $\Delta$-closure as finitary second-order logic, and the $\Delta$-closure preserves Hanf numbers for these logics (cf. Väänänen [5], Lemma 5.3(1), (2)).

Let us turn now to a logic for well-ordered structures. The logic $L(wo)$ is just first-order logic except that there is a special binary relation symbol $<$, and one allows only structures in which $<$ is a well-ordering. A number of logics are roughly equivalent to $L(wo)$ and have the same Hanf number. For example one may add a well-ordering or a well-foundedness quantifier (e.g., see [2], Section 0 for details). It is well-known[3] that $L(wo)$ has a smaller Hanf number than that of second-order logic. Yet if $V = L$, then there is a characterization of the Hanf number of $L(wo)$ that is analogous to Theorem 1, but is easily proved from the definitions. Alternatively, this theorem follows easily[4] from the following two results of Väänänen: the Löwenheim number $\ell^{\Pi}$ (see, e.g., [5], Definition 4.5) of second-order logic is the least $\lambda$ such that $(R_\lambda, \in)$ satisfies the same $\Sigma_2$ sentences as does $(V, \in)$ ([5], Proposition 4.14); and if $V = L$ then $\ell^{\Pi}$ is the Hanf number of $L(wo)$ ([6], Lemma 2).

**Theorem 3** *Assume $V = L$. Then the Hanf number of $L(wo)$ is the least cardinal $\kappa$ such that every $\Sigma_2$ sentence true in $V$ is true in $L_\kappa$.*

Remark 2: It is proved in Silver ([4], Theorem 5.9) that the Hanf number of $L(wo)$ lies properly between the least cardinal $\kappa$ such that $\kappa \to (\alpha)_2^{<\omega}$ for all countable $\alpha$ and the least $\kappa$ such that $\kappa \to (\omega_1)_2^{<\omega}$, if these cardinals exist. Theorem 3 here suggests that a combinatorial *characterization* of the Hanf number of $L(wo)$ is unlikely to exist in $L$; at least, no such characterization can be $\Sigma_2$.[5]

## NOTES

1. We thank Jon Barwise for bringing Friedman's paper to our attention.

2. We will use freely the observation that a predicate is $\Sigma_{i+1}$ iff it is $\Sigma_i(\mathcal{P})$, for all $i \geq 1$.

3. One may write down a sentence $\psi \in L^{\Pi}$ which describes a linear order whose type is some limit cardinal $\kappa$, where for all $x$ there is $\phi_x \in L(wo)$ having a model with domain $[0, y)$ for some $y > x$, where $\phi_x$ does not have models of arbitrarily large power less than $\kappa$. Then $\psi$ has a model whose power equals the Hanf number of $L(wo)$, yet $\psi$ does not have arbitrarily large models (as one can see that $\psi$ has no model whose power is greater than the Hanf number of $L(wo)$, by applying the downward Lowenheim-Skolem theorem for $L(wo)$).

4. Easily, that is, once one checks that $L_\kappa = R_\kappa$ for $\kappa$ as in Theorem 3.

5. We thank J. Väänänen for helpful discussions.

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