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Some Exact Equiconsistency Results in Set Theory

LEO HARRINGTON and SAHARON SHELAH

Since the invention of forcing there have been innumerable examples of consistency results in set theory. These usually show that $ZFC +$ (something) is consistent provided that ZFC itself is consistent; hence they may be viewed as equiconsistency results. More recently there have been many forcing arguments that need more than just the consistency of ZFC; they assume the consistency of certain large cardinals in addition. Sometimes these are still exact equiconsistency results: for example the negation of Kurepa's hypothesis is equiconsistent with an inaccessible cardinal. Sometimes there is a wide gap between consistency strengths: for example, Magidor's model for the failure of the singular cardinal problem (SC) uses somewhat more than a supercompact, while SC itself is only known (by work of Mitchell) to imply inner models with many measurable cardinals. And sometimes there is a gap originally, but the gap is eventually closed: for example, Silver's model for Chang's conjecture uses an Erdos cardinal, and Jensen has shown that Chang's conjecture implies the existence of an Erdos cardinal in the core model. In this paper, we present a few more results exemplifying this last possibility. We show:

Theorem A *The following are equiconsistent* (*modulo ZFC of course*):
(i) *the existence of a Mahlo cardinal*
(ii) *every stationary subset of* $\aleph_2$ *consisting just of cofinality* ω *ordinals is stationary in some ordinal* $< \aleph_2$.

Comments: Baumgartner [1] has shown that (ii) is consistent assuming the existence of a weakly compact cardinal. It is known [2] that $\square_{\omega_1}$ implies that (ii) is false, and it is also known [2] that, unless $\aleph_2$ is Mahlo in L, $\square_{\omega_1}$ holds; thus (ii) implies that $\aleph_2$ is Mahlo in L. So the gap in this case is between Mahlo and weakly compact.

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Theorem B *The following are equiconsistent*:
(i) *the existence of a weakly compact cardinal*
(ii) *Martin's axiom* (*MA*) + *every projective* (*actually* $\underset{\sim}{\Sigma}^1_3$ *is enough*) *set of reals is Lebesque measurable*
(iii) *MA* + *every projective* (*actually* $\underset{\sim}{\Delta}^1_3$ *is enough*) *set of reals has the property of Baire*.

[Here we construe Martin's axiom as implying $\neg CH$.]

Comments: Let LM be the assertion: every set of reals in $L[2^\omega]$ is Lebesque measurable, and let PB be the corresponding statement for the property of Baire. Solovay [6] has used an inaccessible to produce a model of: LM and PB. Kunen has used a weakly compact cardinal to produce a model of: MA, LM, and PB. Recently, Shelah [5] has shown that LM (just for $\underset{\sim}{\Sigma}^1_3$ sets) implies that $\aleph_1$ is inaccessible in L, and that PB is outright consistent. (Shelah also shows that LM for $\underset{\sim}{\Delta}^1_3$ sets is outright consistent.) So the gap, in the case of Theorem B(i) and (ii), is between weakly compact and inaccessible, while for Theorem B(i) and (iii) it is between weakly compact and ZFC.[1]

Actually, our proof of Theorem B breaks up naturally into:

Theorem C *Assuming MA*:
(i) *Either there is a real a such that* $\aleph_1 = (\aleph_1)^{L[a]}$, *or*, $\aleph_1$ *is weakly compact in L*.
(ii) *If* $\aleph_1 = (\aleph_1)^L$, *then there is a* $\underset{\sim}{\Sigma}^1_3$ *set which is not Lebesque measurable and which does not have the property of Baire*.

Comment: Using the above mentioned result of Kunen, Theorem C (with (ii) relativized to L [any real]) yields Theorem B.[2]

The reader interested in brushing up on some of the concepts mentioned above, or on some of the concepts to be mentioned below, would do well to consult [3]. Before starting on the proof of Theorem A, we will mention some facts and definitions about forcing which we will need.

Let M be a model of ZFC, and let Q be a partial ordering (p.o.) in M. We let $\leq_Q$ (or sometimes just $\leq$) denote the ordering of Q. Viewing Q as a set of forcing conditions, we let $p \leq_Q q$ mean that q is at least as strong as p. To avoid superscripts, we let $M(Q)$ be the Boolean valued model (or collection of forcing terms) associated with M and Q. Unless otherwise clarified, truth in $M(Q)$ will mean true with truth value 1. If R is a p.o. of $M(Q)$, we can form $M(Q)(R)$. Of course there is a p.o. $Q * R$ in M such that $M(Q)(R)$ is $M(Q * R)$; define $\langle q, r\rangle \in Q * R$ iff $q \in Q$ and $r \in M(Q)$ and $q \Vdash_Q$ "$r \in R$", and define $(q, r) \leq (q', r')$ iff $q \leq_Q q'$ and $q' \Vdash_Q$ "$r \leq_R r'$". It is sometimes useful to clarify exactly which elements r in $M(Q)$ are allowed in the above definition of $Q * R$; to this end, we will think of R as coming equipped with a subset T of $M(Q)$ (i.e., T is in M, T is a subset of $M(Q)$) and requiring in the above definition that $r \in T$. We place some minimal demands on T, namely: for all $q \in Q$, all $a \in M(Q)$, if $q \Vdash_Q$ "$a \in R$" then for some $q' \geq_Q q$ and some $r \in T$, $q' \Vdash_Q$ "$a = r$". It is usual to choose T to be a representative sample of all possible (up to equivalence) elements of R in $M(Q)$. The above minimal demands ensure that, as we vary through possible T's, the resulting p.o.'s $Q * R$ are essentially the same. Our choice of T, though, can make a difference when we do a transfinite iteration:

In M, given a sequence $\langle R_\alpha \rangle_{\alpha<\gamma}$ (γ an ordinal), and given a regular cardinal κ, we will define for each $\beta \le \gamma$ a p.o. $Q_\beta =$ the iteration with support $<\kappa$ of $\langle R_\alpha \rangle_{\alpha<\beta}$. We define Q_β under the assumption that for all $\alpha < \beta$, R_α is a p.o. in $M(Q_\alpha)$, and so we think of R_α as coming equipped with its own T_α. So: p is in Q_β iff p is a function; the domain of p (*dom p*) is a subset of β of cardinality $<\kappa$; and, for all $\alpha < \beta$ in *dom p*, $p|\alpha$ is in Q_α, $p(\alpha)$ is in T_α and $p|\alpha \Vdash_{Q_\alpha}$ "$p(\alpha) \in R_\alpha$". For p, q in Q_β, $p \le q$ iff *dom p* $\le$ *dom q* and for all $\alpha \in$ *dom p*, $q|\alpha \Vdash_{Q_\alpha}$ "$p(\alpha) \le_{R_\alpha} q(\alpha)$".

For Q a p.o. and for A a dense subset of Q (i.e., $\forall q \in Q \exists a \in A\ a \ge q$), the p.o.'s. Q, A are essentially the same (i.e., they canonically give rise to the same complete Boolean algebra). We say that a p.o. Q is essentially $<\kappa$-closed (κ a regular cardinal) if there is a dense subset A of Q and A is $<\kappa$-closed (i.e., if $p_0 \le \ldots \le p_\alpha \le \ldots$, $(\alpha < \delta,\ \delta < \kappa)$ are all from A, then $\langle p_\alpha \rangle_{\alpha<\delta}$ has a least upper-bound in A). Notice the following:

Fact 1 *Using the notation from above, assume that for each $\beta < \gamma$, R_β is essentially $<\kappa$-closed in $M(Q_\beta)$, and assume in addition that T_β is closed under least upper-bounds (i.e., for $q \in Q_\beta$, and for $X \subseteq T_\beta$, $X \in M$, if $q \Vdash_{Q_\beta}$ "X has a lub in R_β", then for some $p \in T_\beta$, $q \Vdash_{Q_\beta}$ "$p =$ the lub of X"), then Q_γ is essentially $<\kappa$-closed.*

Proof: In $M(Q_\beta)$ let A_β be a $<\kappa$-closed dense subset of R_β. In M, let $B_\beta = \{q \in Q_\beta$: for all $\alpha \in$ *dom q*, $q|\alpha \Vdash_{Q_\beta}$ "$q(\alpha) \in A_\alpha$"$\}$. Clearly our above assumptions ensure that B_β is $<\kappa$-closed. We prove by induction that B_β is dense in Q_β: if *cof* $\beta' \ge \kappa$, then $q \in Q_\beta \Rightarrow q \in Q_{\beta'}$, some $\beta' < \beta$, and so $q \in B_{\beta'} \subseteq B_\beta$. If $\beta = \beta' + 1$ then (by denseness of $B_{\beta'}$ and $A_{\beta'}$) there are $q' \ge q|\beta'$, $q' \in B_{\beta'}$, and $r \in T_{\beta'}$ such that $q' \Vdash_{Q_{\beta'}}$ "$r \in A_{\beta'}$ and $r \ge q(\beta')$". So $q' \cup \{\langle \beta', r\rangle\}$ is in B_β and is $\ge q$. If *cof* $\beta = \lambda < \kappa$, pick $0 = \delta_0 < \ldots < \delta_i < \ldots (i < \lambda)$ a continuous sequence of ordinals with *sup* β. Given $q \in Q_\beta$, build $q = q_0 \le \ldots \le q_i \le \ldots$ $(i \le \lambda)$ so that for all $i < \lambda$, $q_i|\delta_i \in B_{\gamma_i}$, and for all $j \ge i$ and all $\alpha \ge \gamma_j$, $q_i(\alpha) = q_j(\alpha)$. This clearly ensures that for each limit j the lub of $\langle q_i \rangle_{i<j}$ exists and meets the requirements for q_j. Obviously q_λ is in B_β.

We need a few more observations. We define two p.o.'s to be essentially the same if their complete Boolean algebras are isomorphic. A p.o. is atomless if it is nonempty and any element has two incompatible extensions.

Fact 2 (Jech) *Assuming CH, all $\le\omega$-closed, atomless p.o.'s of cardinality $\le\aleph_1$ are essentially the same.*

Proof: Let $P = \omega_1^{<\omega_1} =$ countable sequences of countable ordinals ordered by extension. Let Q be a $\le\omega$-closed, atomless p.o. of *card* $\le \aleph_1$. We claim that Q has a dense subset isomorphic to P. (1) Notice that any element of Q has $\aleph_1$ pairwise incompatible extensions. [*Proof:* Given $q \in Q$, for each $\eta \in 2^{<\omega} =$ a finite sequence of 0's and 1's, find q_η in Q so that $q_\varnothing = q$, $q_\eta \le q_{\eta^\frown i}$, $q_{\eta^\frown 0}$ and $q_{\varnothing^\frown 1}$ are incompatible. For f in 2^ω let $q_f =$ the lub of $\langle q_{f|n} \rangle_{n\in\omega}$.] Next, for each

$\eta \in \omega_1^{<\omega_1}$ find q_η in Q so that: (a) $q_\varnothing$ = least element of Q; (b) $q_\eta \leq q_\tau$ if $\eta \subseteq \tau$; (c) if *dom* η is a limit ordinal α then $q_\eta = \text{lub}\ \langle q_{\eta|\delta};\ \delta < \alpha\rangle$; (d) the $q_{\eta\frown\beta}$ $(\beta < \omega_1)$ are pairwise incompatible and form a maximal antichain above q_η (this is possible by (1) above); and (e) if q is the α^{th} element of Q $(\alpha < \omega_1)$ there is some $\eta \in \omega_1^{\alpha+1}$ such that $q \leq q_\eta$ (by (a), (c), (d) $\{q_\tau\colon \tau \in \omega_1^\alpha\}$ is a maximal antichain, so (e) is possible). The map $\eta \to q_\eta$ sends P isomorphically to a dense subset of Q.

Fact 3 (Baumgartner [1]) *Let M be a model of ZFC, κ a regular cardinal of M, X a subset of κ consisting of cofinality ω ordinals. If there is a $\leq\omega$-closed p.o. P in M such that X is not stationary in $M(P)$, then X is not stationary in M.*

Proof: In $M(P)$ pick C a closed unbounded subset of κ disjoint from X. In M, build D a closed unbounded subset of κ, and for each finite increasing sequence, η, from D, find $p_\eta \in P$, $\alpha_\eta < \kappa$, such that: $\eta \subseteq \tau \Rightarrow p_\eta \leq p_\tau$; $p_\eta \Vdash_{\overline{P}}$ "$\alpha_\eta \in C$"; $\alpha_\eta \geq max(\eta)$; and $\alpha_\eta < min(D \sim (max(\eta) + 1))$. Let δ be a limit ordinal of D. We claim that $\delta \notin X$. This is trivial unless δ has cofinality ω, so pick an increasing function $f\colon \omega \to \delta \cap D$ whose *sup* is δ. So $sup\{\alpha_{f|n}\colon n \in \omega\} = \delta$, and $p_{f|n} \leq p_{f|(n+1)}$. Let $p = \text{lub}\ \{p_{f|n}\colon n \in \omega\}$. So $p \Vdash_{\overline{P}}$ "$\delta \in C$". Thus $\delta \notin X$.

Proof of Theorem A: Let M be a model of *ZFC* which has a Mahlo cardinal κ. For convenience sake we assume *GCH* holds at κ. Let K = the set of cofinality ω ordinals $<\kappa$.

For α an ordinal, let $P\{\alpha\}$ = the Levy collapse of α to $\aleph_1$. So $p \in P\{\alpha\}$ iff $p\colon \delta \to \alpha$ for some $\delta < \aleph_1$. $P\{\alpha\}$ is ordered by extension. For X a set of ordinals, let $P(X)$ = the iteration of $\langle P\{\alpha\}\rangle_{\alpha\in X}$ with countable support. So $q \in P(X)$ iff q is a function, *dom* $q \subseteq X$ has card $\leq \omega$, $q(\alpha) \in P\{\alpha\}$ for each $\alpha \in$ *dom* q. $P(X)$ is ordered pointwise. For $\alpha < \beta$ let $[\alpha, \beta) = \{\gamma\colon \alpha \leq \gamma < \beta\}$. Let $P[\alpha, \beta) = P([\alpha,\beta))$. Notice $P(\alpha) = P(\{\gamma\colon \gamma < \alpha\})$ is different than $P\{\alpha\} = P(\{\alpha\})$. Also notice that for $\alpha < \beta$, $P(\beta) = P(\alpha) \times P[\alpha, \beta)$, and $P[\alpha, \beta) = P\{\alpha\} \times P[\alpha + 1, \beta)$. As is well known, for any set X of ordinals, $P(X)$ is $\leq\omega$-closed; and for ρ a regular cardinal $>\aleph_1$, $P(\rho)$ has the $<\rho$-antichain condition.

Let $P = P(\kappa)$. Working in $M(P)$ we define a sequence $\langle Q_\beta\rangle_{\beta\leq\kappa^+}$ of p.o.'s. To do this we recursively define a sequence $\langle R_\beta\rangle_{\beta<\kappa^+}$ and let Q_β = the iteration with support $<\kappa$ of $\langle R_\alpha\rangle_{\alpha<\beta}$. So, given Q_β we define R_β as follows: in $M(P * Q_\beta)$ pick a subset X_β of K such that for all $\alpha < \kappa$, $X_\beta \cap \alpha$ is not stationary in α. Let $r \in R_\beta$ iff r is a bounded, closed subset of κ disjoint from X, and r is in $M(P)$. Order R_β by end extension, that is, $r \leq_{R_\beta} s$ iff $s \cap (max(r) + 1) = r$. We let T_β be all the elements of $M(P * Q_\beta)$ which, with truth value 1, are equal to a particular closed, bounded subset of κ in $M(P)$. (We define T_β this way, and demand that r be in $M(P)$, because it seems slightly easier. Essentially the same p.o.'s would result if we allowed r to be in $M(P * Q_\beta)$, and if we let T_β be all appropriate elements of $M(P * Q_\beta)$.) Each T_β has cardinality $\leq\kappa$, so each Q_α $(\alpha < \kappa^+)$ has cardinality $\leq\kappa$. Thus, by appropriate choice of the X_β's, we can ensure that for each $\alpha < \kappa^+$, each subset of K in $M(P * Q_\alpha)$ is considered for inclusion among the X_β's. But Q_{κ^+} has the $\leq\kappa$-antichain condition (since, viewing Q_{κ^+} as an iteration, direct limits were taken stationarily often (at the β's of cofinality κ), and each Q_β, $\beta < \kappa^+$, has the $\leq\kappa$-antichain condition). So every subset of K in $M(P * Q_{\kappa^+})$ is actually in $M(P * Q_\alpha)$, some $\alpha < \kappa^+$. By definition of R_β it is clear that X_β is not stationary in $M(P * Q_{\beta+1})$.[3] So in $M(P * Q_{\kappa^+})$, every subset of K, which is not stationary in some ordinal $<\kappa$, is not stationary in κ. So $M(P * Q_{\kappa^+})$ is almost a model of (ii) from Theorem A. It remains to show:

Claim 1 *For each $\beta \leq \kappa^+$, Q_β is $<\kappa$, ∞-distributive in $M(P)$ (i.e., $M(P)$ is closed under $<\kappa$-sequences from $M(P * Q_\beta)$).*

Before proving Claim 1, we need a few more notions. In M, let $\hat{M}$ be a transitive set modeling a rich fragment of ZFC, $\hat{M}$ closed under κ^+ sequences. For $N \subseteq \hat{M}$, let $\bar{N}$ = the transitive collapse of N, and for $a \in N$, let $\bar{a}_N$ = what a collapses to. We will sometimes write $\bar{a}$ for $\bar{a}_N$. We will call $N \subseteq \hat{M}$ rich if it has the following properties: $\langle N, \epsilon\rangle$ is an elementary substructure of $\langle \hat{M}, \epsilon\rangle$; κ, $\langle X_\beta\rangle_{\beta<\kappa^+}$ are in N; $\bar{\kappa}_N \subseteq N$; $\bar{\kappa}_N$ is a regular cardinal, N has cardinality $\bar{\kappa}_N$, and N is closed under $<\bar{\kappa}_N$ sequences and $\bar{\kappa}_N < \kappa$.

It is not hard to find a rich N: Build a continuous sequence $N_0 \subseteq N_1 \subseteq \ldots \subseteq N_i \subseteq \ldots, i < \kappa$, of elementary substructures of $\hat{M}$ all of cardinality $<\kappa$. Pick N_{i+1} so that $sup(N_i \cap \kappa) \subseteq N_{i+1}$, and N_{i+1} is closed under cardinality and N_i-sequences. Let $\kappa_i = \kappa_{N_i}$. Then $\kappa_o < \ldots < \kappa_i < \ldots, i < \kappa$, forms a closed unbounded subset of κ. Since κ is Mahlo, there is an i such that $\kappa_i = i$ is a strongly inaccessible cardinal. Then N_i can be seen to be rich.

The above shows that for all $a \in \hat{M}$, there is a rich N such that $a \in N$.

Fix for the moment a rich subset N of $\hat{M}$. Since κ is in N, so is P. Since $P = P(\kappa)$ and since N is closed under ω-sequences, $\bar{P} = P(\bar{\kappa}) = P \cap N$. So the inclusion $P \cap N \subseteq P$ is the same as the inclusion $P(\bar{\kappa}) \subseteq P(\bar{\kappa}) \times P[\bar{\kappa}, \kappa)$. Thus $N(P)$ is an elementary substructure of $\hat{M}(P)$ (i.e., for ϕ in N, a sentence about $N(P)$, and for $\langle p, q\rangle \in P$, $(\bar{\kappa}) \times P[\bar{\kappa}, \kappa) = P$, if $\langle p, q\rangle \Vdash_{\bar{P}}$ "ϕ is true in $\hat{M}(P)$", then $p \Vdash_{\overline{P(\bar{\kappa})}}$ "ϕ is true in $N(P)$").

Since $\bar{P}$ has the $<\bar{\kappa}$-antichain condition, and since $\bar{N}$ is closed under $<\bar{\kappa}$-sequences from M, we have that $\bar{N}(\bar{P})$ is closed under $<\bar{\kappa}$-sequences from $M(\bar{P})$. For each $\beta \in N \cap \kappa^+$, it is true in $\bar{N}(\bar{P})$ that $\overline{Q_\beta}$ is the iteration with $<\bar{\kappa}$-support of $\langle \overline{R_\alpha}\rangle_{\alpha\in N\cap\beta}$, and so this is still true in $M(\bar{P})$.

Claim 2 *For all $\beta < \kappa^+$, for all rich N such that $\beta \in N$, in $M(\bar{P} * \overline{Q_\beta})$, $\bar{X}_\beta$ is not stationary.*

Claim 3 *For all $\beta < \kappa^+$, for all rich N such that $\beta \in N$, in $M(\bar{P})$, $\bar{Q}_\beta$ is essentially $<\bar{\kappa}$-closed.*

We will now prove Claims 1, 2, and 3 by induction on β.

First notice that Claim 3 for β implies Claim 1 for β. [*Proof:* Given β, let N be a rich subset of $\hat{M}$ such that $\beta \in N$. Since $\hat{M}$ is closed under κ^+-sequences, everything of interest from M is in $\hat{M}$, so we need only show that Claim 1 for β is true in $\hat{M}(P)$, and hence by elementariness, it is enough to show it true in $N(P) \approx \bar{N}(\bar{P})$. But $\bar{Q}_\beta$ is $<\bar{\kappa}$-closed in $M(\bar{P})$, and $\bar{N}$ has cardinality $\bar{\kappa}$, so in $M(\bar{P})$ we can find G a filter on $\bar{Q}_\beta$ generic over $\bar{N}$. So $N(\bar{P})[G]$ is contained in $M(\bar{P})$. But $\bar{N}(\bar{P})$ is closed under $<\bar{\kappa}$-sequences from $M(\bar{P})$, and so $\bar{N}(\bar{P})$ is closed under $<\bar{\kappa}$-sequences from $\bar{N}(\bar{P})[G]$. Since G could be chosen to extend any condition in $\bar{Q}_\beta$, this means that $\bar{N}(\bar{P})$ is closed under $<\bar{\kappa}$-sequences from $\bar{N}(\bar{P})(\overline{Q_\beta})$.]

Next notice that Claims 2 and 3 for all $\beta < \gamma$ imply Claim 3 for γ. [*Proof:* For each $\beta \in N \cap \gamma$, in $M(\bar{P} * \overline{Q_\beta})$ pick C_β a closed unbounded subset of κ disjoint from $\overline{X_\beta}$. Let $A_\beta = \{r \in \overline{R_\beta}\colon max(r) \in C_\beta\}$. Then A_β is a $<\bar{\kappa}$-closed, dense subset of $\overline{R_\beta}$. (*Proof:* If in $M(\bar{P} * \overline{Q_\beta})$, we have $r_0 < \ldots < r_i < \ldots$, $(i < \delta,\ \delta < \bar{\kappa})$ where $r_i \in A_\beta$, then $sup \cup R_i$ is in C_β, and so $r = (\cup r_i) \cup \{sup \cup r_i\}$ is disjoint from $\overline{X_\beta}$ and end-extends each r_i. Since $\overline{Q_\beta}$ is $<\bar{\kappa}$-closed in $M(\bar{P})$, r is in $M(\bar{P})$ and hence in $\bar{N}(\bar{P})$ and so in $\overline{T_\beta}$.) So the A_β's demonstrate that $\langle \overline{R_\beta} \rangle_{\beta \in N \cap \gamma}$ satisfies the hypotheses of Fact 1, and so $\overline{Q_\gamma}$ has a $<\bar{\kappa}$-closed dense subset.]

Finally notice that Claim 3 at β implies Claim 2 at β. (This is the heart of the argument.)

Proof:

Subclaim *In $M(\bar{P})$, $\overline{Q_\beta} \times P[\bar{\kappa}, \kappa)$ is essentially the same as $P[\bar{\kappa}, \kappa)$.*

Proof: In $M(\bar{P} \times P\{\bar{\kappa}\})$, $\bar{\kappa}$ has cardinality $\aleph_1$ and CH holds (since $\bar{\kappa}$ is strongly inaccessible). So by Fact 2 plus Claim 3 for β, $P\{\bar{\kappa}+1\}$ and $P\{\bar{\kappa}+1\} \times \overline{Q_\beta}$ are essentially the same. So in $M(\bar{P})$, $P\{\bar{\kappa}\} \times P\{\bar{\kappa}+1\}$ and $P\{\bar{\kappa}\} \times P\{\bar{\kappa}+1\} \times \overline{Q_\beta}$ are essentially the same. The rest should be clear.

By the subclaim, in $M(P)$ we can find $\bar{G}$, a filter on $\overline{Q_\beta}$, generic over $M(\bar{P})$ such that $M(P)$ is a generic extension of $M(\bar{P})[\bar{G}]$ via the p.o. $P[\bar{\kappa}, \kappa)$. Working in $M(P)$, we can find $G \subset Q_\beta \cap N$ such that $\bar{G} = \{\bar{q}\colon q \in G\}$. G has a lub in Q_β. (*Proof:* Since $\overline{Q_\beta}$ is $\leq\omega$-closed in $M(\bar{P})$, and M$(\bar{P})$ is closed under ω-sequences from $M(P)$, and since G has cardinality $\aleph_1$ in $M(P)$, we can find a sequence $q_0 \leq \ldots \leq q_i \leq \ldots, i < \aleph_1$, of elements of G cofinal in G. It is trivial to see that any sequence like this (of length $\aleph_1$, not ω) has a lub in Q_β). Let $q =$ lub of G. Let Y in $M(\bar{P})[\bar{G}]$ be the interpretation of $\overline{X_\beta}$. Clearly $q \Vdash_{Q_\beta}$ "$X_\beta \cap \bar{\kappa} = Y$". So $q \Vdash_{Q_\beta}$ "in $M(P * Q_\beta)$ Y is not stationary in $\bar{\kappa}$". So in $M(P * Q_\beta)$ there is C a closed, unbounded subset of $\bar{\kappa}$ disjoint from Y. By Claim 1 for β (which follows from Claim 3 for β) C is in $M(P)$. So Y is not stationary in $M(P)$. But $M(P)$ is generic over $M(\bar{P})[G]$ via a $\leq\omega$-closed p.o. So by Fact 3, Y is not stationary in $M(\bar{P})[G]$. Since G could be chosen to extend any condition in $\overline{Q_\beta}$, we have that in $M(\bar{P})(\overline{Q_\beta})$, $\overline{X_\beta}$ is not stationary.

This completes the proof of Theorem A.

Comments: The above argument can be done without first collapsing κ to $\aleph_2$. This would produce a model in which κ is a Mahlo cardinal and every stationary subset of κ of cofinality ω ordinals is stationary in some cardinality $<\kappa$.

Proof of Theorem C(i): Let T be an Aronszajn tree on $\aleph_1$. Baumgartner has shown that there is a ccc p.o. $P(T)$ such that forcing with $P(T)$ makes T into a special Aronszajn tree, namely: $p \in P(T)$ iff $p\colon x \to Q =$ the rationals, where $x \subseteq T$ is finite and p is order preserving; $P(T)$ is ordered by extension. Clearly a generic object for $P(T)$ yields an order-preserving function from T to Q. We also claim that $P(T)$ has ccc.

Proof: Given an uncountable antichain for $P(T)$, by a delta-system argument we can find an antichain $\langle p_\alpha\rangle_{\alpha<\aleph_1}$ such that $dom\ p_\alpha = \hat{x} \cup x_\alpha$ where $\hat{x}$, $\langle x_\alpha\rangle_{\alpha<\aleph_1}$ are pairwise disjoint. By thinning this sequence if necessary, we may assume that for $\alpha < \beta$, ($a \in x_\alpha$, $b \in x_\beta \Rightarrow$ height $a <$ height b). We may also assume that $p_\alpha|\hat{x} = p_\beta|\hat{x}$. Since $p_\alpha \cup p_\beta$ is not a condition of $P(T)$ there must be $a \in x_\alpha$, $b \in x_\beta$ such that $a < b$. We may assume that the x_α's have the same cardinality, say n. Let $a(1, \alpha), \ldots, a(n, \alpha)$ list the elements of x_α. Let U be a uniform ultrafilter on $\aleph_1$, and let "a.e. α" mean "for all α in a set in U". So we have: a.e. α a.e. $\beta\ \exists\ i \leq n\ \exists\ j \leq n\ a(i, \alpha) < a(j, \beta)$. By the finite additivity of U, this becomes: $\exists i \leq n\ \exists\ j \leq n$ a.e. α a.e. β $a(i, \alpha) < a(j, \beta)$. Now pick $X \in U$ such that for all $\alpha \in X$ a.e. β $a(i, \alpha) < a(j, \beta)$. So for $\alpha < \alpha'$ both in X, there are many β such that both $a(i, \alpha)$, $a(i, \alpha')$ are $<a(j, \beta)$. So $a(i, \alpha) < a(i, \alpha')$, i.e. $\{a(i, \alpha)\colon \alpha \in X\}$ is an uncountable branch through T.

We now consider a variant of $P(T)$. Given $\langle d_\alpha\rangle_{\alpha<\aleph_1}$ a sequence of subsets of ω, define a p.o. $P(T, \langle d_\alpha\rangle_{\alpha<\aleph_1})$ by: $p \in P(T, \langle d_\alpha\rangle_{\alpha<\aleph_1})$ iff $p \in P(T)$ and for $a \in dom\ p$, if a has height $\omega\cdot\alpha$ (= the α^{th} limit ordinal) and if $p(a) \in \omega$ (recall that $\omega \subseteq Q$) then $p(a) \in d_\alpha$. To make full use of this definition we assume that T has infinitely many nodes of height 0. $P(T, \langle d_\alpha\rangle_{\alpha<\aleph_1})$ can be seen to have ccc by repeating the above argument for $P(T)$. Clearly a generic object for $P(T, \langle d_\alpha\rangle_{\alpha<\aleph_1})$ gives rise to an order-preserving function $F\colon T \to Q$. But F also has the properties:

(1) $n \in d_\beta$ iff there is a in T of height $\omega\cdot\beta$ such that $F(a) = n$.
(2) if $a \in T$ has the height of an infinite limit ordinal, then $F(a) = sup\{F(b)\colon b < a\}$.

[The "if" direction of (1) follows from genericity (plus the fact that there are infinitely many nodes of height 0); the "only if" direction follows from the definition. (2) clearly follows from genericity.]

We can now prove:

Claim 4 *Assume MA, and let T be an Aronszajn tree on* $\aleph_1$. *Let* $\langle b_\alpha\rangle_{\alpha<\aleph_1}$ *be a sequence of subsets of* ω. *Then there is a real c such that* $\langle b_\alpha\rangle_{\alpha<\aleph_1}$ *is in* $L[T, c]$.

Proof: For $\beta < \aleph_1$, let T_β = the set of nodes of T of height $<\beta$. For each $n \in \omega$, we will define $\langle d_\alpha^n\rangle_{\alpha<\aleph_1}$ and F_n as follows: let $d_\alpha^0 = b_\alpha$. Given $\langle d_\alpha^n\rangle$, let $F_n\colon T \to Q$ come from a reasonably generic filter on $P(T, \langle d_\alpha^n\rangle)$. Let d_α^{n+1} code (in some canonical way) $F_n|T_{\omega\cdot(\alpha+1)}$. Let c be a real which codes $\langle d_0^n\rangle_{n\in\omega}$. We claim that the sequence $\langle d_\alpha^n\rangle_{n\in\omega,\alpha<\aleph_1}$ is in $L[T, c]$: Given $\langle d_\alpha^n\rangle_{n\in\omega,\alpha<\beta}$, $(\beta > 0)$ since d_α^{n+1} codes $F_n|T_{\omega\cdot(\alpha+1)}$, we have $F_n|T_{\omega\cdot\beta}$. Hence by property (2) above, we have $F_n|T_{(\omega\cdot\beta)+1}$. Hence by property (1) above, we have d_β^n.

Using Claim 4, the proof of Theorem C(i) will be complete once we show: if $\aleph_1$ is not weakly compact in L, then there is an Aronszajn tree T on $\aleph_1$ such that $T \in L$. This follows from the following result of Silver's:

Claim 5 *Let κ be a regular cardinal of L, not weakly compact in L. There is a tree T on κ in L such that, in any model of ZFC (extending L), if there is a length κ branch through T then κ has cofinality ω.*

Proof: Work in L. Since κ is not weakly compact, there is a tree T_0 on κ without a length κ branch. To normalize T_0 we assume that a node of T_0 of height α is a function from α to α. Define T by: $\eta = \langle \alpha, M, b\rangle$ is in T iff $\alpha < \kappa$, $M = L_\beta$ some β, $b \in M$, $\alpha \subseteq M$, $M =$ Skolem-Hull of $\alpha \cup \{b\}$ inside M, and b is a function with domain $\supseteq \alpha$, and $b|\alpha$ is in T_0. For $\tau = \langle \alpha', M', b'\rangle$ another node of T, $\tau \geq \eta$ iff $\alpha' \geq \alpha$, $M =$ the transitive collapse of the Skolem-Hull of $\alpha \cup \{b'\}$ inside M', and b' collapses to b. Clearly T is a tree, $\langle \alpha, M, b\rangle$ is a node of height α.

If $\langle \alpha, M_\alpha, b_\alpha\rangle_{\alpha<\kappa}$ is a branch through T, by identifying M_α with the Skolem-Hull of $\alpha \cup \{b_\beta\}$ inside M_β ($\beta \geq \alpha$), we obtain an elementary chain of structures (namely, $\langle M_\alpha, b_\alpha, \epsilon\rangle_{\alpha<\kappa}$). Let $\langle M, b, E\rangle$ be the direct limit of these structures. Clearly, $\langle M, E\rangle \models$ "$V = L + b$ is a function", $\kappa \subseteq M$, and for each $\alpha < \kappa$, $b|\alpha$ is in T_0. So $b|\kappa$ is a branch througn T_0. Thus b is not in L, so M is not well-founded. But M is the direct limit of well-founded structures, so this direct limit must have length of cofinality ω. So κ has cofinality ω.

Proof of Theorem C(ii): (This proof leans heavily on an argument of Roitman.) Recall some definitions: $[X]^2$ = the set of unordered pairs from X; for F: $[X]^2 \to 2$, $Y \subseteq X$, $i = 0$ or 1; Y is called i-homogeneous for F if $F\,[Y]^2 = \{i\}$.

Fix H: $[\aleph_1]^2 \to \omega$ such that for $\alpha < \beta < \gamma$, $H\{\alpha, \gamma\} \neq H\{\beta, \gamma\}$ (so $\alpha \to H\{\alpha, \gamma\}$ is $1-1$ on α's $< \gamma$).

For $d \subseteq \omega$, define F_d: $[\aleph_1]^2 \to 2$ by: $F_d\{\alpha, \gamma\} = 1$ iff $H\{\alpha, \gamma\} \in d$. There is a natural p.o., $P(d)$, which produces 1-homogeneous sets for F_d, namely: $p \in P(d)$ iff $p \subseteq \aleph_1$ is finite, and p is 1-homogeneous for F_d. Order $P(d)$ by inclusion. In general $P(d)$ need not have ccc, but we do have the following. Let M be a model of *ZFC* and pick the above H inside M. Then:

Claim 6 (Roitman) *If d is a real generic over M via either Cohen forcing or random real forcing, then $P(d)$ has ccc in $M[d]$.*

Proof: Let d be Q-generic over M. So Q in M is either Cohen or random real forcing. In either case the following is true (and left to the reader):

If $\langle q_\alpha\rangle_{\alpha<\aleph_1}$ is a sequence of elements of Q, then there is q in Q such that for infinitely many α, $q_\alpha \leq q$.

Assume that in $M[d]$ there is an uncountable antichain $\langle p_\alpha\rangle_{\alpha<\aleph_1}$ for $P(d)$. By a delta-system argument, we may assume that $p_\alpha = \hat{x} \cup x_\alpha$ where $\hat{x}$, $\langle x_\alpha\rangle_{\alpha<\aleph_1}$ are pairwise disjoint. We can also assume that the x_α's have the same cardinality, say $n \in \omega$, and that $\alpha < \beta \Rightarrow max(x_\alpha) < min(x_\beta)$. Since $p_\alpha \cup p_\beta$ is not in $P(d)$, there must be $a \in p_\alpha$, $b \in p_\beta$ such that $H\{a, b\} \notin d$.

In M, for each $\alpha < \aleph_1$ pick $q_\alpha \in Q$, $y_\alpha \subseteq \aleph_1$ such that $q_\alpha \Vdash_Q$ "$x_\alpha = y_\alpha$". By the above property of Q, find an infinite set $Z \subseteq \aleph_1$ and $q' \in Q$ such that $\alpha \in Z \Rightarrow q_\alpha \leq q'$. Pick $\delta > sup\ Z$ and find $y \subseteq \aleph_1$, $q \in Q$ such that $q' \leq q$, $q \Vdash_Q$ "$x_\delta = y$". For $\alpha \in Z$ let $w_\alpha = H$ "$(y_\alpha \times y)$". By thinning Z a bit if necessary, we may assume that the $w_\alpha (\alpha \in Z)$ are pairwise disjoint (possible by the choice of H). Notice that each w_α has cardinality $\leq n^2$. Let $B = \{c \subset \omega: \forall \alpha \in Z(c \not\supseteq w_\alpha)\}$. So $q \Vdash_{\overline{Q}}$ "$d \in B$". But B is a Borel set, and it is both meager and of measure 0. This contradicts our assumption that Q was either Cohen or random forcing.

Let $P^*(d)$ be the finite support product of ω-many copies of $P(d)$. By essentially just repeating the above argument we get: under the same assumptions as Claim 6, $P^*(d)$ has ccc in $M[d]$. Now, for $i = 0$ or 1, let $A_i = \{c \subseteq \omega: \aleph_1$ is the union of countably many sets, each i-homogeneous for $F_c\}$. Clearly forcing with $P^*(d)$ ensures that $d \in A_1$, and forcing with P^* $(\omega \sim d)$ ensures that $d \in A_0$. By the above we have that (for Q either Cohen or random forcing, and for d denoting the Q-generic real) $Q * P^*(d)$ has ccc. So also $Q * P^*$ $(\omega \sim d)$ has ccc (since $\omega \sim d$ is Q-generic whenever d is). So assuming MA, both A_0 and A_1 have members in each nonmeager or nonmeasure zero Borel set (i.e., in each condition from our Q's). But clearly A_0, A_1 are disjoint. So A_0 and A_1 cannot be separated by a measurable or property of Baire set; and so neither A_0 nor A_1 is measurable or has the property of Baire.

Now to conclude the proof of Theorem C(ii), assume (in addition to MA) that $\aleph_1 = \aleph_1^L$. In this case we can choose our function H to be in L, and in fact Δ_1 over $L_{\aleph_1}$. We claim that A_0, A_1 are Σ^1_3. Let $A = A_i$ ($i = 0$ or 1). The definition of A can clearly be put in the normal form: $c \in A$ iff $\exists f: \aleph_1 \to \aleph_1$ $(\langle L_{\aleph_1}, \epsilon, c, f\rangle \models \phi)$, where ϕ is some first-order sentence. By having f absorb some Skolem functions for ϕ, we may assume that ϕ is Π_1. But using MA, any $f: \aleph_1 \to \aleph_1$ can be coded (see [4]) by a real, say a; and the uncoding process is Δ_1 over $\langle L_{\aleph_1}, \epsilon, a\rangle$. So $c \in A$ iff $\exists\, a \subseteq \omega$ (a codes $f: \aleph_1 \to \aleph_1$ and $\langle L_{\aleph_1}, \epsilon, c, f\rangle \models \phi$); and this last expression is seen to be Σ^1_3.

Theorem B follows immediately from the relativized version of Theorem C (i.e., replace L by $L[b]$, b a real) plus the aforementioned result of Kunen:

Claim 7 *If κ in M is weakly compact then there is a generic extension of M in which MA holds and for which every set of reals in $L[2^\omega]$ is measurable and has the property of Baire.*

Proof: For the sake of convenience, we use complete Boolean algebras (cba).

Lemma 1 *If B is a cba with the $<\kappa$-antichain condition ($<\kappa$-cc), and if $X \subseteq B$ has cardinality $<\kappa$, then there is a complete subalgebra (csa) $\bar{B}$ of B such that $X \subseteq \bar{B}$ and $\bar{B}$ has cardinality $<\kappa$.*

Proof: Since B has $<\kappa$-cc (and $\kappa = \kappa^{<\kappa}$) we can find $X \subseteq B' \subseteq B$, B' a csa of B of cardinality $\leq\kappa$. We may assume that $B' \subseteq \kappa$. Let $D =$ the set of maximal antichains of B'. So $D \subseteq \kappa^{<\kappa}$. By Π^1_1 reflection there is $\alpha < \kappa$ such that $B' \cap \alpha$ is a $<\alpha$-complete Boolean algebra and $D \cap \alpha^{<\alpha} =$ the maximal antichains of $B' \cap \alpha$. So $B' \cap \alpha$ is a cba and it is a csa of B'. We may of course pick $\alpha > sup$ X, so $X \subseteq B' \cap \alpha$, and $B' \cap \alpha$ is our desired $\bar{B}$.

Lemma 2 *If P_0, P_1 are two p.o.'s with $<\kappa$-cc, then $P_0 \times P_1$ has $<\kappa$-cc.*

Proof: Let $\langle p^0_\alpha, p^1_\alpha\rangle$ $\alpha < \kappa$ be a sequence of elements of $P_0 \times P_1$. Define F: $[\kappa]^2 \to 2^2$ by: $F(\{\alpha,\beta\})(i) = 0$ iff p^i_α, p^i_β are compatible. Clearly if $X \subseteq \kappa$ is a cardinality κ homogeneous set for F, then for $\alpha, \beta \in X$, p^i_α, p^i_β are compatible $(i = 0, 1)$.

Let θ be a cardinal $>\kappa$ such that $\theta^{<\theta}$ has cardinality θ. As in [7] we can now proceed to build a length θ iteration, with finite support, of cba's with $<\kappa$-cc. Let B be the resulting cba. So B has $<\kappa$-cc. Clearly in $M(B)$, $\aleph_1 = \kappa$ and MA holds.

For $\bar{B}$ a csa of B, in $M(\bar{B})$ we have the factor algebra (which we will denote by $B/\bar{B}$), which is a cba such that $\bar{B} * (B/\bar{B})$ is essentially the same as B. For $\bar{B}$ of cardinality $<\kappa$, in $M(\bar{B})$ we can build a cba $Q_{\bar{B}}$ in the same way that B was built in M. Now using the fact that Lemma 2 is true in $M(\bar{B})$ ($\bar{B}$ of cardinality $<\kappa$), and assuming that the iterations which build B and $Q_{\bar{B}}$ are reasonably repetitious (which they can be by Lemma 2) we get:

(*) In $M(\bar{B})$, $Q_{\bar{B}}$ is homogeneous and $B/\bar{B}$ is essentially the same as $Q_{\bar{B}}$.

Finally, in $M(B)$ let A be a set of reals in $L[2^\omega]$. So for some real b in $M(B)$, A is definable over $M(B)$ using b plus some parameters from M. Thus there is a formula ϕ in M such that in $M(B)$: for all reals a, $a \in A$ iff $M(B) \models \phi\ (a, b)$. By Lemma 1, there is B' a csa of B, B' or cardinality $<\kappa$ such that $b \in M(B')$. In $M(B')$ let R be either Cohen forcing or random real forcing, and let $\bar{B} = B' * R$. Let $Q = Q_{\bar{B}}$. So, by (*) above, in $M(B)$, if a is R-generic over $M(B')$ then: $a \in A$ iff $M(B')[a](Q) \models \phi\ (a, b)$. But Q is homogeneous in $M(B')(R)$. So now we can proceed as in [6].

The observant reader will have noticed that we have not yet quite done all that was implied in Theorem B. To justify the equiconsistency of (i) and (iii) we should show:

Claim 8 *Assuming MA plus $\aleph_1 = \aleph_1^L$, there is a Δ^1_3 set without the property of Baire. (The parallel result with a real parameter holds.)*

This follows from a result of Shelah: If c is Cohen generic over M, M a model of ZFC, then in $M[c]$ there is a Souslin tree, $T(c)$.

The definition of $T(c)$ can be extended to make sense for all reals c. So, let $B_0 = \{c \subseteq \omega$: $T(c)$ has an uncountable branch$\}$, and let $B_1 = \{c \subseteq \omega$: $T(c)$ is a special Aronszajn tree$\}$. Assuming MA, $B_1 = 2^\omega \sim B_0$, and by Shelah's result both B_0 and B_1 have members in any nonmeager Borel set. So B_0 does not have the property of Baire. If $\aleph_1 = \aleph_1^L$, then $T(c)$ can be constructed to be Δ_1 over $\langle L_{\aleph_1}, \epsilon, c\rangle$. So as in the above proof of Theorem C(ii), B_0 and B_1 are both Σ^1_3 and hence Δ^1_3.

NOTES

1. Magidor proved that if in (ii) of Theorem A we reflect two stationary sets simultaneously, then the statement is equiconsistent with the existence of weakly compact cardinals (one implication is from Baumgartner [1]).
2. The results were obtained in the spring of 1978 (and announced in *The Notices of the American Mathematical Society*).
3. If it is appropriate in $M[P * Q_{\kappa^+}]$, it is actually appropriate in $M[P * Q_\alpha]$ (trivially

for large enough α, but as the forcing conditions are $(\leq\omega)$-closed no such restriction is needed).

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L. Harrington
Department of Mathematics
University of California
Berkeley, California 94720

S. Shelah
The Hebrew University
Jerusulem, Israel