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A Note on Satisfaction Classes

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1 Introduction Let L be the language of Peano Arithmetic (PA). If M is a nonstandard model of PA then any arithmetization of L determines a nonstandard language Form (M), consisting of the formulas in the sense of M. Investigation of this language was begun in Robinson's [11]. Satisfaction classes were introduced by Krajewski in [9] in a study of the semantics of Form (M).

It is a common practice in the literature to give only a "rough idea" of satisfaction class and refer for a precise definition to [9]. This sometimes leads to misunderstandings, especially when one has to distinguish full satisfaction classes from those which are not full. For the reader's convenience in what follows, we present Krajewski's definition with some minor changes on which we comment later.

In the definition below, the symbols $\neg$, $\vee$, $\&$ denote functions on Form (M) or (Form $(M))^2$, respectively, given by the arithmetization. This applies also to symbols $\exists v_k$, $\forall v_k$, where v_k is a variable of Form (M).

We say that $\Phi \subseteq \text{Form}(M)$ is *closed under immediate subformulas* if whenever any of the formulas $\neg\phi$, $\exists v_k \phi$, $\forall v_k \phi$ is in Φ, then ϕ is in Φ, and whenever $\phi \vee \psi$ or $\phi \& \psi$ is in Φ, then so are ϕ and ψ.

Satisfaction classes on M are certain sets of pairs of the form $\langle \phi, a \rangle$, where $\phi \in \text{Form}(M)$ and a is a valuation for ϕ. So a is a sequence of elements of M with domain corresponding to the set of free variables of ϕ. Using an arithmetical coding of finite sequences we treat satisfaction classes as subsets of M.

1.1 Definition If M is a model of PA, a subset S of M is a *satisfaction class* iff

a. every $x \in S$ is of the form $\langle \phi, a \rangle$, where $\phi \in \text{Form}(M)$ and a is a valuation for ϕ
b. the class $\Phi(S) = \{\phi \in \text{Form}(M) : \exists a \langle \phi, a \rangle \in S \vee \forall a$ (a is a valuation for ϕ) $\Rightarrow \langle \neg\phi, a \rangle \in S\}$ is closed under immediate subformulas
c. if $M \models \phi a$ and $\ulcorner \phi \urcorner$ is the Gödel number of ϕ, then $\langle \ulcorner \phi \urcorner, a \rangle \in S$

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d. if $\neg\phi \in \Phi(S)$ and a is a valuation for ϕ, then $\langle \neg\phi, a\rangle \in S \Leftrightarrow \langle \phi, a\rangle \notin S$
e. if $\phi \vee \psi \in \Phi(S)$ and a is a valuation for $\phi \vee \psi$, then $\langle \phi \vee \psi, a\rangle \in S \Leftrightarrow \langle \phi, a'\rangle \in S \vee \langle \phi, a''\rangle \in S$, where a' and a'' are suitable valuations for ϕ and ψ, respectively, obtained from a; similarly for $\phi \& \psi$
f. If $\exists v_k\phi \in \Phi(S)$, $\langle \exists v_k\phi, a\rangle \in S \Leftrightarrow [(v_k$ is a free variable of ϕ and $\exists b\langle \phi, ab\rangle \in S)$ or (v_k is not a free variable of ϕ and $\langle \phi, a\rangle \in S)]$, where ab is a suitable valuation for ϕ obtained from a and b, similarly for $\forall v_k$.

In part c of the above definition, ϕ ranges over all formulas of L, while in Krajewski's original definition ϕ ranges over atomic formulas of L only. This change allows us to say just "satisfaction class" where otherwise we would have to say "satisfaction class deciding all standard formulas". Krajewski's definition is intended to be as general as possible. In special cases the definition for satisfaction classes can be made much simpler or at least shorter (cf. [7]; see also [12] for a stronger but useful definition).

1.2 Definition A satisfaction class S on M is *full* if for every $\phi \in \mathrm{Form}(M)$ and every valuation a for ϕ we have $\langle \phi, a\rangle \in S$ or $\langle \neg\phi, a\rangle \in S$.

Notice that S is a full satisfaction class iff $\Phi(S)$ consists of all formulas of $\mathrm{Form}(M)$.

In what follows we will often write $S(\phi, a)$ instead of $\langle \phi, a\rangle \in S$.

There are a number of interesting results on full satisfaction classes. The most significant is the following theorem.

1.3 Theorem

a. (Kotlarski, Krajewski, Lachlan [7]) *Every resplendent model of PA possesses a full satisfaction class.*
b. (Lachlan [8]) *Every nonstandard model of PA possessing a full satisfaction class is recursively saturated.*

Let L_S be L together with an additional predicate symbol S. Let $PA(S)$ be a recursive theory consisting of PA, the induction schema for all formulas of L_S and a set of L_S sentences saying that S is a satisfaction class.

If $(M, S) \models PA(S)$, then we say that S is an *inductive satisfaction class* on M.

1.4 Theorem

a. *Every resplendent model of PA possesses an inductive satisfaction class.*
b. *Every model of PA possessing an inductive satisfaction class is recursively saturated.*

From Theorems 1.3 and 1.4 it follows that a countable nonstandard model M of PA possesses a full satisfaction class iff it possesses an inductive satisfaction class and both these things may happen if and only if M is recursively saturated.

While the proof of Theorem 1.3 is rather difficult and uses the whole power of satisfaction classes, the proof of 1.4 is quite easy. In particular for the proof of 1.4b we need only c and d of Definition 1.1 and a weak form of induction (cf. [3] or [4]).

There are uncountable recursively saturated models without full satisfaction classes and without inductive satisfaction classes. A rather classless model of Kaufmann (cf. [2]) serves as an example here. Again, while it is rather obvious that Kaufmann's model does not have any inductive satisfaction classes, the proof that it does not have any full satisfaction classes, given recently by Smith (cf. [14]), is not so easy.

If we mix together two sorts of satisfaction classes mentioned above we will obtain full inductive satisfaction classes, i.e., those inductive satisfaction classes which are full.

Let $\mathbf{N}$ be the standard model and let $S_{tr} = \{\langle \ulcorner\phi\urcorner, a\rangle : \mathbf{N} \models \phi(a) \ \&\ \phi$ is a formula of $L\}$. S_{tr}, of course, is a full inductive satisfaction class on $\mathbf{N}$. In fact, it is the only satisfaction class on $\mathbf{N}$. If (M,S) is a proper elementary extension of $(\mathbf{N}, S_{tr})$, then by Theorems 1.3 or 1.4, M is recursively saturated and, of course, S is a full inductive satisfaction class on M. But not all countable recursively saturated models have full inductive satisfaction classes. This follows from the next proposition.

1.5 Proposition (Krajewski [9]) *If $M \models PA$ possesses a full inductive satisfaction class, then $M \models Con(PA)$.*

For other interesting results see also [10] and a recent paper [6].

2 Statement of the results In this note we present three results concerning the variety of satisfaction classes on countable recursively saturated models of Peano Arithmetic. The main aim is to show a special method of dealing with inductive satisfaction classes which allows us to use standard techniques developed for the study of models of PA based on the existence of universal truth formulas for Σ_n formulas of the language of PA. Our results are analogues of the following three classical theorems on models of PA.

2.1 Theorem A (Jensen, Ehrenfeucht) *For every nonstandard countable model M of PA there is a continuum of pairwise elementarily inequivalent initial segments of M which are models of PA.*

Let L be the language of PA. Let L^* be any countable language extending L and let $PA(L^*)$ be PA together with the full induction schema for L^*.

2.2 Theorem B (Gaifman) *For every nonstandard countable model M of $PA(L^*)$ there is a continuum of pairwise nonisomorphic, countable, elementary end extensions of M.*

A subset S of $\mathbf{N}$ is said to be *representable* in a complete theory T if there is a formula ϕ such that $X = \{n \in \mathbf{N} : \phi(n) \in T\}$.

2.3 Theorem C (Scott) *For every countable Scott set χ there is a consistent, complete extension T of PA such that χ is the family of sets representable in T.*

From general model theoretic facts it follows that on every countable recursively saturated model of PA there is a continuum of distinct inductive satisfaction classes. Our results go a bit further. Let us say that satisfaction classes S_1, S_2 on a model M are *isomorphic* (*elementarily equivalent*) if the structures (M,S_1), (M,S_2) are isomorphic (elementarily equivalent).

2.4 Theorem A1 *On every countable recursively saturated model of PA there is a continuum of inductive, pairwise nonelementarily equivalent satisfaction classes.*

2.5 Theorem B1 *For every countable recursively saturated model M of PA and every inductive satisfaction class S on M there is a continuum of pairwise nonisomorphic satisfaction classes on M which are elementarily equivalent to S.*[1]

2.6 Theorem C1 *Let T be a consistent, complete extension of PA and let* χ *be a countable family of subsets of* $\mathbf{N}$ *which contains the set of Gödel numbers of the sentences in T. Then* χ *is a Scott set if and only if there exists a consistent, complete extension* $T(S)$ *of* $PA(S) \cup T$ *such that* χ *is the family of sets representable in* $T(S)$.

We shall give an extended sketch of a proof of Theorem A1 which is based on the proof of Theorem A from [1]. The proof of Theorem C1 is a bit more complicated. One can give a proof of Theorem C1 using ideas behind the proof of Theorem A1 and following a proof of Theorem C. The proof of Theorem C from [1] is excellent for this purpose but one can use the original proof of Scott from [13] as well.

Theorem B1 is an easy corollary of Theorem B and the following basic isomorphism theorem (cf. [15]).

2.7 Theorem *Any two countable recursively saturated, elementarily equivalent models of PA with the same standard systems are isomorphic.*

Proof of Theorem B1: Let M be a countable model of PA and let S be an inductive satisfaction class on M. By Theorem B we have continuum many pairwise nonisomorphic elementary end extensions of (M,S). If (M_1,S_1), (M_2,S_2) are two such extensions, then M_1 and M_2 are recursively saturated, elementarily equilvalent to M and have the same, standard systems as M. So M_1 and M_2 are isomorphic to M. This finishes the proof of Theorem B1.

3 Proof of Theorem A1 For the proof of Theorem A1 it will be convenient to use the following Σ_n^S hierarchy of formulas of L_S.

3.1 Definition Let Σ_0^S be the set of formulas of L_S containing S and all the formulas of L, which is closed under $\neg$, $\vee$, $\exists x < y$ and $\forall x < y$. Let $\Pi_0^S = \Sigma_0^S$ and for $n \geq 0$ let Σ_{n+1}^S be the set of formulas of the form $\exists x \phi$ where $\phi \in \Pi_n^S$ and let Π_{n+1}^S be the set of formulas of the form $\forall x \phi$ where $\phi \in \Sigma_n^S$.

Let B_n^S be the set of Boolean combinations of Σ_n^S and Π_n^S formulas and let Δ_n^S be the set of those formulas which are equivalent in $PA(S)$ to some Σ_n^S and Π_n^S formulas.

Observe that every formula of L is in Σ_0^S.

The next lemma is crucial for our considerations.

3.2 Lemma *For every $n \in \mathbf{N}$ there is a Δ^S_{n+1} formula Tr^S_n such that for every formula ϕ of B^S_n we have*

$$PA(S) \vdash \forall x[\phi(x) \Leftrightarrow Tr^S_n(\ulcorner\phi\urcorner, x)],$$

where $\ulcorner\phi\urcorner$ is the Gödel number of ϕ.

One constructs truth formulas Tr^S_n as in the case of truth formulas for Σ_n formulas of PA. For the construction of Σ^S_0 we use the following fact.

For every formula ϕ of L:

$$PA(S) \vdash \forall x\phi(x) \Leftrightarrow S(\langle\ulcorner\phi\urcorner, x\rangle).$$

We say that a set of sentences A of L_S is B^S_n-complete if for every sentence ϕ of B^S_n we have $\phi \in A$ or $\neg\phi \in A$.

Let M be a countable recursively saturated model. For every $\delta \in 2^{<\omega}$ we will define inductively a set of L_S sentences T_δ such that for every δ, δ_1, $\delta_2 \in 2^{<\omega}$:

1. T_δ is consistent with $PA(S)$
2. $Th(M) \subseteq T_\varnothing$ and if $\delta_1 \subseteq \delta_2$ then $T_{\delta_1} \subseteq T_{\delta_2}$
3. If $lh\delta = n$ then T_δ is B^S_n-complete
4. T_δ is coded in $SSy(M)$
5. If $\delta_1 \neq \delta_2$ then $T_{\delta_1} \neq T_{\delta_2}$.

For every $\delta \in 2^{<\omega}$ we will first define a certain theory $T'_\delta \subseteq B^S_{lh\delta}$ which is consistent with $PA(S)$ and coded in $SSy(M)$. Then we will take T_δ to be any $B^S_{lh\delta}$-complete extension of T'_δ which is consistent with $PA(S)$ and coded in $SSy(M)$. The construction uses the following lemma due to Scott (cf. [13]).

3.3 Lemma *If T is a consistent theory coded in a Scott set χ then there is a complete consistent theory $\bar{T}$ which extends T and is coded in χ.*

Let $T_\varnothing$ be any B^S_0-complete extension of $Th(M)$, which is consistent with $PA(S)$ and coded in $SSy(M)$. Assume that we have T_δ for some $\delta \in 2^{<\omega}$ and $lh\delta = n$. With the help of Tr^S_n we may obtain a Π^S_{n+1}-sentence that says "I am unprovable from $PA(S)$ and whatever B^S_n-sentences hold". Then $PA(S) \cup T_\delta \cup \{\alpha_n\}$ and $PA(S) \cup T_\delta \cup \{\neg\alpha_n\}$ are consistent (cf. [1] Lemma 2.5) and we put

$$T'_\delta 0 = T_\delta \cup \{\alpha_n\} \text{ and } T'_\delta 1 = T_\delta \cup \{\neg\alpha_n\}.$$

If F is an infinite branch of $2^{<\omega}$ then $T_F = \bigcup_{\delta \in F} T_\delta$ is a consistent, complete extension of $PA(S) \cup Th(M)$. Let (N, S) be a minimal model of T_F for some branch F. We may safely exclude the case when N is standard, then we have:

1. N is recursively saturated.
2. $SSy(N) \subseteq SSy(M)$, since for any set $A \in SSy(N)$, A is represented in T_F; hence A is recursive in $B^S_n \cap T_F$ for some $n \in \mathbf{N}$ and by the construction $B^S_n \cap T_F$ is coded in $SSy(M)$.
3. $N \equiv M$.

By the above and a suitable variant of the basic isomorphism theorem we may assume that $N \prec M$ (cf. [15]).

Now we will use the following result due to Kotlarski [5] and Schmerl [12].

3.4 Theorem *If M, $N \models PA$, $X \subseteq N$, the structure (N,X) satisfies the induction schema and M is a cofinal extension of N then there exists $\bar{X} \subseteq M$ such that $(N,X) \prec (M,\bar{X})$.*

Let $\bar{N} = \{x \in M : \exists y \in N, x \leq y\}$. Let $\bar{S} \subseteq \bar{N}$ be such that $(N,S) \prec (\bar{N},\bar{S})$. So $(\bar{N},\bar{S}) \models T_F$ and by the basic isomorphism theorem $\bar{N}$ is isomorphic to M. This finishes the proof of Theorem A1.

Let us also mention in this section the following corollary of Theorem C1.

3.5 Theorem *For every countable recursively saturated model M there exists a satisfaction class S on M such that (M,S) is a minimal model (i.e., has no proper elementary submodels).*

Proof: Let $T = Th(M)$ and $\chi = SSy(M)$. Let $T(S)$ be a consistent, complete extension of $PA(S) \cup T$ such that χ is the family of sets representable in $T(S)$. Let (N,S) be a minimal model of $T(S)$. Then, except for the case when N is standard, N is recursively saturated, so N is isomorphic to M, which finishes the proof.

4 Nonisomorphic pairs of models Smoryński in [16] applies certain initial segment constructions to show that for every recursively saturated model M of PA there exists a continuum of pairwise nonisomorphic structures of the form (M,M_0) where $M_0 \prec_e M$ and M_0 is recursively saturated. Here we show a different proof of this result which gives also some additional information.

4.1 Lemma *Let M be a countable model of PA and let $\mathfrak{A}$ be an uncountable family of subsets of M such that for X_1, $X_2 \in \mathfrak{A}$ if $X_1 \neq X_2$ then $(M,X_1) \not\equiv (M,X_2)$. If for every $X \in \mathfrak{A}$, $M(X)$ is a countable elementary end extension of M in which X is coded then the family $\{(M(X),M) : X \in \mathfrak{A}\}$ contains an uncountable family of pairwise nonisomorphic pairs.*

Proof: It is easy to see that if f is an isomorphism of (M_1,M) with (M_2,M) where $M \prec_e M_i$, $i = 1,2$, then the image under f of every subset of M coded in M_1 must be coded in M_2. Since for every $X \in \mathfrak{A}$ the family of subsets of M coded in $M(X)$ is countable, it follows that for every $X \in \mathfrak{A}$ the set $\{Y \in \mathfrak{A} : (M(Y),M) \cong (M(X),M)\}$ is countable, so the result follows.

Now, let $\mathfrak{A}$ be an uncountable family of pairwise nonisomorphic inductive satisfaction classes on a countable model M of PA. For every $S \in \mathfrak{A}$ let $(M(S),\bar{S})$ be a countable elementary extension of (M,S), such that $M \prec_e M(S)$. Obviously, S is coded in $M(S)$ and by Lemma 4.1 we have an uncountable family of pairwise nonisomorphic pairs of the form $(M(S),M)$. Moreover, since $\bar{S}$ is an inductive satisfaction class on $M(S)$, $M(S)$ is recursively saturated. Hence, it is isomorphic to M. Thus we obtain an uncountable family of nonisomorphic pairs of the form (M,M_0), where $M_0 \prec_e M$ and M_0 is recursively saturated.

There are many ways of constructing $(M(S), \bar{S})$ from (M, S). Using well-known ultrapower constructions we may get pairs $(M(S), M)$ satisfying various combinatorial properties (M can be semiregular, regular, strong, etc.... in $M(S)$). Hence we have uncountable families of nonisomorphic pairs (M, M_0) with these properties.

Let us recall that an extension $M \prec N$ is said to be *conservative* if every subset of M which is coded in N is definable in M. By the MacDowell-Specker theorem every model of $PA(L^*)$ has an elementary end extension which is conservative.

Suppose that a countable model $M \models PA$ has a full, inductive satisfaction class. Then by Theorem B1 there is an uncountable family $\mathfrak{B}$ of pairwise nonisomorphic full inductive satisfaction classes on M. If for every $S \in \mathfrak{B}$, $(M(S), \bar{S})$ is a conservative extension of (M, S), then it is easy to show that in fact for all S_1, $S_2 \in \mathfrak{B}$, such that $S_1 \neq S_2$, $(M(S_1), M)$ and $(M(S_2), M)$ are nonisomorphic. Now we shall prove a slightly stronger result.

Let us say that a pair (M, M_0) is *not embeddable* in (N, N_0) if for every elementary embedding f of M into N the image of M_0 under f, $f * M_0$, is not cofinal in N_0.

4.2 Theorem *If a countable model M of PA possesses a full, inductive satisfaction class then there exists a continuum of pairwise not embeddable structures of the form* (M, M_0), *where* $M_0 \prec_e M$ *and* M_0 *is recursively saturated.*

For the proof of this theorem we shall need the following fact.

4.3 Lemma (Krajewski [9]) *Let* S_1 *and* S_2 *be full satisfaction classes on M. If the structure* (M, S_1, S_2) *satisfies the induction schema (or even the* Σ_1*-induction schema) then* $S_1 = S_2$.

In particular it follows that if S is a full inductive satisfaction class, then the family of sets which are L_S*-definable in* (M, S) *contains no other full satisfaction class.*

Now the proof of Theorem 4.2 is quite easy. If M has a full, inductive satisfaction class, then in the proof of Theorem A1 we may replace $PA(S)$ by the theory $PA(S)$ + "S is full" and obtain an uncountable family of pairwise nonelementarily equivalent full inductive satisfaction classes. Let $\mathfrak{C}$ be such a family. For every $S \in \mathfrak{C}$ let $(M(S), \bar{S})$ be a countable, conservative extension of (M, S). Suppose that for S_1, $S_2 \in \mathfrak{C}$, $S_1 \neq S_2$, f is an elementary embedding of $M(S_1)$ into $M(S_2)$. We show that $f * M$ cannot be cofinal in M. Suppose, on the contrary, that it is. Then by the Kotlarski-Schmerl theorem we have $S_3 \subseteq M$ such that $(f * M, f * S_1) \prec (M, S_3)$. The set $f * S_1$ is coded in $f * M(S_1)$. From the definition of S_3 (cf. [5] or [12]) it is clear that S_3 must be coded in $M(S_2)$. But then S_3 is definable in (M, S_2), hence $S_3 = S_2$. This gives a contradiction and finishes the proof.

NOTE

1. It should be acknowledged that both these results (Theorems A1 and B1) were suggested to the author by Henryk Kotlarski.

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