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Individual Concepts as Propositional Variables in $ML^{\nu+1}$

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1 Introduction The modal languages ML^{ν} and ML^{ν}_{*} of Bressan (to be described in more detail in the second part of this introduction) are presented in [4] and [5]; substantially, ML^{ν}_{*} is obtained from ML^{ν} by adding propositional variables and constants. For every positive integer ν, the modal language ML^{ν} is based on a *type-system* τ^{ν} which has ν types $(1, \ldots, \nu)$ for *individual terms* and, accordingly, the semantical structures for ML^{ν} (the ML^{ν}-interpretations) are constructed starting from ν *individual domains* $D_1, \ldots, D_\nu$ and a set Γ of (elementary) *possible cases* (elsewhere called *worlds* or *points*), briefly, Γ-cases. The individual terms of type r of ML^{ν} are assumed to range over *individual concepts* (of type r) which are functions from Γ into D_r. This holds similarly for the ML^{ν}_{*}-interpretations, where, in addition, the propositional variables range over sets of possible cases. In every interpretation for ML^{ν} (or ML^{ν}_{*}) the conceivability relation between possible cases is $\Gamma \times \Gamma$ and, hence, the corresponding calculi MC^{ν} and MC^{ν}_{*} are based on Lewis's S5.

If we consider an $ML^{\nu+1}$-interpretation in which $D_{\nu+1}$ is a two-element set, then the individual concepts of type $\nu + 1$ can be considered as characteristic functions of subsets of Γ and hence they serve to represent propositions. In this paper this representation is used to reduce the concepts of *ML^{ν}_{*}-validity* and *general ML^{ν}_{*}-validity* (see Definition 2.2) to the analogous concepts for $ML^{\nu+1}$. In this way, the completeness of the calculus MC^{ν}_{*} (with respect to general ML^{ν}_{*}-interpretations) can be deduced from that of $MC^{\nu+1}$, which is proved in [14]. In particular, in Section 3 a correspondence between $ML^{\nu+1}$-interpretations (in which $D_{\nu+1}$ is $\{0,1\}$) and ML^{ν}_{*}-interpretations is defined, which becomes a bijection when restricted to general interpretations. In Section 4 it is proved that a formula p of ML^{ν}_{*} is valid (or valid in a general sense) iff the same holds for a suitable correspondent of it in $ML^{\nu+1}$. Furthermore, in Section 5,

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the syntactical counterparts of these results are proved and this yields the completeness of MC_*^ν.

The general interpreted modal calculus MC^ν was conceived, at the beginning, in order to provide a logical basis suitable for the axiomatization of physical theories. In particular, MC^ν aims at improving the axiomatizations of classical particle mechanics performed in [10] and [3], as far as modalities or quantification of possible worlds is concerned. In [10] some counterfactual conditionals (considered troublesome by the author himself) have an essential role. In [3] a generalized version of Painlevé's axiomatization is presented and, in order to treat the above conditionals rigorously, an unusual extensional language is employed.[1] The work [4] on MC^ν allows us to base [3] on a usual modal language, e.g., the (unformalized) one used in [10]. The considerations above, however, are concerned only with one aspect of Bressan's work, since, actually, the ideas developed in [4] also have considerable relevance with respect to general issues concerning the introduction of quantifiers into intensional contexts.

For every individual term Δ we can consider the individual concept $\tilde{\Delta}$ corresponding to it (in a given interpretation), which is a function from Γ-cases to individuals. Then, following Carnap, we say that the *extension* of Δ in the possible case γ is the individual $\tilde{\Delta}(\gamma)$; thus the *intension* of Δ is represented by $\tilde{\Delta}$, which Bressan calls the quasi-intension (briefly, *QI*) of Δ. The sentences of ML^ν have subsets of Γ as *QI*s and their extensions are truth values.[2] Let $\Delta_1, \ldots, \Delta_n$ be terms in ML^ν of the types $t_1, \ldots, t_n$, respectively, then $t = \langle t_1, \ldots, t_n, 0\rangle$ is a (relational) type in τ^ν and $\Delta(\Delta_1, \ldots, \Delta_n)$ is a (well-formed) formula whenever Δ is a term of type t. In [4], N7, Bressan shows that a correct use of the predication in modal context must be *nonextensional*, which means that, in general, the truth value of $\Delta(\Delta_1, \ldots, \Delta_n)$ in $\gamma(\in\Gamma)$ does not depend only on the extensions of $\Delta_1, \ldots, \Delta_n$ in γ, but on the whole intensions of these terms. In particular, the extension of Δ in a Γ-case is a set of n-tuples of quasi-intensions. This holds similarly for functional terms and is one of the most important innovative features of ML^ν.

Let us remark that the above considerations do not exclude the treatment of extensional relations (or functions) in ML^ν. This can be done since identity is interpreted contingently: $\Delta_1 = \Delta_2$ holds in γ iff Δ_1 and Δ_2 have the same extension in γ. Thus, for every relational term Δ (that we assume to be unary for the sake of simplicity) we can define the *extensionalization* $\Delta^{(e)}$ of Δ by $\Delta^{(e)}(x) \equiv_D (\exists y)(\Delta(y) \wedge x = y)$ and, for $t = \langle t_1, 0\rangle$, we can define the property Ext_t (which has the type $\langle t, 0\rangle$) of being an extensional relation of type t: $Ext_t(R) \equiv_D R = R^{(e)}$, where R is a variable of type t.

Contingent identity has also an essential role in the interpretation of definite descriptions, which are treated in a unified way (that is, without any a priori distinction between intensional and extensional contexts) and for which Frege's method is adopted.[3] The Church lambda-abstraction is defined in ML^ν by means of the description operator: $(\lambda x_1, \ldots, x_n)p =_D (\imath R)(\forall x_1, \ldots, x_n)(R(x_1, \ldots, x_n) \equiv p)$; and it is proved to have the usual properties.

Another important notion developed in [4] is that of "absoluteness". A (unary) attribute F is *absolute* if it is *modally constant* (it has the same extension in every Γ-case) and *modally separated* (if ξ and η belong to the extension of F, then their extensions coincide in every or in no possible case). Bressan shows, among other things, that the notion of absolute attributes is essential in using natural numbers—which are defined in ML^ν according with the Frege-Russell definition—for instance, in order to distinguish "9" from "the number of known planets".[4] Absolute attributes can be viewed as determining "criteria" for *transworld identification*. We can say that ξ at γ is equal to ξ' at γ' (with respect to the absolute attribute F) whenever a QI η falls under F such that the extension of η is the same as that of ξ in γ and as that of ξ' in γ'. With respect to this, Belnap claims that "Bressan's notion of absoluteness is the proper foundation for an adequate understanding of essentialism, essential predication, and the *de dicto/de re* distinction" (cf. [2], p. xxiv), and in [1] it is shown that Thomason's quantification over substances (i.e., constant individual concepts, see [13]) can be expressed by quantification (over arbitrary individual concepts) restricted to suitably chosen absolute attribute.[5]

In [4], NN47–49, it is shown that Γ-cases can be represented within ML^ν itself; in particular, the formula $El(u)$ and $|_u$ are defined, to be read respectively as "u represents a possible case" and "the possible case (represented by) u actually holds". This provides a very remarkable growth of the expressive power of ML^ν; for instance, several conceivability relations between (representatives of) possible cases can be defined simultaneously in ML^ν, together with the corresponding modal operators.[6]

The language ML^ν was designed with a view to its use over *standard* interpretations. Unfortunately, the intended use faces a significant difficulty: the correlative concept of validity is nonaxiomatizable.[7] The difficulty is remedied in [14], where the familiar ideas of Henkin [8] are applied to define the *general* interpretations for ML^ν. It is with respect to general interpretations that the completeness of MC^ν is provable.[8] (See Section 2 for a complete discussion of interpretations.)

In [5] Bressan considers the problem of axiomatizing and formalizing probability theories (e.g., on the basis of Reichenbach's work [12]). He observes that, in order to deal with these theories, one should be able to express functions and relations having propositional arguments; hence he defines ML^ν_*. Let us remark that propositional variables (and quantification over them) are expressible in ML^ν itself: propositions correspond to sets of Γ-cases, which can be represented in ML^ν (see above). The use of ML^ν_*, instead of ML^ν, is motivated in [5] by the quite natural treatment of probability concepts allowed by it. In any case, the reducibility of ML^ν_* to simpler languages can be used to derive technical results.

2 Preliminaries The modal language ML^ν_* (where $\nu \in Z^+$, the set of positive integers) is based on the *type-system* τ^ν_*, which is the smallest set such that $\{0, 1, \ldots, \nu\} \subseteq \tau^\nu_*$ and $\langle t_1, \ldots, t_n, t_0\rangle \in \tau^\nu_*$ whenever $n > 0$ and $\{t_0, t_1, \ldots, t_n\} \subseteq \tau^\nu_*$. We call $\langle t_1, \ldots, t_n, t_0\rangle$ a type for *relations* or *functions* (and we denote it by $(t_1, \ldots, t_n)$ or $(t_1, \ldots, t_n, t_0)$, respectively) according to whether or not $t_0 = 0$.

For every $t \in \tau^\nu_*$ and every $n \in Z^+$, the *constant* c_{tn} and the *variable* v_{tn} are primitive symbols of ML^ν_*, in addition to the usual logical symbols: $=$, $\sim$, $\wedge$, $\Box$, $\imath$, comma, and left and right parentheses. The set $\mathcal{E}^*_t$ of the *designators* or *wfes* (*well-formed expressions*) of type t ($\in \tau^\nu_*$) for ML^ν_* is defined recursively by means of the *formation rules* (f_1) to (f_8) below, where $n \in Z^+$ and t, t_0, $t_1, \ldots, t_n$ run over τ_*^ν.

(f₁) $c_{tn}, v_{tn} \in \mathcal{E}_t^*$

(f₂) $\Delta_1, \Delta_2 \in \mathcal{E}_t^* \Rightarrow (\Delta_1 = \Delta_2) \in \mathcal{E}_0^*$

(f₃) $\Delta \in \mathcal{E}^*_{\langle t_1, \ldots, t_n, t_0 \rangle}$ and $\Delta_i \in \mathcal{E}^*_{t_i} (i = 1, \ldots, n) \Rightarrow$ $(\Delta(\Delta_1, \ldots, \Delta_n)) \in \mathcal{E}^*_{t_0}$

(f₄₋₇) $p, q \in \mathcal{E}_0^* \Rightarrow (\sim p), (p \wedge q), ((v_{tn})p), (\Box p) \in \mathcal{E}_0^*$

(f₈) $p \in \mathcal{E}_0^* \Rightarrow ((\iota v_{tn})p) \in \mathcal{E}_t^*$.

The elements of $\mathcal{E}_t^*$ are called wff (*well-formed formulas*) for $t = 0$ and *terms* for $t \neq 0$. The symbols $\vee$, $\supset$, $(\exists v_{tn})$, $\Diamond$, $\equiv$, and other metalinguistic abbreviations are understood to be defined in the usual way. In particular, (ιx), $=$, (x), $\sim$, $\wedge$, $\vee$, $\supset$, and $\equiv$ have decreasing cohesive powers and $(\exists_1 x)p$ will stand for $(\exists x)[p \wedge (y)(p[x/y] \supset x = y)]$. Furthermore, in order to avoid spelling out the types of all the expressions used, henceforth we assume every such expression to be well formed.

The type system τ^ν, on which the language ML^ν is based, is the smallest subset of τ_*^ν such that $\{1, \ldots, \nu\} \subseteq \tau^\nu$ and $\langle t_1, \ldots, t_n, t_0 \rangle \in \tau^\nu$ whenever $\{t_1, \ldots, t_n\} \subseteq \tau^\nu$ and $t_0 \in \bar{\tau}^\nu = \tau^\nu \cup \{0\}$. The set $\mathcal{E}_t$ of the wfes of type t $(\in \bar{\tau}^\nu)$ for ML^ν is defined by (f_1) to (f_8), where $t, t_1, \ldots, t_n$ run over τ^ν and t_0 runs over $\bar{\tau}^\nu$.

The basic axiom schemes for the calculus MC^ν (which is based on ML^ν) are MA3.1–3.18 in [15]. In particular, we point out that the *indiscernibility of identicals* (MA3.9) concerns itself with necessary identity, that is:

(2.1) $\Box(\Delta_1 = \Delta_2) \supset \Delta[z/\Delta_1] = \Delta[z/\Delta_2]$,

and that the axioms for *descriptions* (MA3.14, 15) are:

(2.2) I. $(\exists_1 v_{tn})p \wedge p[v_{tn}/x] \supset x = (\iota v_{tn})p$

II. $\sim(\exists_1 v_{tn})p \supset (\iota v_{tn})p = a_t^*$

(where a_t^* denotes $(\iota v_{t1})(v_{t1} \neq v_{t1})$) and

(2.3) I. $\sim a_t^*(x_1, \ldots, x_n)$ $\quad (t = (t_1, \ldots, t_n))$

II. $a_t^*(x_1, \ldots, x_n) = a_{t_0}^*$ $\quad (t = (t_1, \ldots, t_n : t_0))$.

The axioms of MC_*^ν are the instances in ML_*^ν of the axioms of MC^ν and, in addition,

A*1 $\sim a_0^*$

A*2 $(\phi \equiv \psi) \equiv (\phi = \psi)$

(where ϕ and ψ are variables of type 0).

The *deduction rules* in both MC^ν and MC_*^ν are *Modus Ponens*, the *Generalization Rule*, and the *Necessitation Rule*, that is:

(2.4) $\dfrac{p \supset q, p}{q}, \quad \dfrac{p}{(x)p}, \quad \dfrac{p}{\Box p}$.

The definitions of "*provable in*..." and "*deducible from K in*..." are the usual ones and we shall write $\models_{\overline{*}}$ and $\models_{\overline{\nu}}$ as abbreviations for $\models_{\overline{MC^\nu_*}}$ and $\models_{\overline{MC^\nu}}$, respectively.

For every choice of ν sets $D_1, \ldots, D_\nu$ of *individuals* and every set Γ of *possible cases* we say that $\mathcal{S} = \{\mathcal{QI}_t : t \in \tau^\nu_*\}$ is an *ML^ν_*-structure* if the following conditions hold:

(2.5) $\mathcal{QI}_r \subseteq (\Gamma \to D_r)(r = 1, \ldots, \nu)$, $\mathcal{QI}_0 \subseteq \mathcal{P}(\Gamma)$
(2.6) $\mathcal{QI}_{\langle t_1, \ldots, t_n, t_0 \rangle} \subseteq ((\Pi^n_i \mathcal{QI}_{t_i}) \to \mathcal{QI}_{t_0})$,

where Π^n_i denotes the Cartesian product with the index i running from 1 to n, $\mathcal{P}$ denotes the power set, and $A \to B$ is the set of functions from A to B.

An *ML^ν_*-interpretation* is an ordered triple $\langle \mathcal{S}, a^\nu, \mathcal{I} \rangle$ $(= \mathcal{I})$ in which $\mathcal{S}$ is an ML^ν_*-structure, a^ν is a function of domain τ^ν_* such that $a^\nu_t \in \mathcal{QI}_t$ for all $t \in \tau^\nu_*$, and $\mathcal{I}$ is a function assigning every constant c_{tn} of ML^ν_* an element of $\mathcal{QI}_t (t \in \tau^\nu_*)$. If, in an ML-interpretation $\mathcal{I}$, (2.5) and (2.6) hold as equalities, then $\mathcal{I}$ is said to be *standard*. For every $t \in \tau^\nu_*$, the elements of $\mathcal{QI}_t$ are called *quasi-intensions* (briefly *QI*s) of type t and a^ν_t is said to be the *nonexisting* object of type t. A *valuation* $\mathcal{V}$ of the variables of ML^ν_* in the ML^ν_*-interpretation $\mathcal{I}$ (briefly, an $\mathcal{I}$-valuation) is a function such that $\mathcal{V}(v_{tn}) \in \mathcal{QI}_t$ for every $t \in \tau^\nu_*$ and every $n \in Z^+$.

If ξ and η are *QI*s (of type $t \in \tau^\nu_*$) of the ML^ν_*-interpretation $\mathcal{I}$ and γ is a possible case (of $\mathcal{I}$), then we say that ξ and η are *γ-equivalent* (briefly, $\xi =_\gamma \eta$) in $\mathcal{I}$ when

(2.7) $t \in \{1, \ldots, \nu\}$ *and* $\xi(\gamma) = \eta(\gamma)$, *or*
$t = 0$ *and* $\xi \cap \{\gamma\} = \eta \cap \{\gamma\}$, *or*
$t = \langle t_1, \ldots, t_n, t_0 \rangle$ *and* $\xi(\alpha) =_\gamma \eta(\alpha)$ *for all* $\alpha \in \Pi^n_i \mathcal{QI}_{t_i}$.

In [4] (see Theorem 10.2) it is proved that $\xi = \eta$ iff $\xi =_\gamma \eta$ for all $\gamma \in \Gamma$.

For every wfe Δ of ML^ν_*, the *designatum* $des_{\mathcal{I}\mathcal{V}}(\Delta)$ of Δ (with respect to the ML^ν_*-interpretation $\mathcal{I}$ and the $\mathcal{I}$-valuation $\mathcal{V}$) is determined by the rules (d_1) to (d_8) below, in which $\mathcal{V}(x/\xi)$ is the $\mathcal{I}$-valuation just like $\mathcal{V}$ except $\mathcal{V}(x/\xi)(x) = \xi$, and $\tilde{\Delta}'$ denotes $des_{\mathcal{I}\mathcal{V}}(\Delta')$ for every subexpression Δ' of Δ. Note that $des_{\mathcal{I}\mathcal{V}}(\Delta)$ may fail to be a *QI* of $\mathcal{I}$; this is unsatisfactory, but it does not happen when $\mathcal{I}$ is general.

(d_1) $des_{\mathcal{I}\mathcal{V}}(v_{tn}) = \mathcal{V}(v_{tn})$, $des_{\mathcal{I}\mathcal{V}}(c_{tn}) = \mathcal{I}(c_{tn})$
(d_2) $des_{\mathcal{I}\mathcal{V}}(\Delta_1 = \Delta_2) = \{\gamma : \tilde{\Delta}_1 =_\gamma \tilde{\Delta}_2\}$
(d_3) $des_{\mathcal{I}\mathcal{V}}(\Delta'(\Delta_1, \ldots, \Delta_n)) = \tilde{\Delta}'(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n)$
($d_{4,5}$) $des_{\mathcal{I}\mathcal{V}}(\sim p) = \Gamma \backslash \tilde{p}$, $des_{\mathcal{I}\mathcal{V}}(p \wedge q) = \tilde{p} \cap \tilde{q}$
(d_6) $des_{\mathcal{I}\mathcal{V}}((v_{tn})p) = \bigcap_{\xi \in \mathcal{QI}_t} des_{\mathcal{I}\mathcal{V}'}(p)$, where $\mathcal{V}' = \mathcal{V}(v_{tn}/\xi)$
(d_7) $des_{\mathcal{I}\mathcal{V}}(\Box p) = \Gamma$ if $\tilde{p} = \Gamma$, $\varnothing$ otherwise
(d_8) $des_{\mathcal{I}\mathcal{V}}((\iota v_{tn})p) =$ the only *QI* ξ of type t such that:
(a) $\gamma \in des_{\mathcal{I}\mathcal{V}}((\exists_1 v_{tn})p)$ and $\gamma \in des_{\mathcal{I}\mathcal{V}'}(p)$ for $\mathcal{V}' = \mathcal{V}(v_{tn}/\eta) \Rightarrow \xi =_\gamma \eta$
(b) $\gamma \notin des_{\mathcal{I}\mathcal{V}}((\exists_1 v_{tn})p) \Rightarrow \xi =_\gamma a^\nu_t$.

The exact uniqueness of the *QI* ξ fulfilling (a) and (b) can be easily proved (cf. Theorem 11.1 in [4]) and, in particular, $des_{\mathcal{I}\mathcal{V}}(a^*_t) = a^\nu_t$ for every $t \in \tau^\nu_*$ (cf. Theorem 11.2 in [4]).

Definition 2.1 The *QI* ξ of type $\langle t_1, \ldots, t_n, t_0\rangle$ $(\in \tau_*^\nu)$ is said to be definable (in the ML_*^ν-interpretation $\mathcal{J}$) if there exist: (i) an $\mathcal{J}$-valuation $\mathcal{V}$; (ii) an n-tuple $X = \langle x_1, \ldots, x_n\rangle$ of variables of type $t_1, \ldots, t_n$, respectively; and (iii) a wfe Δ of type t_0 such that

(2.8) $\xi = d(\Delta, X, \mathcal{J}, \mathcal{V}) = \{\langle\langle\xi_1, \ldots, \xi_n\rangle, des_{\mathcal{J}\mathcal{V}'}(\Delta)\rangle :$
$\xi_i \in \mathcal{Q}\mathcal{J}_{t_i}\,(i = 1, \ldots, n)$ *and* $\mathcal{V}' = \mathcal{V}(x_1/\xi_1, \ldots, x_n/\xi_n)\}$.

Definition 2.2 An ML_*^ν-interpretation $\mathcal{J}$ is said to be general if every *QI* of type t $(\in \tau_*^\nu)$ definable in $\mathcal{J}$ belongs to $\mathcal{Q}\mathcal{J}_t$.

Let us remark that in a general ML_*^ν-interpretation $\mathcal{J}$, $des_{\mathcal{J}\mathcal{V}}(\Delta)$ is a *QI* of $\mathcal{J}$ for every wfe Δ and every $\mathcal{J}$-valuation $\mathcal{V}$.

The definitions of structure, interpretation, and general interpretation for ML^ν are identical to those given above for ML_*^ν, where, of course, the types are suitably assumed to run over τ^ν or $\bar{\tau}^\nu$ (cf. Definitions 3.1 and 3.2 in [15]). In the sequel the same symbols will be used to denote ML^ν-interpretations as well as ML_*^ν-interpretations; it will be clear from the context which kind of interpretation will be referred to.

If $\mathcal{J}$ is a general ML_*^ν- (or ML^ν-) interpretation, then $\varnothing \in \mathcal{Q}\mathcal{J}_0$ and every constant function (from $\Pi_i^n\, \mathcal{Q}\mathcal{J}_{t_i}$ into $\mathcal{Q}\mathcal{J}_{t_0}$) belongs to $\mathcal{Q}\mathcal{J}_{\langle t_1, \ldots, t_n, t_0\rangle}$; so that we can include the following (useful) assumption (2.9) in the definition of general ML_*^ν- (or ML^ν-) interpretation:

(2.9) I. $a_0^\nu = \varnothing$
II. *The image of* $a^\nu_{\langle t_1, \ldots, t_n, t_0\rangle}$ *is* $\{a^\nu_{t_0}\}$.

Furthermore, in [15] (cf. Theorem 4.2 and Hypothesis 5.1) it is shown that no loss of generality takes place if we assume that

(2.10) I. a_r^ν $(r \in \{1, \ldots, \nu\})$ *is a constant function*
II. $\mathcal{Q}\mathcal{J}_0 = \{\eta : \eta = des_{\mathcal{J}\mathcal{V}}(p)$, *for some wff p and some $\mathcal{J}$-valuation* $\mathcal{V}\}$.

(Let us remark that (2.10)I can be assumed even if $\mathcal{J}$ is not general.)

As usual we say that a formula p of ML_*^ν is *true* in the ML_*^ν-interpretation $\mathcal{J}$ if $des_{\mathcal{J}\mathcal{V}}(p) = \Gamma$ for every $\mathcal{J}$-valuation $\mathcal{V}$; p is said to be *ML_*^ν-valid* [*g-ML_*^ν-valid*] (briefly, $\models_* p\,[\overset{g}{\models}_* p]$) if it is true in every [every general] ML_*^ν-interpretation. *ML^ν-validity* and *g-ML^ν-validity* are defined in the same way.

The *completeness* of MC^ν with respect to general ML^ν-interpretations is proved in [14], whereas the *soundness* of the general ML_*^ν-interpretations, i.e.:

(2.11) $K \vdash_* p \Rightarrow K \overset{g}{\models}_* p \quad (p \in \mathcal{E}_0^*,\ K \subseteq \mathcal{E}_0^*)$,

is provable by an induction on the complexity of p.

3 $ML^{\nu+1}$-interpretations Let $\mathcal{J} = \langle \mathcal{S}, a^{\nu+1}, \mathcal{J}\rangle$ be an $ML^{\nu+1}$-interpretation in which we assume $D_{\nu+1}$ to be a two-element set and $\mathcal{Q}\mathcal{J}_{\nu+1}$ to contain at least two necessarily distinct *QI*s. This holds iff the formula

(3.1) $(\exists x, y)(\Box(x \neq y) \wedge (z)(z = x \vee z = y))$,

where x, y, z are distinct variables of type $\nu + 1$, is true in $\mathcal{J}$.[9]

If $\mathcal{I}$ is as above, then every *QI* ξ of type $\nu+1$ in $\mathcal{I}$ (which is a function from Γ into $D_{\nu+1}$) corresponds to a subset of Γ and hence the variables of type $\nu+1$ can adequately represent (in $\mathcal{I}$) propositional variables. Furthermore, this correspondence can be extended in a natural way to the *QI*s of higher type-level, so that an ML^{ν}_{*}-interpretation (which we shall denote by $\mathcal{I}^*$) can be represented in $\mathcal{I}$. The details of the construction of $\mathcal{I}^*$ are as follows.

The subset $\tau^{\nu+1}_{-}$ of $\tau^{\nu+1}$ is the smallest set such that $\{1,\ldots,\nu+1\} \subseteq \tau^{\nu+1}_{-}$ and $\langle t_1,\ldots,t_n,t_0\rangle \in \tau^{\nu+1}_{-}$ whenever $\{t_1,\ldots,t_n,t_0\} \subseteq \tau^{\nu+1}_{-}$. For every *QI* ξ of type t $(\epsilon\tau^{\nu+1}_{-})$ in $\mathcal{I}$, the correspondent ξ^{σ} of ξ is defined by

$$\begin{aligned}
(3.2)\quad & \xi^{\sigma} = \xi \textit{ if } t \in \{1,\ldots,\nu\} \\
& \xi^{\sigma} = \{\gamma : \xi(\gamma) \neq a^{\nu+1}_{\nu+1}(\gamma)\} \textit{ if } t = \nu+1 \\
& \xi^{\sigma} = \{\langle\langle \xi^{\sigma}_1,\ldots,\xi^{\sigma}_n\rangle, \xi^{\sigma}_0\rangle : \langle\langle \xi_1,\ldots,\xi_n\rangle, \xi_0\rangle \in \xi\} \textit{ if } t = \langle t_1,\ldots,t_n,t_0\rangle.
\end{aligned}$$

The correspondence $\xi \to \xi^{\sigma}$ is trivially one-to-one. If ξ has type t $(\epsilon\tau^{\nu+1}_{-})$ then the type of ξ^{σ} (that we shall denote by t^{σ}) is obtained from t by substituting 0 for every occurrence of $\nu+1$ in it. Therefore, the correspondence $t \to t^{\sigma}$ is a bijection between $\tau^{\nu+1}_{-}$ and τ^{ν}_{*}, and the sets

$$(3.3)\quad \mathcal{QI}^*_{t\sigma} = \{\xi^{\sigma} : \xi \in \mathcal{QI}_t\} \quad (t \in \tau^{\nu+1}_{-})$$

constitute an ML^{ν}_{*}-structure that we call $\mathcal{S}^*$ (cf. (2.5) and (2.6)).

The ML^{ν}_{*}-interpretation $\mathcal{I}^* = \langle \mathcal{S}^*, a^{\nu}, \mathcal{I}^*\rangle$ is defined by means of (3.4) and (3.5) below (in addition to (3.3)).

$$\begin{aligned}
(3.4)\quad & a^{\nu}_{t\sigma} = (a^{\nu+1}_{t})^{\sigma} (t \in \tau^{\nu+1}_{-}) \\
(3.5)\quad & \mathcal{I}^*(c_{t\sigma n}) = (\mathcal{I}(c_{tn}))^{\sigma} (t \in \tau^{\nu+1}_{-}, n \in Z^+).
\end{aligned}$$

It is obvious that a^{ν} fulfills the condition (2.9) on the nonexisting object whenever the same is for $a^{\nu+1}$.

From now on, in order to avoid tedious specifications, by "$ML^{\nu+1}$-interpretation" we shall mean an $ML^{\nu+1}$-interpretation in which $D_{\nu+1} = \{0,1\}$, $\mathcal{QI}_{\nu+1}$ has two elements which are necessarily distinct, and $a^{\nu+1}_{\nu+1}(\gamma) = 0$ for every possible case γ (cf. (2.10)I).

We shall denote by $\mathcal{E}_{-} (= \mathcal{E}^{\nu+1}_{-})$ the set of the wfes (of $ML^{\nu+1}$) in which only variables and constant of types in $\tau^{\nu+1}_{-}$ occur (note that the wfes in $\mathcal{E}_{-}$ have types in $\tau^{\nu+1}_{-} \cup \{0\}$). For every wfe $\Delta \in \mathcal{E}_{-}$, we let Δ^* be the wfe (of ML^{ν}_{*}) obtained by replacing every variable v_{tn} [constant c_{tn}] in Δ with $v_{t\sigma n}[c_{t\sigma n}]$. Thus, if Δ is a formula then so is Δ^*, whereas Δ^* has type t^{σ} when Δ has type $t \neq 0$.

Before proving the following theorem we need to note that, for every couple ξ_1, ξ_2 of *QI*s (of type t) in $\mathcal{I}$,

$$(3.6)\quad \xi_1 =_{\gamma} \xi_2 \textit{ iff } \xi^{\sigma}_1 =_{\gamma} \xi^{\sigma}_2 \ (\gamma \in \Gamma);$$

this can be proved, on the basis of (3.2) and (2.7), by an induction on the complexity of t.

Theorem 3.1 *Assume that* (1) $\mathcal{I}$ *is an* $ML^{\nu+1}$*-interpretation*, (2) $\mathcal{V}$ *is an* $\mathcal{I}$*-valuation, and* (3) $\mathcal{V}^*$ *is the* $\mathcal{I}^*$*-valuation defined by*

$$\mathcal{V}^*(v_{t\sigma n}) = (\mathcal{V}(v_{tn}))^{\sigma} \ (t \in \tau^{\nu+1}_{-}, n \in Z^+) \ .$$

Then, for every wfe $\Delta \in \mathcal{E}_-$, $des_{\mathcal{G}^*\mathcal{V}^*}(\Delta^*) = (des_{\mathcal{G}V}(\Delta))^\sigma$ *(where the equality* $\eta = \eta^\sigma$ *is understood for every* $\eta \subseteq \Gamma$*).*

Proof: We use an induction on the number $\imath_\Delta$ of occurrences of $\imath$ in Δ, and, in correspondence with a given value of $\imath_\Delta$, we use an induction on the complexity of Δ. The second part does not depend on $\imath_\Delta$, therefore it is considered only for $\imath_\Delta = 0$, as follows.

If Δ is a constant or a variable, then the thesis holds trivially.

Let Δ be $\Delta_1 = \Delta_2$. Δ^* is $\Delta_1^* = \Delta_2^*$, thus, by the inductive hypothesis, the thesis is a consequence of (3.6).

The proofs in the cases where Δ is $F(\Delta_1, \ldots, \Delta_n)$, or $\sim p$, or $p \wedge q$, or $\Box p$ are straightforward applications of the inductive hypothesis.

Let Δ be $(x)p$ (where x has type t). Then, for every $\gamma \in \Gamma$, $\gamma \in des_{\mathcal{G}\mathcal{V}}(\Delta) \Leftrightarrow \gamma \in des_{\mathcal{G}\mathcal{V}(x/\xi)}(p)$ for all $\xi \in \mathcal{Q}\mathcal{G}_t \Leftrightarrow$ (by the inductive hypothesis) $\gamma \in des_{\mathcal{G}^*\mathcal{V}^*(x^*/\xi^\sigma)}(p^*)$ for all $\xi \in \mathcal{Q}\mathcal{G}_t \Leftrightarrow$ (by (3.3)) $\gamma \in des_{\mathcal{G}^*\mathcal{V}^*}((x^*)p^*)$.

Now, let Δ be $(\imath x)p$ (where x has type t) and let us assume inductively that the thesis holds for every wfe Δ' such that $\imath_{\Delta'} < \imath_\Delta$. We denote $des_{\mathcal{G}\mathcal{V}}(\Delta)$ and $des_{\mathcal{G}^*\mathcal{V}^*}(\Delta^*)$ by ξ and ξ^*, respectively.

If $\gamma \notin des_{\mathcal{G}\mathcal{V}}((\exists_1 x)p) (= des_{\mathcal{G}^*\mathcal{V}^*}((\exists_1 x^*)p^*)$, then $\xi =_\gamma a_t^{\nu+1}$ and $\xi^* =_\gamma a_{t\sigma}^\nu$, that is (by (3.4) and (3.6)) $\xi^\sigma =_\gamma \xi^*$.

If $\gamma \in des_{\mathcal{G}\mathcal{V}}((\exists_1 x)p)$ and $\gamma \in des_{\mathcal{G}\mathcal{V}(x/\eta)}(p)$, then (by the inductive hypothesis) $\gamma \in des_{\mathcal{G}^*\mathcal{V}^*(x^*/\eta^\sigma)}(p^*)$. Hence, $\xi =_\gamma \eta$ and $\xi^* =_\gamma \eta^\sigma$, that is $\xi^\sigma =_\gamma \xi^*$.

Therefore, $\xi^* = \xi^\sigma$ since they are γ-equivalent for every $\gamma \in \Gamma$.

4 Reduction of general ML_*^ν-validity to general $ML^{\nu+1}$-validity In ML_*^ν there are infinitely many wfes that are not $*$-translations of wfes in $\mathcal{E}_-$ (e.g., $v_{01} \wedge v_{01}$ is one of them). In this section we complete the characterization of ML_*^ν-validity in terms of $ML^{\nu+1}$-validity by proving that every wfe Δ' in ML_*^ν is equivalent to a wfe of the form Δ^*, with Δ in $\mathcal{E}_-$.

Definition 4.1 (1) An occurrence $\bar{p}$ of the wff p in the wfe Δ' (of ML_*^ν) is said to be a formula-occurrence (briefly, an f-occurrence) (of p) if $\sim\bar{p}$, or $\bar{p} \wedge q$, or $(x)\bar{p}$, or $\Box\bar{p}$, or $(\imath x)\bar{p}$ is a subexpression of Δ'. Otherwise we say that $\bar{p}$ is a term-occurrence (briefly, a t-occurrence) of p.

(2) A wff p (of ML_*^ν) is said to be a t-wff if it is v_{0n}, or c_{0n}, or $F(\Delta_1, \ldots, \Delta_n)$, or $(\imath v_{0n})p$. Otherwise we say that p is an f-wff.

Lemma 4.1 *For every wfe* Δ' *of* ML_*^ν, *a wfe* $\Delta \in \mathcal{E}_-$ *exists such that* $\Delta' = \Delta^*$ *iff no f-wff has a t-occurrence in* Δ' *and no t-wff has an f-occurrence in* Δ'.

Proof: If p $(\in \mathcal{E}_0^*)$ is q^* and p is a t-wff, then q is a term of $ML^{\nu+1}$, whereas q is a formula if p is an f-wff. Therefore, Δ' (in ML_*^ν) is not Δ^* (for any $\Delta \in \mathcal{E}_-$) whenever some t-wff has an f-occurrence in Δ' or some f-wff has a t-occurrence in Δ'.

Conversely, assume that, for every formula p, if $\bar{p}$ is an f-occurrence [a t-occurrence] of p in Δ', then p is an f-wff [a t-wff]. In order to prove that a wfe $\Delta \in \mathcal{E}_-$ exists such that Δ' is Δ^*, we use an induction on the complexity of Δ'.

If Δ' is a variable or a constant, then the thesis holds trivially.

Let Δ' be $\sim p$. By the inductive hypothesis p is q^* for a suitable wfe $q \in \mathcal{E}_-$. The formula p has (only) an f-occurrence in Δ' and hence it is an f-wff; thus, q is a formula and Δ' is $(\sim q)^*$.

In cases Δ' is $p \wedge p_1$, or $(x)p$, or $\Box p$, or $(\iota x)p$ we proceed exactly as above.

Let Δ' be $F'(\Delta'_1, \ldots, \Delta'_n)$ where F' is a term of type $\langle t_1, \ldots, t_n, t_0 \rangle$. Then, by the inductive hypothesis, $F, \Delta_1, \ldots, \Delta_n \in \mathcal{E}_-$ exist such that F^* is F' and, for $i = 1$ to n, Δ_i^* is Δ'_i. If Δ'_i is a term then Δ_i is also a term. If $\Delta'_i \in \mathcal{E}_0^*$, then it is a t-wff since it has a t-occurrence in Δ' and hence Δ_i is a term. Therefore, $F(\Delta_1, \ldots, \Delta_n)$ is well formed and Δ' is $(F(\Delta_1, \ldots, \Delta_n))^*$.

The case in which Δ' is $\Delta'_1 = \Delta'_2$ is similar to the previous one.

Theorem 4.1 *Assume that* (1) $\mathcal{J}$ *is an* ML^ν_**-interpretation in which* $a_0^\nu = \varnothing$, *and* (2) Δ' *is a wfe of* ML^ν_*, *then a wfe* $\Delta \in \mathcal{E}_-$ *exists such that, for every* $\mathcal{J}$*-valuation* $\mathcal{W}$, $des_{\mathcal{J}\mathcal{W}}(\Delta') = des_{\mathcal{J}\mathcal{W}}(\Delta^*)$.

Proof: We first remark that, for every wff p of ML^ν_*, the equivalence $p \equiv (p \neq a_0^*)$ is true in $\mathcal{J}$, as well as the equality $p = (\iota\psi)((\psi \neq a_0^*) \equiv p)$ (with ψ not free in p).

If $\bar{p}$ is a t-occurrence of the f-wff p in Δ', then the wfe obtained by replacing (in Δ') $\bar{p}$ with $(\iota\psi)((\psi \neq a_0^*) \equiv p)$ has the same designatum as Δ' (with respect to every $\mathcal{J}$-valuation) and has fewer t-occurrences of f-wffs than Δ'. Likewise, the wfe obtained by replacing an f-occurrence $\bar{p}$ of the t-wff p in Δ' with $p \neq a_0^*$ is equivalent to Δ' and has fewer f-occurrences of t-wffs than it has.

By applying the above substitutions finitely many times, we obtain a wfe of the form Δ^* for which the thesis holds.

Now, in order to consider ML^ν_*- and $ML^{\nu+1}$-validity in the general sense (i.e., with respect to general interpretations), we first prove that the structure for a general $ML^{\nu+1}$-interpretation is determined by the set $\{\mathcal{QJ}_t : t \in \tau_-^{\nu+1}\}$. For every $t \in \bar{\tau}^{\nu+1}$ we let t' be the element of $\tau_-^{\nu+1}$ obtained by substituting $\nu + 1$ for every occurrence of 0 in t, and let ρ_t be the function, of domain $\mathcal{QJ}_{t'}$ (in a given $ML^{\nu+1}$-interpretation), defined by

$$(4.1) \quad \begin{aligned} &\rho_t(\xi) = \{\gamma : \xi(\gamma) = 1\}, \textit{for } t = 0 \\ &\rho_t(\xi) = \xi, \textit{for } t \in \{1, \ldots, \nu + 1\} \\ &\rho_t(\xi) = \{\langle\langle\rho_{t_1}(\xi_1), \ldots, \rho_{t_n}(\xi_n)\rangle, \rho_{t_0}(\xi_o)\rangle : \\ &\qquad \langle\langle\xi_1, \ldots, \xi_n\rangle, \xi_0\rangle \in \xi\}, \textit{for } t = \langle t_1, \ldots, t_n, t_0\rangle. \end{aligned}$$

Note that, when $t \in \tau_-^{\nu+1}$, $t' = t$ and ρ_t is the identity on $\mathcal{QJ}_t$.

Lemma 4.2 *If* $\mathcal{J}$ *is a general* $ML^{\nu+1}$*-interpretation, then, for every* $t \in \bar{\tau}^{\nu+1}$, ρ_t *is a bijection between* $\mathcal{QJ}_{t'}$, *and* $\mathcal{QJ}_t$.

Proof: By induction on the complexity of t. By (4.1) ρ_t is injective for every $t \in \bar{\tau}^{\nu+1}$, therefore we have to prove that, for every $\xi \in \mathcal{QJ}_{t'}$ [$\eta \in \mathcal{QJ}_t$], $\rho_t(\xi)$ [$\rho_t^{-1}(\eta)$] is a definable QI in $\mathcal{QJ}_t$ [$\mathcal{QJ}_{t'}$]. Because of $(4.1)_3$, the wfe defining $\rho_t(\xi)$ (for $t = \langle t_1, \ldots, t_n, t_0\rangle$) must include wfes expressing equalities of the form $\rho_{t_i}(\xi_i) = \eta_i$; this will be supplied by defining, at every step of the induction, a wff $\Phi_t(\Delta, \Delta')$ (where Δ and Δ' have the type t and t', respectively) such that, for every $\mathcal{I}$-valuation $\mathcal{V}$, $des_{\mathcal{I}\mathcal{V}}(\Phi_t(\Delta,\Delta')) = \Gamma[\varnothing]$ if $des_{\mathcal{I}\mathcal{V}}(\Delta) = [\neq]\ \rho_t(des_{\mathcal{I}\mathcal{V}}(\Delta'))$.

For $t \in \{1, \ldots, \nu+1\}$, ρ_t is trivially bijective and $\Phi_t(\Delta,\Delta')$ is $\Box(\Delta = \Delta')$.

Let $t = 0$. If $\xi \in \mathcal{Q}\mathcal{I}_{\nu+1}$ and $\mathcal{V}(x) = \xi$, then $des_{\mathcal{I}\mathcal{V}}(x \neq a^*_{\nu+1})$ is $\rho_0(\xi)$ and belongs to $\mathcal{Q}\mathcal{I}_0$. Conversely, if $\eta \in \mathcal{Q}\mathcal{I}_0$, $des_{\mathcal{I}\mathcal{V}}(p) = \eta$ (cf. (2.9)II), and the variable x (of type $\nu+1$) is not free in p, then $des_{\mathcal{I}\mathcal{V}}((\exists_1 x)(p \equiv x \neq a^*_{\nu+1})) = \Gamma$ and $\xi = des_{\mathcal{I}\mathcal{V}}((\iota x)(p \equiv x \neq a^*_{\nu+1}))$ (which is in $\mathcal{Q}\mathcal{I}_{\nu+1}$) fulfills: $\xi(\gamma) = 1$ iff $\gamma \in \eta$; that is, $\rho_0(\xi) = \eta$. $\Phi_0(\Delta,\Delta')$ is $\Box(\Delta \equiv \Delta' \neq a^*_{\nu+1})$.

Let now t be $\langle t_1, \ldots, t_n, t_0\rangle$. We can assume inductively that the thesis holds and that $\Phi_t(\Delta,\Delta')$ is defined for $t \in \{t_0, t_1, \ldots, t_n\}$.

Case 1. $t_0 = 0$. Assume that: (1) F is any variable of type t'; (2) p is the formula

$$(\exists y_1, \ldots, y_n)\left[\bigwedge\nolimits_i^n \Phi_{t_i}(x_i,y_i) \wedge F(y_1, \ldots, y_n) \neq a^{\nu+1}_{\nu+1}\right] ;$$

and (3) $\eta = d(p, \{x_1, \ldots, x_n\}, \mathcal{I}, \mathcal{V})$, where $\mathcal{V}(F) = \xi$ $(\in \mathcal{Q}\mathcal{I}_{t'})$. For every n-tuple $\langle \eta_1, \ldots, \eta_n\rangle \in \Pi_i^n \mathcal{Q}\mathcal{I}_{t_i}$, $\gamma \in \eta(\eta_1, \ldots, \eta_n)$ iff an n-tuple $\langle \xi_1, \ldots, \xi_n\rangle (\in \Pi_i^n \mathcal{Q}\mathcal{I}_{t_i'})$ exists such that $\rho_{t_i}(\xi_i) = \eta_i$ $(i = 1, \ldots, n)$ and $\xi(\xi_1, \ldots, \xi_n)(\gamma) = 1$. By the inductive hypothesis, every η_i can be written as $\rho_{t_i}(\xi_i)$ and hence $\eta(\rho_{t_1}(\xi_1), \ldots, \rho_{t_n}(\xi_n)) = \{\gamma : \xi(\xi_1, \ldots, \xi_n)(\gamma) = 1\}$, which is $\rho_0(\xi_1, \ldots, \xi_n)$. That is, $\eta = \rho_t(\xi)$.

Conversely, assume that: (1) R is a variable of type t; (2) Δ is $(\iota x)(\exists x_1, \ldots, x_n)\left[\bigwedge_i^n \Phi_{t_i}(x_i,y_i) \ .\wedge.\ R(x_1, \ldots, x_n) \equiv x \neq a^*_{\nu+1}\right]$; and (3) $\xi = d(\Delta, \{y_1, \ldots, y_n\}, \mathcal{I}, \mathcal{V})$, where $\mathcal{V}(R) = \eta (\in \mathcal{Q}\mathcal{I}_t)$. Let ξ_0 be $\xi(\xi_1, \ldots, \xi_n)$; then $\xi_0(\gamma) = 1$ iff $\gamma \in \eta(\rho_{t_1}(\xi_1), \ldots, \rho_{t_n}(\xi_n))$, that is $\rho_0(\xi_0) = \eta(\rho_{t_1}(\xi_1), \ldots, \rho_{t_n}(\xi_n))$. Since this holds for every n-tuple, $\langle \xi_1, \ldots, \xi_n\rangle \in \Pi_i^n \mathcal{Q}\mathcal{I}_{t_i'}, \rho_t(\xi) = \eta$.

It is now easy to verify that $\Phi_t(\Delta,\Delta')$ is

$$(\forall x_1, \ldots, x_n, y_1, \ldots, y_n)$$
$$\left[\bigwedge\nolimits_i^n \Phi_{t_i}(x_i,y_i) \supset \Box(\Delta(x_1, \ldots, x_n) \equiv \Delta'(y_1, \ldots, y_n) \neq a^{\nu+1}_{\nu+1})\right].$$

Case 2. $t_0 \neq 0$. The proof (similar to that of Case 1) is left to the reader: it is a straightforward application of the definition of ρ_t.

This result proves that the correspondence $\mathcal{I} \to \mathcal{I}^*$ is substantially injective, when restricted to general $ML^{\nu+1}$-interpretations. In fact, if $\mathcal{I}$ and $\mathcal{I}_1$ are general $ML^{\nu+1}$-interpretations and $\mathcal{I}^* = \mathcal{I}_1^*$, then: (1) $\mathcal{I}$ and $\mathcal{I}_1$ have the same $ML^{\nu+1}$-structure (by Lemma 4.2); (2) the nonexisting objects are the same in $\mathcal{I}$ and $\mathcal{I}_1$ (by (2.9)); and (3) $\mathcal{I}(c_{tn}) = \mathcal{I}_1(c_{tn})$ whenever $t \in \tau^{\nu+1}_{-}$ (by (3.5)); therefore, $\mathcal{I}$ and $\mathcal{I}_1$ differ (at most) in the valuations of some constant of a type in $\tau^{\nu+1} \backslash \tau^{\nu+1}_{-}$.

In order to prove that the $*$-correspondence is a bijection (in the sense above) between the set of all general $ML^{\nu+1}$-interpretations and that of all general ML^{ν}_{*}-interpretations (cf. Theorem 4.2 below), we need to express the designatum of any wfe in a (general) $ML^{\nu+1}$-interpretation $\mathcal{I}$ by means of a suitable designatum in $\mathcal{I}^*$. Note that the converse of this is given by Theorems 3.1 and 4.1.

For every wfe Δ of $ML^{\nu+1}$, we denote by Δ° the wfe (of ML^ν_*) obtained by substituting $v_{t'\sigma n}$ $[c_{t'\sigma n}]$ for every variable v_{tn} [constant c_{tn}] in Δ. The correspondence $\Delta \to \Delta^\circ$ is not injective (e.g., $v^\circ_{\langle\nu+1,0\rangle 1} = v^\circ_{\langle\nu+1,\nu+1\rangle 1}$) which can cause some trouble in the proofs; thus, we assume that in every wfe considered in the sequel, variables with different types have different indices.

Lemma 4.3 *Assume that*: (1) $\Delta \in \mathcal{E}_t$ $(t \in \bar{\tau}^{\nu+1})$ *and no constant occurs in* Δ; (2) $\mathcal{I}$ *is an* $ML^{\nu+1}$*-interpretation in which, for every* $t \in \bar{\tau}^{\nu+1}$, ρ_t *is a bijection between* $\mathcal{Q}\mathcal{I}_{t'}$ *and* $\mathcal{Q}\mathcal{I}_t$; (3) $\mathcal{V}$ *is an* $\mathcal{I}$*-valuation*; *and* (4) $\mathcal{V}^\circ$ *is any* $\mathcal{I}^*$*-valuation such that, for every variable* $x \in \mathcal{E}_s$ *occurring in* Δ, $\mathcal{V}^\circ(x^\circ) = (\rho_s^{-1}\mathcal{V}(x))^\sigma$. *Then*

$$des_{\mathcal{I}^*\mathcal{V}^\circ}(\Delta^\circ) = \xi^\sigma \textit{ iff } \rho_t(des_{\mathcal{I}\mathcal{V}}(\Delta)) = \xi \ ,$$

(*where the equalities* $\sigma\eta = \eta = \rho_0\eta$ *are understood for every* $\eta \subseteq \Gamma$).

Proof: Let us first remark that $\mathcal{V}^\circ(x^\circ)$ is well defined since, by the assumption above, $x^\circ = y^\circ$ holds for no variable y in Δ different from x. Now the proof proceeds exactly like that of Theorem 3.1. We have only to note that, for every $u \in \bar{\tau}^{\nu+1}$, $\xi_1, \xi_2 \in \mathcal{Q}\mathcal{I}_{u'}$, and $\gamma \in \Gamma$, $\xi_1 =_\gamma \xi_2$ iff $\rho_u(\xi_1) =_\gamma \rho_u(\xi_2)$ (which follows from (3.6) and (4.1)) and that, for every $u \in \tau^{\nu+1}$, $\sigma\rho_u^{-1}$ (the composition map) is a membership preserving bijection from $\mathcal{Q}\mathcal{I}_u$ onto $\mathcal{Q}\mathcal{I}^*_{u'\sigma}$.

Theorem 4.2 (a) *For every general* $ML^{\nu+1}$*-interpretation* $\mathcal{I}$, $\mathcal{I}^*$ *is general.* (b) *For every general* ML^ν_**-interpretation* $\mathfrak{I} = \langle\{\widetilde{\mathcal{Q}\mathcal{I}}_t : t \in \tau^\nu_*\}, a^\nu, \mathfrak{I}\rangle$, *there exists a general* $ML^{\nu+1}$*-interpretation* $\mathcal{I}$ *such that* $\mathcal{I}^* = \mathfrak{I}$.

Proof: (a). By Theorem 4.1 we can consider only *QI*s defined in $\mathcal{I}^*$ by wfe of the form Δ^*. Every variable of ML^ν_* is x^* for a suitable $x \in \mathcal{E}_-$ and every $\mathcal{I}^*$-valuation is $\mathcal{V}^*$ for a suitable $\mathcal{I}$-valuation $\mathcal{V}$; thus, the thesis follows from (3.3) and (3.4) and the equality

$$d(\Delta^*, \{x_1^*, \ldots, x_n^*\}, \mathcal{I}^*, \mathcal{V}^*) = (d(\Delta, \{x_1, \ldots, x_n\}, \mathcal{I}, \mathcal{V}))^\sigma$$

which is a straightforward consequence of Theorem 3.1.

(b). For every $t \in \tau_-^{\nu+1}$, we can let $\mathcal{Q}\mathcal{I}_t$ be the only set of *QI*s for $ML^{\nu+1}$ such that $\mathcal{Q}\mathcal{I}^*_{t\sigma} = \widetilde{\mathcal{Q}\mathcal{I}}_{t\sigma}$ (cf. (3.3)). For $t \in \bar{\tau}^{\nu+1} \backslash \tau_-^{\nu+1}$ we set $\mathcal{Q}\mathcal{I}_t = \{\rho_t(\xi) : \xi \in \mathcal{Q}\mathcal{I}_{t'}\}$. Furthermore, we let $a^{\nu+1}$ be determined by (2.9) (hence (3.4) holds) and, as far as the valuation of the constants is concerned, we let $\mathcal{I}$ be any valuation for which $\mathcal{I}^* = \mathfrak{I}$ (cf. (3.6)).

Let ξ be the *QI* (of type $t = \langle t_1, \ldots, t_n, t_0\rangle$) $d(\Delta, \{x_1, \ldots, x_n\}, \mathcal{I}, \mathcal{V})$. We can assume (without loss of generality) that no constant occurs in Δ, since, otherwise, we could replace the constants with new variables and change $\mathcal{V}$ suitably; hence Lemma 4.3 can be used. We consider the *QI* (for ML^ν_*) $\xi^\circ = \mathrm{d}(\Delta^\circ, \{x_1^\circ, \ldots, x_n^\circ\}, \mathfrak{I}, \mathcal{V}^\circ)$ where, for every variable x of type s occurring in Δ, $\mathcal{V}^\circ(x^\circ) = \sigma\rho_s^{-1}(\mathcal{V}(x))$. ξ° is in $\widetilde{\mathcal{Q}\mathcal{I}}_{t'\sigma}$ and hence there is an $\eta \in \mathcal{Q}\mathcal{I}_{t'}$ such that $\xi^\circ = \eta^\sigma$. Now, by Lemma 4.3, for every $\langle\langle\eta_1, \ldots, \eta_n\rangle, \eta_0\rangle \in (\Pi_i^n \mathcal{Q}\mathcal{I}_{t_i'}) \times \mathcal{Q}\mathcal{I}_{t_0'}$, $\langle\langle\eta_1^\sigma, \ldots, \eta_n^\sigma\rangle, \eta_0^\sigma\rangle \in \xi^\circ$ iff $\langle\langle\rho_{t_1}(\eta_1), \ldots, \rho_{t_n}(\eta_n)\rangle, \rho_{t_0}(\eta_0)\rangle \in \xi$. This is equivalent to $\rho_t(\eta) = \xi$ and hence $\xi \in \mathcal{Q}\mathcal{I}_t$.

By Theorems 3.1 and 4.2, the "reduction" theorem holds:

Theorem 4.3 *For every wff p of ML^{ν}_{*}, p is g-ML^{ν}_{*}-valid iff it is (equivalent to) the $*$-translation of a g-$ML^{\nu+1}$-valid wff of $ML^{\nu+1}$.*

5 ***Completeness of MC^{ν}_{*}: Concluding remarks*** The completeness of MC^{ν}_{*}, with respect to general ML^{ν}_{*}-interpretations, is a consequence of the syntactical counterparts of Theorems 3.1 and 4.1.

Let Δ' be any wfe of ML^{ν}_{*} and let Δ_1 be the wfe (of the form Δ^*) obtained from Δ' by means of the substitutions considered in Theorem 4.1. Then, by Theorem 32.4 in [4],

(5.1) $\vdash_{*} \Box(\Delta' = \Delta_1)$

is a consequence of the following lemma.

Lemma 5.1 *For every wff p of ML^{ν}_{*} and every variable ψ of type* 0, *not free in p:* (a) $\vdash_{*} (p \equiv (p \neq a_0^*))$, *and* (b) $\vdash_{*} (p = (\imath\psi)((\psi \neq a_0^*) \equiv p))$.

Proof: By A*1, $p \equiv \sim(p \equiv a_0^*)$ is an instance of a tautology and hence (a) holds by A*2.

By (a), $\vdash_{*} ((\psi \equiv (p \neq a_0^*)) \supset (\psi = p))$ and $\vdash_{*} ((\exists\psi)(\psi \equiv (p \neq a_0^*))$; that is $\vdash_{*} ((\exists_1\psi)(\psi \equiv (p \neq a_0^*)))$, which yields (b) by (a) and (2.2)I.

Henceforth, by $MC^{\nu+1}$ we mean the calculus endowed with (3.1) in addition to the usual axioms; that is, since MC^{ν} is complete with respect to general ML^{ν}-interpretations, $MC^{\nu+1}$ axiomatizes the concept of g-$ML^{\nu+1}$-validity considered in this paper.

Lemma 5.2 *For every wff q of $ML^{\nu+1}$, $\vdash_{\nu+1} q$ implies $\vdash_{*} q^{\circ}$.*

Proof: The derivation rules (for $MC^{\nu+1}$) are preserved by the correspondence $\Delta \rightarrow \Delta^{\circ}$ (that is, $((x)p)^{\circ}$ is $(x^{\circ})p^{\circ}$, $(p_1 \supset p_2)^{\circ}$ is $p_1^{\circ} \supset p_2^{\circ}$, and $(\Box p)^{\circ}$ is $\Box p^{\circ}$); therefore, we have to prove that $\vdash_{*} q^{\circ}$ whenever q is an axiom of $MC^{\nu+1}$.

If q is one of the axioms MA3.1–MA3.18 in [15] then q° is an axiom of MC^{ν}_{*} (actually, q and q° are instances of the same axiom schema).

Now let q be (3.1). Then q° is

$$(\exists\psi, \phi)(\Box(\phi \neq \psi) \wedge (\forall\theta)(\psi = \theta \vee \phi = \theta)) \ ,$$

where ψ, ϕ, and θ are distinct variables of type 0. Let ψ' be $p \wedge \sim p$ and ϕ' be $p \vee \sim p$ (where p is any closed wff). $\psi' = \theta \vee \phi' = \theta$ and $\psi' \neq \phi'$ are tautologies, and $(\forall\theta)(\psi' = \theta \vee \phi' = \theta)$ and $\Box(\psi' \neq \phi')$ can be derived by necessitation and generalization. Then $\vdash_{*} q^{\circ}$ follows from $\vdash_{*} \Box(\psi' \neq \phi') \wedge (\forall\theta)(\psi' = \theta \vee \phi' = \theta)$ by the rule $\vdash_{*} p \Rightarrow \vdash_{*} (\exists x)p$.

Theorem 5.1 *For every wff p of ML^{ν}_{*}, $\vdash_{*} p$ iff $\models^{g}_{*} p$.*

Proof: By (2.10) we have only to prove the implication from right to left. By Theorem 4.1 and Lemma 5.1 we can assume p to be q^* ($q \in \mathcal{E}_{-}$). Then (by Theorem 4.3) $\models^{g}_{\nu+1} q$, which is equivalent to $\vdash_{\nu+1} q$. Hence the thesis follows from the equality $q^* = q^{\circ}$ and Lemma 5.2.

In the Introduction we observed that the propositional variables are expressible in ML^ν by means of the representatives of possible cases. It is worth recalling in this connection that the use of the formulas $EL(u)$ and $|_u$ (from a syntactical point of view) requires MC^ν to be endowed with the additional axiom AS12.19 (cf. Note (7)). This axiom has no role in the embedding of ML^ν_* into $ML^{\nu+1}$ considered in previous sections and, actually, it provides something more: not only can the subsets of Γ be represented by it, but also the possible cases one by one.

If we want to achieve in ML^ν itself a construction like that considered in this work without referring to AS12.19, then we can use one of the following two methods. The first one requires the assumption that, in every ML^ν-interpretation, $\mathcal{QI}_1$ contains two *QI*s ξ_1 and ξ_2 which are necessarily distinct. In this way, every individual concept ξ such that $\xi =_\gamma \xi_1$ or $\xi =_\gamma \xi_2$, for all $\gamma \in \Gamma$, represents a subset of Γ (that is, $\{\gamma : \xi =_\gamma \xi_1\}$). The second method (which requires no particular assumption) consists in representing subsets of Γ by *QI*s of type (1): $\eta(\subseteq\Gamma)$ corresponds to $\eta' = \{\langle \xi, \gamma \rangle : \xi \in \mathcal{QI}_1, \gamma \in \eta\}$. Let us remark that, no matter what method is adopted to embed ML^ν_* into ML^ν, the technical details of the whole construction are very complex. For instance, the representatives of all *QI*s for ML^ν_* are to be defined in addition to those of the subsets of Γ, and, for every $t \in \tau^\nu_*$, there must be built up a formula $\mathcal{R}_t(\Delta)$ meaning "Δ represents an expression of type t of ML^ν_*". Of course, the quantifications and the descriptions (in ML^ν_*) turn out to be expressed by quantifications and descriptions (in ML^ν) restricted by $\mathcal{R}_t$, for suitable t.

NOTES

1. A class, PMC, of possible mechanical cases is a primitive notion in [3]. It affects various other notions. For instance, instead of the ordinary notion of position, one uses the position $\mathcal{P}_\xi(M, \theta, \gamma)$ of the mass point M (in the kinematic space ξ) at the instant θ, in the case $\gamma \in$ PMC. Thus in various cases an ordinary assertion p is replaced by an assertion p_γ containing γ explicitly; and $(\forall\gamma \in \text{PMC})p_\gamma$ $[(\exists\gamma \in \text{PMC})p_\gamma]$ stands for $\Box p[\Diamond p]$.

2. For every sentence p, $\tilde{p}$ is the set of possible cases in which p holds and the extension of p in γ $(\in\Gamma)$ is T or F according to whether p holds in γ or not.

3. For every type t, a *QI* a^ν_t is fixed to represent the "nonexisting" object of that type and, for every $\gamma \in \Gamma$ the extension of $(\imath x)p$ in γ is defined as follows. If a *QI* ξ exists such that: (1) p is true in γ when the interpretation of x is ξ, and (2) every *QI* ξ' with the property (1) has the same extension of ξ in γ, then the extension of $(\imath x)p$ in γ is that of ξ; otherwise, the extension of $(\imath x)p$ in γ is that of the nonexisting object of the same type of x.

4. Let ξ be the number of known planets. Then $\xi = 9$ holds in the actual case γ_R, whereas it is natural to assume that $\Box(\xi = 9)$ does not. Now, the property Nn of being a natural number is absolute and $Nn(9)$ holds in every Γ-case. Thus, $Nn^{(e)}(\xi)$ and $Nn(\xi)$ are respectively true and false in γ_R.

5. Note that the existence of this attribute must be explicitly asserted in MC^ν (cf. AS25.1 in [4]).

6. In order to prove in MC^ν the main results concerning the representatives of the possible cases, we must endow MC^ν with the strong axiom AS12.19 in [4], which asserts that, for every relational *QI* ξ and every $\gamma \in \Gamma$, there is a modally constant *QI* ξ' having (in every possible case) the extension of ξ in γ. This holds iff the *QI*s of the formulas of ML^ν constitute an atomic Boolean subalgebra of $\mathcal{P}(\Gamma)$ (cf. [9], Section 4, and [15], Theorem 5.1) and, indeed, the independence of AS12.19 can be proved by considering a different semantics for ML^ν, in which the formulas take values on a complete, but not atomic, Boolean algebra (cf. [7], Section 15). On the basis of this remark, in MC_*^ν AS12.19 is equivalent to $(\exists\phi)(\phi \wedge (\psi)(\psi \supset \Box(\phi \supset \psi)))$, where ϕ and ψ are propositional variables (cf. [6], p. 338).
7. This is a consequence of the well-known results on the (extensional) theory of types, but the same holds for the first-order part of ML^ν (see [11]).
8. A similar result is proved in [11] for the first-order part of MC^ν, deprived of descriptions.
9. In the strict sense, this formula is true in an $ML^{\nu+1}$-interpretation $\mathcal{I}$ whenever $\mathcal{Q}\mathcal{I}_1$ contains two elements, ξ_1 and ξ_2, necessarily distinct and, for every possible case γ and every $\xi \in \mathcal{Q}\mathcal{I}_1$, $\xi(\gamma) = \xi_1(\gamma)$ or $\xi(\gamma) = \xi_2(\gamma)$. This does not imply that $D_{\nu+1}$ has exactly two elements, but it can be proved that an $ML^{\nu+1}$-interpretation exists, which is isomorphic to $\mathcal{I}$ and where $D_{\nu+1}$ has the required property (cf. Theorem 4.2 in [15]).

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