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Finitary Consistency of a Free Arithmetic

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The axioms of the theory *FA* are all the instances of the following schemata:

(A0) A, *if A is a tautology*
(A1) $\forall x(A \supset B) \supset (\forall xA \supset \forall xB)$
(A2) $A \supset \forall xA$, *if x is not free in A*
(A3) $\forall y(\forall xA \supset A(y/x))$
(A4) $t = t$
(A5) $t = t' \supset (A \supset A(t'/t))$
(A6) $\forall x{\sim}(s(x) = 0)$
(A7) ${\sim}\, t = t' \supset {\sim}s(t) = s(t')$
(A8) $t + 0 = t$
(A9) $t + s(t') = s(t + t')$
(A10) $t \cdot 0 = 0$
(A11) $t \cdot s(t') = (t \cdot t') + t$
(A12) $\exists x(x = t)$, *if t is a numeral.*

The rules of *FA* are:

(R1) $$\frac{\vdash A \qquad \vdash A \supset B}{\vdash B}$$

(R2) $$\frac{\vdash A}{\vdash \forall xA}$$

(R3) $$\frac{\vdash A(0/x) \qquad \vdash A \supset A(s(x)/x)}{\vdash A(t/x)}$$

FA is weaker than standard arithmetic in two senses. First, in *FA* one cannot prove the schemata:

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(1) $\exists x(x = t) \supset \exists x(x = s(t))$
(2) $(\exists x(x = t) \wedge \exists x(x = t')) \supset \exists x(x = t + t')$
(3) $(\exists x(x = t) \wedge \exists x(x = t')) \supset \exists x(x = t \cdot t')$.

Second, since *FA* is based on a free quantification theory (and therefore lacks the schema

(A3′) $\forall xA \supset A(t/x)$),

the axiom schema (A6) does not imply

(A6′) $\sim s(t) = 0$,

as would be the case in standard logic, but only the weaker

(A6″) $\exists x(x = t) \supset \sim s(t) = 0$.

Since $\vdash \exists x(x = t)$ for all numerals t, (1)–(3) and (A6′) are provable when t and t' are closed terms; but they are not provable when t or t' contain variables. Both these differences would disappear if we added (a free variant of) the omega rule, but with the omega rule our proof theory would cease to be finitary.

The consistency of *FA* is a simple consequence of the following:

Lemma *Let FAn be the subtheory of FA which results by eliminating all the axioms of the form (A12), where $n < t$. FAn is consistent.*

For suppose that a contradiction is provable in *FA*. Then the proof will use only a finite number of instances of (A12). So the proof will also be a proof in some *FAn*.

As for the proof of the lemma, let a (free) model $M = \langle D, D', f \rangle$ be defined as follows:

D (the inner domain) $= \{m : m \leq n\}$
D' (the outer domain) $= \{n + 1\}$
$f(0) = 0$
$f(s)(m) =$ the remainder of $m + 1/n + 2$
$f(+)(j,k) =$ the remainder of $j + k/n + 2$
$f(\cdot)(j,k) =$ the remainder of $j \cdot k/n + 2$.

(Intuitively, the inner domain is the set of existing objects, and the outer domain the set of nonexisting ones. For further details on this style of free semantics, the reader may consult [1].)

An assignment v is a function from the set of variables into D. The denotation function W_M^v for M and v is such that

$W_M^v(0) = f(0)$
$W_M^v(x) = v(x)$
$W_M^v(s(t)) = f(s)(W_M^v(t))$
$W_M^v(t + t') = f(+)(W_M^v(t),\ W_M^v(t'))$
$W_M^v(t \cdot t') = f(\cdot)(W_M^v(t),\ W_M^v(t'))$.

The auxiliary valuation V_M^v for M and v is defined as usual, and so is the valuation V_M for M.

It can be proved that all axioms of *FAn* are true in M and all rules of *FAn* are truth-preserving in M. Since M is a finite model, this establishes the consistency of *FAn* (and consequently of *FA*) by finitary methods. For the sake of illustration, I will now show that (A6) is true in M.

Suppose that $V_M(\text{A6}) = F$. Then, for some v, $V_M^v(\sim s(x) = 0) = F$. Then $W_M^v(s(x)) = W_M^v(0)$. Then $f(s)(W_M^v(x)) = f(0)$. Then $f(s)(v(x)) = 0$. Then the remainder of $v(x) + 1/n + 2 = 0$. But $v(x)$ is a natural number $\leq n$, and hence $v(x) + 1 < n + 2$. Therefore it is impossible that the remainder of $v(x) + 1/n + 2$ be 0.

REFERENCE

[1] Leblanc, H., *Existence, Truth, and Provability*, State University of New York Press, Albany, 1982.

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