Notre Dame Journal of Formal Logic
Volume 25, Number 3, July 1984

A Model Theoretic Proof of Feferman's Preservation Theorem

DAVID MARKER

Let L be a countable first-order language containing a binary relation symbol $\lhd$. If $\mathfrak{A}$ and $\mathfrak{B}$ are L-structures and $\mathfrak{A} \subseteq \mathfrak{B}$, then we say $\mathfrak{B}$ is a *faithful extension* of $\mathfrak{A}$ if and only if for any $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$ if $\mathfrak{B} \models b \lhd a$, then $b \in \mathfrak{A}$. Thus if $\lhd$ is a linear order on $\mathfrak{A}$, $\mathfrak{B}$ is a faithful extension if and only if it is an end extension.

In [2] Feferman gives a very natural classification of the formulas which are preserved under faithful extensions. His proof uses a many-sorted interpolation theorem proved by a cut elimination argument. With the introduction of recursively saturated models Barwise and Schlipf [1], and Schlipf [5] attempted to give a unified framework for many preservation and definability theorems. In this note I will give an instructive model theoretic proof of Feferman's theorem. (I should note that Stern [8] and Guichard [4] have given model theoretic proofs of Feferman's theorem using model-theoretic forcing and consistency properties, respectively, but neither of these approaches matches the elegance of [5].)

The proof given here is directly inspired by Friedman's theorem [3] that every countable model of Peano Arithmetic is isomorphic to a proper initial segment of itself and the related embedding results presented in Smoryński [6]. In fact, independently of the author, Smorynski [7] uses Friedman's theorem to prove Feferman's result in the special case that $\mathfrak{A}$ and $\mathfrak{B}$ are models of Peano Arithmetic.

1 Embedding recursively saturated models

Definition 1.1 Let L be as above. We inductively define Σ a class of L-formulas as follows:

(i) If $\varphi(\bar{v})$ is quantifier free, then $\varphi(\bar{v})$ is in Σ.
(ii) If $\varphi(\bar{v})$ and $\psi(\bar{v})$ are in Σ, then $\varphi(\bar{v}) \wedge \psi(\bar{v})$ and $\varphi(\bar{v}) \vee \psi(\bar{v})$ are in Σ.

Received August 8, 1983; revised December 2, 1983

(iii) If $\varphi(u,v,\bar{w})$ is in Σ, then $\exists v\,\varphi(v,\bar{w})$ is in Σ.
(iv) If $\varphi(u,v,\bar{w})$ is in Σ, then $\forall v(v \lhd u \to \varphi(u,v,\bar{w}))$ is in Σ.

We abbreviate $\forall v(v \lhd u \to \varphi(u,v,\bar{w}))$ as $\forall v \lhd u\,\varphi(u,v,\bar{w})$.

Definition 1.2 If $\mathfrak{A}$ and $\mathfrak{B}$ are L-structures, an embedding $f:\mathfrak{A}\to\mathfrak{B}$ is *faithful* iff f is one to one and $\mathfrak{B}$ is a faithful extension of the image of $\mathfrak{A}$.

Lemma 1.3 *Suppose $\varphi(\bar{v})$ is a Σ-formula and $f:\mathfrak{A}\to\mathfrak{B}$ is a faithful embedding. If $\bar{a}\in\mathfrak{A}$ and $\mathfrak{A}\models\varphi(\bar{a})$, then $\mathfrak{B}\models\varphi(f(\bar{a}))$.*

Proof: By a simple induction on the complexity of Σ-formulas.

Our goal is to provide a partial converse to Lemma 1.3. We might first introduce a bit of notation.

Definition 1.4 We define Π a class of L-formulas containing the duals of Σ-formulas. That is, Π is the smallest class of L-formulas containing the quantifier-free formulas and closed under conjunction, disjunction, universal quantification, and, if $\varphi(u,v,\bar{w})\in\Pi$, then $\exists v(v\lhd u \wedge \varphi(u,v,\bar{w}))\in\Pi$. (Again $\exists v(v\lhd u\wedge\varphi(u,v,\bar{w}))$ will be denoted $\exists v\lhd u\varphi(u,v,\bar{w})$.)

If $\mathfrak{A}\models T$ and $\bar{a}\in\mathfrak{A}$, the Σ-*type* of $\bar{a}$ in $\mathfrak{A}$ is the collection of all Σ-formulas $\varphi(\bar{v})$ such that $\mathfrak{A}\models\varphi(\bar{a})$. We define Π-types similarly.

We can now prove the main result.

Theorem 1.5 *Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are countable L-structures and the pair $(\mathfrak{A},\mathfrak{B})$ is recursively saturated. Assume further that if $\varphi\in\Sigma$ is a sentence and $\mathfrak{A}\models\varphi$, then $\mathfrak{B}\models\varphi$. We may conclude that there is a faithful embedding $f:\mathfrak{A}\to\mathfrak{B}$.*

Proof: Let a_0, a_1, $a_2,\ldots$ list $\mathfrak{A}$ and let b_0, $b_1,\ldots$ list $\mathfrak{B}$. We build f by finite stages. Our inductive assumption is that if f has been defined on domain $\bar{a}$, the Σ-type of $\bar{a}$ in $\mathfrak{A}$ is contained in the Σ-type of $f(\bar{a})$ in $\mathfrak{B}$ (or equivalently, the Π-type of $f(\bar{a})$ in $\mathfrak{B}$ is contained in the Π-type of $\bar{a}$ in $\mathfrak{A}$). Note our assumptions on $\mathfrak{A}$ and $\mathfrak{B}$ give the induction hypothesis for the initial case $\bar{a}=\phi$.

Step n. (1) Let f be defined on $\bar{a}$. (We allow the possibility that $n=0$ and $\bar{a}=\phi$.) Let i be minimal so that $a_i\notin\bar{a}$. Let $\Gamma(v)=\{\theta(v,f(\bar{a})):\theta\in\Sigma$ an L-formula and $\mathfrak{A}\models\theta(a_i,\bar{a})\}$.

Claim 1 It is consistent that $\mathfrak{B}$ realizes $\Gamma(v)$.

Let $\theta_0,\ldots,\theta_n\in\Gamma$. Then $\mathfrak{A}\models\exists v\bigwedge_{i=1}^{n}\theta_i(v,\bar{a})$. As the Σ-type of $\bar{a}$ in $\mathfrak{A}$ is contained in the Σ-type of $f(\bar{a})$ in $\mathfrak{B}$, $\mathfrak{B}\models\exists v\bigwedge_{i=1}^{n}\theta_i(v,f(\bar{a}))$. Thus realizing $\Gamma(v)$ in $\mathfrak{B}$ is consistent.

Claim 2 $\Gamma(v)$ is realized in $\mathfrak{B}$.

Realizing $\Gamma(v)$ in $\mathfrak{B}$ is equivalent to realizing $\Gamma^*(v)$ in $(\mathfrak{A},\mathfrak{B})$ where $\Gamma^*(v)=\{v\in\mathfrak{B}\}\cup\{\theta^{\mathfrak{A}}(a_i,\bar{a})\to\theta^{\mathfrak{B}}(v,f(\bar{a})):\theta\in\Sigma$ an L-formula$\}$ and $\theta^{\mathfrak{A}}$, $\theta^{\mathfrak{B}}$ denote the formulas obtained by replacing all quantifiers $\exists v$ and $\forall v$ by $\exists v\in\mathfrak{A}$, $\forall v\in\mathfrak{A}$ and $\exists v\in\mathfrak{B}$, $\forall v\in\mathfrak{B}$, respectively. But then $\Gamma^*(v)$ is a consistent recursive type and thus must be realized. Let b realize $\Gamma^*(v)$. Clearly b realizes $\Gamma(v)$.

Let $f(a_i) = b$. By choice of Γ our induction hypothesis is preserved.

(2) Suppose b_i is least so that $b_i \notin f(\bar{a})$ and for some $b \in f(\bar{a})$ $\mathfrak{B} \models b_i < b$. We must ensure b_i is in the range of f to make f faithful. Let $\Gamma(v) = \{\theta(v,\bar{a}): \theta \in \Pi$ an L-formula and $\mathfrak{B} \models \theta(b_i,f(\bar{a}))\}$. Let $\theta_0,\ldots,\theta_n \in \Gamma(V)$. Then $\mathfrak{B} \models \exists v < b \bigwedge_{i=1}^{n} \theta(v,f(\bar{a}))$. Since the Π-type of $f(\bar{a})$ in $\mathfrak{B}$ is contained in the Π-type of $\bar{a}$ in $\mathfrak{A}$, $\mathfrak{A} \models \exists v < f^{-1}(b) \bigwedge_{i=1}^{n} \theta(v,\bar{a})$. Thus it is consistent to realize Γ in $\mathfrak{A}$. As in Claim 2 above, Γ must be realized by some $a \in \mathfrak{A}$. Let $f(a) = b_i$. Again it is clear that the induction hypothesis is maintained.

This concludes step n.

It is easy to see that Part (1) of the construction ensures f is a total function embedding $\mathfrak{A}$ to $\mathfrak{B}$. Part (2) of the construction guarantees that if $b \in \text{range}(f)$ and $c < b$, then $c \in \text{range}(f)$. Hence f is faithful.

2 Feferman's theorem Feferman's theorem follows from Theorem 1.5 and the following lemma. Fix T an L-theory.

Lemma 2.1 *Let φ be a consistent L-sentence which is not provably equivalent to a Σ-sentence in T; then there are $\mathfrak{A}$, $\mathfrak{B} \models T$ such that $\mathfrak{A} \models \varphi$ and $\mathfrak{B} \models \neg\varphi \cup Th_\Sigma(\mathfrak{A})$ (where $Th_\Sigma(\mathfrak{A})$ denotes the Σ-sentences true in $\mathfrak{A}$).*

Proof: Suppose not. If $\mathfrak{A} \models \varphi$, let $\Gamma_{\mathfrak{A}} = \{\psi \in \Sigma : \mathfrak{A} \models \psi\}$. If $\mathfrak{B} \models \Gamma_{\mathfrak{A}}$ then $\mathfrak{B} \models \varphi$ as otherwise we would be done. Thus there is $n_{\mathfrak{A}}$ and $\psi_1^{\mathfrak{A}},\ldots,\psi_{n_{\mathfrak{A}}}^{\mathfrak{A}} \in \Gamma_{\mathfrak{A}}$ so that $T \vdash \bigwedge_{i=1}^{n_{\mathfrak{A}}} \psi_i^{\mathfrak{A}} \rightarrow \varphi$. Let $\theta^{\mathfrak{A}}$ denote $\bigwedge_{i=1}^{n_{\mathfrak{A}}} \psi_i^{\mathfrak{A}}$.

Let $\Delta = \{\neg\theta^{\mathfrak{A}} : \mathfrak{A} \models \varphi\}$. $\Delta \cup \{\varphi\}$ is inconsistent since if $\mathfrak{A} \models \Delta \cup \{\varphi\}$, $\mathfrak{A} \models \theta^{\mathfrak{A}}$ and $\neg\theta^{\mathfrak{A}}$. Thus there are $\mathfrak{A}_1 \ldots \mathfrak{A}_n$ such that $T \vdash \varphi \rightarrow \bigvee_{i=1}^{n} \theta^{\mathfrak{A}_i}$. Since $T \vdash \theta^{\mathfrak{A}_i} \rightarrow \varphi$, $T \vdash \varphi \leftrightarrow \bigvee_{i=1}^{n} \theta^{\mathfrak{A}_i}$. But $\bigvee_{i=1}^{n} \theta^{\mathfrak{A}_i} = \bigvee_{i=1}^{n} \bigwedge_{j=1}^{n_{\mathfrak{A}_i}} \psi_j^{\mathfrak{A}_i}$, a Σ-formula. Hence φ is provably equivalent to a Σ-formula.

Corollary 2.2 (Feferman's theorem) *An L-formula $\varphi(\bar{v})$ is preserved under faithful extensions of models of T iff $\varphi(\bar{v})$ is provably equivalent to a Σ-formula.*

Proof: ($\Leftarrow$) This is Lemma 1.3.

($\Rightarrow$) Without loss of generality assume φ is a sentence. If φ is not provably equivalent to a Σ-sentence we can use Lemma 2.1 to find countable $\mathfrak{A}_0 \models \varphi \cup T$ and $\mathfrak{B}_0 = T \cup \neg\varphi \cup Th_\Sigma(\mathfrak{A}_0)$. Form the pair $(\mathfrak{A}_0,\mathfrak{B}_0)$ and let $(\mathfrak{A},\mathfrak{B}) \succ (\mathfrak{A}_0,\mathfrak{B}_0)$ be a countable recursively saturated extension. By Theorem 1.5 there is a faithful embedding of $\mathfrak{A}$ into $\mathfrak{B}$. Thus φ is not preserved under faithful extensions.

REFERENCES

[1] Barwise, J. and J. Schlipf, "An introduction to recursively saturated and resplendent models," *The Journal of Symbolic Logic*, vol. 41 (1976).

[2] Feferman, S., "Persistent and invariant formulas for outer extensions," *Compositio Mathematica*, vol. 20 (1968).

[3] Friedman, H., "Countable models of set theories," in *Cambridge Summer School in Mathematical Logic*, ed., A. R. D. Mathias and H. Rogers, Springer-Verlag, 1973.

[4] Guichard, D., "A many sorted interpolation theorem for L(Q)," *Proceedings of the American Mathematical Society*, vol. 80 (1980).

[5] Schlipf, J., "Towards model theory through recursive saturation," *The Journal of Symbolic Logic*, vol. 43 (1978).

[6] Smorynski, C., "Recursively saturated nonstandard models of arithmetic," *The Journal of Symbolic Logic*, vol. 46 (1981).

[7] Smorynski, C., "Lectures on nonstandard models of arithmetic," *Proceedings of the Florence Meeting*, North-Holland, Amsterdam, to appear.

[8] Stern, J., "A new look at the interpolation problem," *The Journal of Symbolic Logic*, vol. 40 (1975).

Department of Mathematics
University of California at Berkeley
Berkeley, California 94720