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The Connective of Necessity of Modal Logic S_5 is Metalogical

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Let a,b be formulas of the language of the classical propositional calculus and let the first of them be a classical thesis while the second is not. This fact is often denoted as follows: $\vdash a$, $\dashv b$. In a certain sense the operations $\vdash$ and $\dashv$ are inconsistent and we will write informally $\vdash\ = \neg \dashv$ ($\neg$ being negation). We can consider the operation $\vdash$ as a connective A of some propositional calculus containing the classical one and containing formulas Aa and $\neg Ab$ as theses. Among its theses would be the formulas $Aa \equiv p \rightarrow p$ and $Ab \equiv \neg(p \rightarrow p)$ (p being a propositional variable). It seems that by such a definition (i.e., $Aa = p \rightarrow p$ iff a is a thesis and $Aa = \neg(p \rightarrow p)$ iff a is not a thesis) this new logic could be obtained. This is not so, however, for among the expressions $\neg Ap$, $\neg A(p \rightarrow p)$ the first would be a thesis and the second a nonthesis, which would not allow us to treat p as a variable. We are thus led to consider the greatest such set of formulas closed under substitution, i.e., the set S defined below. This is an intuitive way to summarize the problem of this paper, i.e., the problem of building a system using the connective of assertion A and containing the classical logic.

This system will be shown to be identical with the system of modal logic S_5. The manner of introducing the connective A suggests it possesses a metalogical character in comparison with the classical connectives. This allows us to suppose that in S_5 it will be possible to "express" certain metalogical properties of the logic obtained by omitting the connective A, i.e., classical logic. Indeed, Pogorzelski's Theorem on structural completeness of classical propositional calculus (Theorem 2) is "expressed" as a rule of S_5. We also note the fact that Stone's Theorem is in the same sense equivalent to a rule of S_5^* (see below). Eventually we give some fragmentary methods of rejection of formulas in S_5 based only on classical logic and on the manner of reading the connective A.[1]

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Let L be a set of formulas formed by means of the classical connectives: $\neg$(negation), $\wedge$(conjunction), $\vee$(disjunction), $\rightarrow$(implication), and $\equiv$(equivalence). L^A is the extended set of formulas generated by a one-argument connective A. By the symbols CPC, e, t, f we denote: the set of classical theses, any substitution, some fixed classical thesis, and some fixed classical counterthesis.

Let $^*:L^A \rightarrow L$ be an interpretation which preserves variables, all classical connectives, and, in what concerns A, is defined as follows

$$(Aa)^* = \begin{cases} t \text{ if } a^* \in CPC \\ f \text{ otherwise} \end{cases} \qquad (a \in L^A).$$

Since for $a \in L$, $a^* = a$ then for every formula $a \in L$

$$(Aa)^* = t \text{ iff } a \in CPC$$
$$(Aa)^* = f \text{ iff } a \notin CPC \ .$$

This suggests the manner of reading the connective A. We will read it as "is asserted".

Let $a \in L^A$. $a \in S$ iff for every substitution $e:L^A \rightarrow L^A$ $(ea)^* \in CPC$.

Lemma 1 *Let* $a, b \in L^A$,

(i) $Aa \rightarrow a, A(a \rightarrow b) \rightarrow (Aa \rightarrow Ab), \neg A \neg Aa \rightarrow Aa \in S$
(ii) *If* $a \in S$ *then* $Aa \in S$
(iii) *The set* S *is closed under substitutions and modus ponens.*

Proof: Simple.

Let $\dashv$ be the symbol of rejection related to the set S.

Lemma 2 *Let* $a,b,c \in L$. *The set* S *is closed under the following rules*

$$r_1 \ \frac{\dashv a \rightarrow b}{\dashv Aa \rightarrow Ab} \qquad r_2 \ \frac{\dashv a \rightarrow b, \dashv a \rightarrow c}{\dashv Aa \rightarrow Ab \vee Ac} \ .$$

Proof: We prove our lemma only for r_2 (for r_1 the proof is easier). Suppose that $\dashv a \rightarrow b$ and $\dashv a \rightarrow c$. Since $a,b,c \in L$, then $a \rightarrow b \notin CPC$ and $a \rightarrow c \notin CPC$. Let v_0, v_1 be $0-1$ valuations such that $v_0(a \rightarrow b) = v_1(a \rightarrow c) = 0$. *Let* $e:L \rightarrow L$ be a substitution such that

$$eq = \begin{cases} t \text{ if } v_0q = v_1q = 1 \\ f \text{ if } v_0q = v_1q = 0 \\ p \text{ if } v_0q = 0 \text{ and } v_1q = 1 \\ \neg p \text{ if } v_0q = 1 \text{ and } v_1q = 0 \end{cases}$$

where q is any variable and p any fixed variable. Since $v_0a = v_1a = 1$ then $ea \in CPC$, and since $v_0b = v_1c = 0$ then $eb \notin CPC$ and $ec \notin CPC$. Hence, $(Aea)^* = t$ and $(Aeb)^* = (Aec)^* = f$. Hence $(Aea)^* \rightarrow (Aeb)^* \vee (Aec)^* \notin CPC$. Hence, $\dashv Aa \rightarrow Ab \vee Ac$.

Lemma 3 *Let* $a, a_1, \ldots, a_i \in L$. *The set of formulas* S *is closed under the following rule*

$$\frac{\dashv a \rightarrow a_1, \ldots, \dashv a \rightarrow a_i}{\dashv Aa \rightarrow Aa_1 \vee \ldots \vee Aa_i} \ .$$

Proof: We prove our lemma for $i = 3$ (for every natural i we can obtain the proof by an easy generalization of our proof). Notice that

1. $AAb \equiv Ab \in S$
2. $A(Ab \vee Ac) \equiv Ab \vee Ac \in S.$ $\quad (b,c \in L^A)$

The proof can be illustrated as follows

$$\dfrac{\dfrac{\dfrac{\dfrac{\dashv a \to a_1, \dashv a \to a_2}{\dashv Aa \to Aa_1 \vee Aa_2}r_2 \quad \dfrac{\dashv a \to a_3}{\dashv Aa \to Aa_3}r_1}{\dashv AAa \to A(Aa_1 \vee Aa_2) \vee AAa_3}r_2}{\dashv AAa \to Aa_1 \vee Aa_2 \vee AAa_3}2}{\dashv Aa \to Aa_1 \vee Aa_2 \vee Aa_3}1 .$$

Let S_5 be the set of modal theses of the modal calculus S_5 (where we assume that the symbol of necessity is denoted by the letter A).

Theorem 1 $\quad S = S_5.$

Proof: By Lemma 1 we have $S_5 \subseteq S$. By [3] and Lemma 3 we have $L - S_5 \subseteq L - S$.

Lemma 4 *For all $a,b \in L$*

(i) *$Aa \in S$ iff $a \in CPC$*
(ii) *$Aa \to Ab \in S$ iff for every substitution $e{:}L \to L$ if $ea \in CPC$ then $eb \in CPC$.*

Proof: (i) Trivial. (ii) ($\to$): Suppose that $Aa \to Ab \in S$ and for some substitution $e{:}L \to L$, $ea \in CPC$. Hence $(e(Aa \to Ab))^* \in CPC$ and $(Aea)^* = t \in CPC$. Since $(e(Aa \to Ab))^* = (Aea)^* \to (Aeb)^*$ and the set CPC is closed under modus ponens, then $(Aeb)^* \in CPC$. Hence $eb = (eb)^* \in CPC$. (ii) ($\leftarrow$): Suppose that $Aa \to Ab \notin S$. It follows that there is a substitution $e{:}L^A \to L^A$ such that $(e(Aa \to Ab))^* \notin CPC$, i.e., $(Aea)^* \to (Aeb)^* \notin CPC$. Hence $(Aea)^* = t$ and $(Aeb)^* = f$. And hence $(ea)^* \in CPC$ and $(eb)^* \notin CPC$. Let $e_0{:}L \to L$ be a substitution such that $e_0p = (ep)^*$ for every variable p. Notice that $e_0a = (ea)^*$ and $e_0b = (eb)^*$. Then $e_0a \in CPC$ and $e_0b \notin CPC$.

In a quite similar way we can prove the following

Lemma 4′ *For all $a,a_1, \ldots, a_i \in L$*

(i) *$A(a_1 \wedge \ldots \wedge a_i \to a) \in S$ iff $a_1 \wedge \ldots \wedge a_i \to a \in CPC$*
(ii) *$Aa_1 \wedge \ldots \wedge Aa_i \to Aa \in S$ iff for every substitution $e{:}L \to L$ if $ea_1, \ldots, ea_i \in CPC$ then $ea \in CPC$.*

Let r be the rule described by the following schema

$$\frac{\vdash Aa_1 \wedge \ldots \wedge Aa_i \to Aa}{\vdash A(a_1 \wedge \ldots \wedge a_i \to a)} \qquad (a,a_1, \ldots, a_i \in L)$$

($\vdash a$ iff $a \in S$).

By Lemma 4′ this rule expresses the following implication: if for every substitution $e{:}L \to L$ if $ea_1, \ldots, ea_i \in CPC$ then $ea \in CPC$, then $a_1 \wedge \ldots \wedge a_i \to a \in CPC$.

This implication is equivalent to Pogorzelski's Structural Completeness Theorem of classical propositional calculus with modus ponens as the rule (cf. [1]).

Theorem 2 *The classical propositional calculus (with modus ponens as the sole rule) is structurally complete if and only if* S_5 *is closed under the rule r.*

Now we extend our sets of formulas to the infinite formulas and we denote by r^* the following rule

$$\frac{\vdash Aa_1 \wedge \ldots \wedge Aa_i \wedge \ldots \to Aa}{\vdash A(a_1 \wedge \ldots \wedge a_i \wedge \ldots \to a)} \qquad (a, a_1, \ldots, a_i, \ldots \in L).$$

Let us denote by S_5^* the set of formulas obtained by the above extension. So by [2] and by the analogous Lemma 4′ (for infinite formulas) we have the following:

Theorem 3 S_5^* *is closed under the rule* r^* *if and only if Stone's Representation Theorem for Boolean algebras holds.*

If we know the algebra of S_5 then we can show that the following formulas are not theses:

1. $\neg Ap$
2. $\neg A \neg p \to p$
3. $A(Ap \to q) \to A(p \to Aq)$
4. $A(p \to q) \vee A(q \to p)$.

By the definitions of S, A, and * the reader can easily see that the schemas below present the proofs of rejections of the respective formulas (we omit the asterisk which should stay beside each of the formulas).

1′. $\neg A(p \to p)$
$\neg t$
f

2′. $\neg A \neg p \to p$
$\neg f \to p$
$t \to p$
p

3′. $A(Ap \to p) \to A(p \to Ap)$
$A(f \to p) \to A(p \to f)$
$At \to A \neg p$
$t \to f$
f

4′. $A(p \to q) \vee A(q \to p)$
$f \vee f$
f

We want to note that in verifying formulas by the algebra of S_5 the logical value of a formula Aa depends on the logical value of a. But in the method presented here the value of formula Aa (t or f) depends on the fact that formula a is a thesis or is not a thesis. This, once again, assures us that this manner of reading the connective A is proper. Thus the title of this paper is accurate and, I think, suggests that system S_5 should not be treated as a modal system.

NOTE

1. The attempt to treat the notion of rejection as a kind of connective was first carried out in "System of rejection propositions on the basis of Leśniewski's protothetics" (in preparation), by my colleague Toshiharu Waragai.

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