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On the Equivalence Between the Calculi MC^ν and $EC^{\nu+1}$ of A. Bressan

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Part I General interpretations for the modal language ML^ν

1 Introduction The modal calculus MC^ν (based on the language ML^ν) and the extensional calculus $EC^{\nu+1}$ (based on $EL^{\nu+1}$) are presented and investigated in [2]; and in Section 15 of that work the translation $\Delta \to \Delta^\eta$ of ML^ν into $EL^{\nu+1}$ is defined (on the basis of the semantical rules for ML^ν). The main result concerning the function η is proved (syntactically) in [2] (Theorem 63.1). The theorem asserts that, for a suitable version of MC^ν,

(1.1) $\quad \vdash_{\overline{MC^\nu}} p$ iff $\vdash_{\overline{EC^{\nu+1}}} p^\eta$, for every formula p of ML^ν.

Obviously, the only relevant part of (1.1) is the implication from right to left, since its converse is the very goal aimed at in defining η.

Now, in [8] MC^ν is proved to be complete with respect to general ML^ν-interpretations (cf. Section 3) and an analogous result for $EC^{\nu+1}$ can be easily achieved by adapting the proof of Theorem 2 in [4]. Therefore (1.1) is a trivial consequence of

(1.2) $\quad \models^{g}_{\overline{MC^\nu}} p$ iff $\models^{g}_{\overline{EC^{\nu+1}}} p^\eta$, for every formula p of ML^ν,

where $\models^{g}_{\overline{MC^\nu}} p$ [$\models^{g}_{\overline{EC^{\nu+1}}} p^\eta$] expresses that $p[p^\eta]$ is true in every general model of the considered version of $MC^\nu[EC^{\nu+1}]$.

In this work the structures of the general interpretations for ML^ν and

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$EL^{\nu+1}$ are investigated, mainly in order to give a simpler proof of (1.1) and in order to extend it to a stronger result (Theorem 10.3). This theorem asserts that every given maximal consistent set K of (totally closed) formulas of ML^ν can be embedded, by means of η, in exactly one maximal consistent set of (closed) formulas of $EL^{\nu+1}$, in which the constants (of $EL^{\nu+1}$) occur only in formulas of the form $q^\eta (q \in K)$.

The language ML^ν is based on a type system τ^ν and a QI structure for an ML^ν-interpretation $\mathcal{J}$ is a set $\mathscr{J} = \{ \mathcal{QI}_t : t \in \bar{\tau}^\nu\}$ where $\mathcal{QI}_t$ is the set of the designata (in $\mathcal{J}$) of the expressions of type t. The semantical rules for ML^ν are expressed in an extensional metalanguage and every element of $\mathcal{QI}_t$—quasi-intension of type t—($t \in \bar{\tau}^\nu$) turns out to be an (extensional) object of type t^η ($\in \tau^{\nu+1}$, the type system on which $EL^{\nu+1}$ is based). Thus every QI-structure $\mathscr{J}$ can be obtained from a suitable Ob-structure $S = \{Ob_s : s \in \bar{\tau}^{\nu+1}\}$ for an $EL^{\nu+1}$-interpretation I, by setting $\mathcal{QI}_t = Ob_{t^\eta}$, for all $t \in \bar{\tau}^\nu$. In this case we say that $\mathscr{J}$ is the QI-structure induced by S ($\mathscr{J} = S^i$). Furthermore, the correspondence $S \to S^i$ can be extended in a natural way to a correspondence $I \to I^i$ between $EL^{\nu+1}$-interpretations and ML^ν-interpretations (cf. Definition 2.1).

The version of MC^ν considered in this work is the minimal calculus for which (1.1) is provable (cf. Section 7); it has two additional axioms—MA4.1 and MA5.1—besides the basic ones (MA3.1-3.18) which are true in every general ML^ν-interpretation.[1] In Part I, Sections 4 and 5, necessary and sufficient conditions on a given general ML^ν-interpretation $\mathcal{J}$ are determined for MA4.1 and MA5.1 to be true in it. In particular, if these axioms are true in $\mathcal{J}$, then certain quasi-intensions can be considered in the QI-structure $\mathscr{J}$ of $\mathcal{J}$, which can represent the sets $D_1, \ldots, D_\nu$ of individuals and the set Γ of possible cases on which $\mathscr{J}$ (and the Ob-structure S determining $\mathscr{J}$) are based. Therefore, the construction of S and the correspondence $S \to S^i$ can be expressed in $\mathscr{J}$, and the restriction of this correspondence to objects of type $t^\eta (t \in \bar{\tau}^\nu)$ turns out to be expressed by an embedding of $\mathscr{J}$ into itself (cf. Section 8).[2] This is substantially a modal analogue of the embedding $Ob_t \to Ob_{t^\eta}$ ($= \mathcal{QI}_t$) which mirrors the transition from an extensional semantics (or object system) to our modal analogue.

In Part II, Sections 6 and 7, the translation η of ML^ν into $EL^{\nu+1}$ is considered, and some useful results are stated, which, for any expression Δ of ML^ν, relate the designatum of Δ^η in an $EL^{\nu+1}$-interpretation I, with the designatum of Δ in I^i. In particular, in Section 7 the easier part of (1.1) is proved.

The other part is a consequence of the following statement: for every general ML^ν-interpretation $\mathcal{J}$, a general $EL^{\nu+1}$-interpretation I exists such that $\mathcal{J} = I^i$. This is proved in Sections 8 and 9 substantially by defining I inside $\mathcal{J}$. Let us remark that the abovementioned possibility of embedding the QI-structure of $\mathcal{J}$ into itself is strongly used in the construction of I, as well as in the proofs that $\mathcal{J} = I^i$ and that I is general.

The procedures introduced in Section 8 are also used in Section 10 in order to prove Theorem 10.1, on which the proof of the abovementioned uniqueness result is based.

2 Semantics for the languages ML^ν and $EL^{\nu+1}$ The language ML^ν ($\nu \in Z^+$, the set of positive integers) is based on a *type system* τ^ν which is the smallest set such that $\{1, \ldots, \nu\} \subseteq \tau^\nu$ and $\langle t_1, \ldots, t_n, t_0\rangle \in \tau^\nu$ whenever $t_1, \ldots, t_n \in \tau^\nu$ and $t_0 \in \bar{\tau}^\nu = \tau^\nu \cup \{0\}$. For every $t \in \tau^\nu$, the *constants* c_{tn} and the *variables* $v_{tn}(n \in Z^+)$ are primitive symbols of ML^ν in addition to the usual logical symbols: $=$, $\sim$, $\wedge$, $\Box$, ι, comma, and left and right parentheses. The set $\mathscr{E}_t(t \in \bar{\tau}^\nu)$ of the *well-formed expressions* (wfes) having type t of ML^ν, is defined recursively according to the following rules (f_1)-(f_8), where $t, t_1, \ldots, t_n$ run over τ^ν and $t_0[n]$ runs over $\bar{\tau}^\nu[Z^+]$.

(f_1) $v_{tn} \in \mathscr{E}_t$ and $c_{tn} \in \mathscr{E}_t$
(f_2) if $\Delta_1, \Delta_2 \in \mathscr{E}_t$, then $\Delta_1 = \Delta_2 \in \mathscr{E}_0$
(f_3) if $\Delta_i \in \mathscr{E}_{t_i}$ $(i = 1, \ldots, n)$ and $\Delta \in \mathscr{E}_{\langle t_1,\ldots,t_n,t_0\rangle}$, then $(\Delta(\Delta_1, \ldots, \Delta_n)) \in \mathscr{E}_{t_0}$
$(f_{4\text{-}7})$ if $p, q \in \mathscr{E}_0$, then $(\sim p)$, $(p \wedge q)$, $((v_{tn})p)$, and $(\Box p) \in \mathscr{E}_0$
(f_8) if $p \in \mathscr{E}_0$, then $(\iota v_{tn})p \in \mathscr{E}_t$.[3]

The extensional language $EL^{\nu+1}$ can be defined as the extensional part of $ML^{\nu+1}$; that is, the wfes of $EL^{\nu+1}$ are those of $ML^{\nu+1}$ in which the modal operator $\Box$ does not appear. The set of wfes of type $s(\in \bar{\tau}^{\nu+1})$ in $EL^{\nu+1}$ is denoted by E_s.

Following Carnap we denote $\langle\theta_1, \ldots, \theta_n, \theta_0\rangle$ (in τ^ν or $\tau^{\nu+1}$) by $(\theta_1, \ldots, \theta_n)$ or $(\theta_1, \ldots, \theta_n: \theta_0)$ according to whether θ_0 is 0 or not. For every $\theta \in \bar{\tau}^\nu[\bar{\tau}^{\nu+1}]$ the elements of $\mathscr{E}_\theta[E_\theta]$ are called *well-formed formulas* (wffs) when $\theta = 0$, *relation terms* when $\theta = (\theta_1, \ldots, \theta_n)$, *function terms* when $\theta = (\theta_1, \ldots, \theta_n: \theta_0)$, and *individual terms* when $\theta \in \{1, \ldots, \nu\}[\{1, \ldots, \nu + 1\}]$.

The symbols $\vee$, $\supset$, $(\exists v_{tn})$, $\Diamond$, and other metalinguistic abbreviations are understood to be introduced in the usual way. In particular, we often write $\Delta \in \Delta'$ instead of $\Delta'(\Delta)$, and $(\exists_1 x)p$, $(\forall x \in F)p$, and $(\iota x \in F)p$ will stand respectively for $(\exists x)(p \wedge (y)(p[x/y] \supset x = y))$, $(x)(x \in F \supset p)$, and $(\iota x)(x \in F \wedge p)$, in both ML^ν and $EL^{\nu+1}$. Furthermore, every expression used in the sequel is assumed to be well formed. This makes several explanations unnecessary.

A formula p (in ML^ν) will be said to be *modally closed* if it is constructed from wffs $\Box p_1, \ldots, \Box p_n$ by means of $\sim$, $\wedge$, (v_{tn}), and $\Box$; if p is also (extensionally) closed, then we say it is *totally closed.*

For every choice of $\nu + 1$ sets $D_1, \ldots, D_{\nu+1}$ we say that the set $S = \{Ob_s: s \in \bar{\tau}^{\nu+1}\}$ is an Ob-*structure* (for $EL^{\nu+1}$) in case the following conditions (2.1-2.3) hold.[4]

(2.1) $Ob_0 = \{0, 1\}$; $Ob_r = D_r$ $(r = 1, \ldots, \nu + 1)$
(2.2) $Ob_{(s_1,\ldots,s_n)} \subseteq \mathscr{P}(\Pi_i^n Ob_{s_i})$
(2.3) $Ob_{(s_1,\ldots,s_n:s_0)} \subseteq ((\Pi_i^n Ob_{s_i}) \to Ob_{s_0})$.

If $a^{\nu+1}$ is a function, of domain $\tau^{\nu+1}$, such that $(a_s^{\nu+1} =_d) a^{\nu+1}(s) \in Ob_s$ (for all $s \in \tau^{\nu+1}$), then we say that $\langle S, a^{\nu+1}\rangle$ is an *Ob-system.* $a_s^{\nu+1}$ is called the *nonexisting object* of type s, since it will be assumed to be the designatum of every description (in E_s) which does not fulfill its exact uniqueness condition (cf. rule (d_8) below).

An $EL^{\nu+1}$-*interpretation* is an ordered triple $\boldsymbol{I} = \langle S, a^{\nu+1}, I\rangle$ in which I is a *valuation of the constants* of $EL^{\nu+1}$ in S, that is, a function assigning each $c_{sn} \in E_s$ an object $I(c_{sn})$ in Ob_s. If in (2.2, 3) the relation $\subseteq$ holds as an equality, then $\boldsymbol{I}$ is said to be *standard.*

The set of all valuations of the variables of $EL^{\nu+1}$ in I (briefly, I-valuations) will be denoted by Val_I. The following rules (d_1)-(d_8) define the *designatum* $des_{IV}(\Delta)$ of the arbitrary wfe Δ in $EL^{\nu+1}$ in correspondence with the $EL^{\nu+1}$-interpretation I and the I-valuation V. In these rules we assume $n \in Z^+$, $s \in \tau^{\nu+1}$, and $des_{IV}(\Delta') = \bar{\Delta}'$ for every subexpression Δ' of Δ; furthermore, in $(d_3)[(d_4)]$ $R(\Delta_1, \ldots, \Delta_n)[f(\Delta_1, \ldots, \Delta_n)]$ denotes a formula [a term]. Note that $V(v_{sn}/\xi)$ is an I-valuation just like V except $V(v_{sn}/\xi)(v_{sn}) = \xi$.

(d_1) $des_{IV}(v_{sn}) = V(v_{sn})$, $des_{IV}(c_{sn}) = I(c_{sn})$

(d_2) $des_{IV}(\Delta_1 = \Delta_2) = 1$ if $\bar{\Delta}_1 = \bar{\Delta}_2$, 0 otherwise

(d_3) $des_{IV}(R(\Delta_1, \ldots, \Delta_n)) = 1$ if $\langle\bar{\Delta}_1, \ldots, \bar{\Delta}_n\rangle \in \bar{R}$, 0 otherwise

(d_4) $des_{IV}(f(\Delta_1, \ldots, \Delta_n)) = \bar{f}(\bar{\Delta}_1, \ldots, \bar{\Delta}_n)$

$(d_{5\text{-}6})$ $des_{IV}(\sim p) = 1 - \bar{p}$; $des_{IV}(p \wedge q) = \bar{p} \cdot \bar{q}$

(d_7) $des_{IV}((v_{sn})p) = min_{\xi \in Ob_s}(des_{IV_\xi}(p))$ where $V_\xi = V(v_{sn}/\xi)$

(d_8) $des_{IV}((\imath v_{sn})p)$ = the only $\xi \in Ob_s$ such that $des_{IV'}(p) = 1$ for $V' = V(v_{sn}/\xi)$, if such a unique ξ exists, $a_s^{\nu+1}$ otherwise.

As usual a formula p is said to be *true* [*satisfiable*] in I if $des_{IV}(p) = 1$ for every [some] $V \in Val_I$. It is a matter of routine to prove that $des_{IV}(\Delta)$ does not depend on V when Δ is a closed wfe; in this case we shall often write $des_I(\Delta)$ for $des_{IV}(\Delta)$.

In order to define the semantics for ML^ν we first consider the translation η of $\bar{\tau}^\nu$ into $\tau^{\nu+1}$:

$$(2.4) \quad \begin{cases} 0^\eta = (\nu + 1), \quad r^\eta = (\nu + 1 : r) \quad (r = 1, \ldots, \nu) \\ (t_1, \ldots, t_n)^\eta = (t_1^\eta, \ldots, t_n^\eta, \nu + 1) \\ (t_1, \ldots, t_n : t_0)^\eta = (t_1^\eta, \ldots, t_n^\eta : t_0^\eta); \end{cases}$$

and, secondly, we say that a *structure of quasi-intensions* (briefly, *QI-structure*) for ML^ν is a set $\mathscr{J} = \{\mathscr{Q}\mathscr{I}_t : t \in \bar{\tau}^\nu\}$ such that, for some Ob-structure $S = \{Ob_s : s \in \tau^{\nu+1}\}$,

$$(2.5) \quad \mathscr{Q}\mathscr{I}_t = Ob_{t^\eta}, \text{ for all } t \in \bar{\tau}^\nu.$$

The $(\nu + 1)$th basic set for a QI-structure, i.e., $D_{\nu+1}$, is denoted by Γ and its elements are called *possible cases.* Furthermore, for all $t \in \bar{\tau}^\nu$, every $\xi \in \mathscr{Q}\mathscr{I}_t$ is said to be a *quasi-intension* (briefly, QI) of type t.

Analogously to the extensional case, an *ML^ν-interpretation* is an ordered triple $\mathbf{\mathscr{I}} = \langle \mathscr{J}, a^\nu, \mathscr{I} \rangle$ in which: (1) $\mathscr{J}$ is a QI-structure, (2) a^ν is a function of domain τ^ν, such that $a_t^\nu \in \mathscr{Q}\mathscr{I}_t$ for all $t \in \tau^\nu$, and (3) $\mathscr{I}$ is a valuation of the constants of ML^ν in $\mathscr{J}$.[5]

Definition 2.1 Let $I = \langle S, a^{\nu+1}, I \rangle$ be an $EL^{\nu+1}$-interpretation and let $\mathbf{\mathscr{I}} = \langle \mathscr{J}, a^\nu, \mathscr{I} \rangle$ be the ML^ν-interpretation determined by the equalities (2.5) and: $a_t^\nu = a_{t^\eta}^{\nu+1}, \mathscr{I}(c_{tn}) = I(c_{t^\eta n})$, for all $t \in \tau^\nu$ and $n \in Z^+$. Then we denote $\mathbf{\mathscr{I}}$ by I^i and say that: (i) $\mathbf{\mathscr{I}}$ is induced by I, and (ii) I is an extensional correspondent of $\mathbf{\mathscr{I}}$.

Definition 2.2 Let $\mathbf{\mathscr{I}}$ be an ML^ν-interpretation and let $\xi, \zeta \in \mathscr{Q}\mathscr{I}_t (t \in \bar{\tau}^\nu)$. Then ξ and ζ are said to be equivalent in the case $\gamma (\in \Gamma)$ (briefly $\xi =_\gamma \zeta$) if one of the following conditions holds:

(a, b) $t = 0$ and $\xi \cap \{\gamma\} = \zeta \cap \{\gamma\}$, $t \in \{1, \ldots, \nu\}$ and $\xi(\gamma) = \zeta(\gamma)$,

(c) $t = (t_1, \ldots, t_n)$ and $\xi \cap ((\Pi_i^n \mathscr{Q}\mathscr{I}_{t_i}) \times \{\gamma\}) = \zeta \cap ((\Pi_i^n \mathscr{Q}\mathscr{I}_{t_i}) \times \{\gamma\})$,

(d) $t = (t_1, \ldots, t_n\colon t_0)$ and, for all $\alpha \in \Pi_i^n \mathscr{Q}\mathscr{I}_{t_i}$, $\xi(\alpha) =_\gamma \zeta(\alpha)$.

Let $\mathscr{J}(= \langle \mathscr{J}, a^{\nu}, \mathscr{I}\rangle)$ be an ML^{ν}-interpretation and let $\mathscr{V} \in Val_{\mathscr{J}}$. Then the designatum $des_{\mathscr{J}\mathscr{V}}(\Delta)$ of the arbitrary wfe Δ (with respect to $\mathscr{J}$ and $\mathscr{V}$) is defined by means of the following Rules (δ_1) to (δ_9), where the analogues of the hypotheses for (δ_1) to (δ_8) are assumed.

(δ_1) $des_{\mathscr{J}\mathscr{V}}(v_{tn}) = \mathscr{V}(v_{tn})$, $des_{\mathscr{J}\mathscr{V}}(c_{tn}) = \mathscr{I}(c_{tn})$

(δ_2) $des_{\mathscr{J}\mathscr{V}}(\Delta_1 = \Delta_2) = \{\gamma \in \Gamma\colon \bar{\Delta}_1 =_\gamma \bar{\Delta}_2\}$

(δ_3) $des_{\mathscr{J}\mathscr{V}}(R(\Delta_1, \ldots, \Delta_n)) = \{\gamma \in \Gamma\colon \langle \bar{\Delta}_1, \ldots, \bar{\Delta}_n, \gamma\rangle \in \bar{R}\}$

(δ_4) $des_{\mathscr{J}\mathscr{V}}(f(\Delta_1, \ldots, \Delta_n)) = \bar{f}(\bar{\Delta}_1, \ldots, \bar{\Delta}_n)$

($\delta_{5\text{-}6}$) $des_{\mathscr{J}\mathscr{V}}(\sim p) = \Gamma - \bar{p}$; $des_{\mathscr{J}\mathscr{V}}(p \wedge q) = \bar{p} \cap \bar{q}$

(δ_7) $des_{\mathscr{J}\mathscr{V}}((v_{tn})p) = \bigcap_{\xi \in \mathscr{Q}\mathscr{I}_t} des_{\mathscr{J}\mathscr{V}_\xi}(p)$, where $\mathscr{V}_\xi = \mathscr{V}(v_{tn}/\xi)$

(δ_8) $des_{\mathscr{J}\mathscr{V}}(\Box p) = \Gamma[\phi]$ if $\bar{p} = \Gamma[\bar{p} \neq \Gamma]$

(δ_9) $des_{\mathscr{J}\mathscr{V}}((\iota v_{tn})p)$ = the only QI ζ such that
(a) $\gamma \in des_{\mathscr{J}\mathscr{V}}((\exists_1 v_{tn})p)$ and $\gamma \in des_{\mathscr{J}\mathscr{V}'}(p)$ for $\mathscr{V}' = \mathscr{V}(v_{tn}/\xi) \Rightarrow \zeta =_\gamma \xi$,
(b) $\gamma \in des_{\mathscr{J}\mathscr{V}}(\sim(\exists_1 v_{tn})p) \Rightarrow \zeta =_\gamma a_t^{\nu}$.

The exact uniqueness of the QI ζ fulfilling (a) and (b) is proved in [2], N11; let us remark however that ζ (as well as other designata) may fail to be in $\cup \mathscr{J}$. In any case this unsatisfactory situation does not happen when general ML^{ν}-interpretations (cf. Definition 3.2) are dealt with, and these are substantially the only ones we shall investigate.

A formula p is said to be *true* in $\mathscr{J}$ if $des_{\mathscr{J}\mathscr{V}}(p) = \Gamma$ for every $\mathscr{V} \in Val_{\mathscr{J}}$. If a $\mathscr{V} \in Val_{\mathscr{J}}$ exists such that $des_{\mathscr{J}\mathscr{V}}(q) \neq \phi$, then q is said to be *satisfiable* in $\mathscr{J}$.

From now on we assume that every interpretation of $ML^{\nu}[EL^{\nu+1}]$ fulfills the following (usual) conditions on $a^{\nu}[a^{\nu+1}]$:

(A) $a^{\nu}_{(t_1,\ldots,t_n)}[a^{\nu+1}_{(s_1,\ldots,s_n)}]$ is the empty set

(B) The image of $a^{\nu}_{(t_1,\ldots,t_n:t_0)}[a^{\nu+1}_{(s_1,\ldots,s_n:s_0)}]$ is $\{a^{\nu}_{t_0}\}[\{a^{\nu+1}_{s_0}\}]$.

These assumptions are conventional and could be chosen otherwise; we adopt them substantially because they will render certain proofs simpler.

3 The calculi $EC^{\nu+1}$ and MC^{ν}, general interpretations, and completeness theorems for $EC^{\nu+1}$ and MC^{ν} The following list shows the basic axioms for the calculus $EC^{\nu+1}$, based on $EL^{\nu+1}$. In it p and q denote wffs, Δ denotes a term, and $x, y, z, x_1, \ldots, x_n, F, G, f$, and g denote variables of suitable types; furthermore, in EA3.14,15 we use the symbol a_s^* as an abbreviation for $(\iota v_{sn})(v_{sn} \neq v_{sn})$.

EA3.1-6	The axioms of the predicate calculus
EA3.7-9	$x = x; x = y \wedge y = z \supset x = z; x = y \supset \Delta[z/x] = \Delta[z/y]$
EA3.10	$F = G \equiv (\forall x_1, \ldots, x_n)(F(x_1, \ldots, x_n) \equiv G(x_1, \ldots, x_n))$
EA3.11	$f = g \equiv (\forall x_1, \ldots, x_n) f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$
EA3.12	$(\exists F)(\forall x_1, \ldots, x_n)(F(x_1, \ldots, x_n) \equiv p)$ (F not free in p)
EA3.13	$(\exists f)(\forall x_1, \ldots, x_n) f(x_1, \ldots, x_n) = \Delta$ (F not free in Δ)
EA3.14	(a) $(\exists_1 v_{sn})q \wedge q[v_{sn}/y] \supset y = (\iota v_{sn})q$
	(b) $\sim(\exists_1 v_{sn})q \supset (\iota v_{sn})q = a_s^*$

EA3.15 (a) $\sim a_s^*(x_1, \ldots, x_n)$ (where $s = (s_1, \ldots, s_n)$)
(b) $a_s^*(x_1, \ldots, x_n) = a_{s_0}^*$ (where $s = (s_1, \ldots, s_n : s_0)$).

If we regard the symbols occurring in EA3.1-EA3.15 as ranging over ML^ν, then (for $i = 1$ to 15) EA3.i also represents a schema of wffs of ML^ν, which we denote by EA3.i′. Thus the axiom schemes for the calculus MC^ν (based on ML^ν) can be introduced briefly as follows.

MA3.i EA3.1′ for $i \in \{1, \ldots, 8, 10, \ldots, 15\}$.
MA3.9 $\Box x = y \supset \Delta[z/x] = \Delta[z/y]$.
MA3.16,17 $\Box(p \supset q) .\supset. \Box p \supset \Box q$; $\Box p \supset p$.
MA3.18 $p \supset \Box p$, where p is modally closed.

In addition to EA3.1-EA3.15 [MA3.1-MA3.18], we assume:

(i) the closure [total closure] of an axiom of $EC^{\nu+1}$ [MC^ν] is an axiom of $EC^{\nu+1}$[MC^ν]
(ii) the only deduction rule in $EC^{\nu+1}$[MC^ν] is the modus ponens.

Definition 3.1 The *QI* $\xi[\zeta]$ of type $(t_1, \ldots, t_n)[(t_1, \ldots, t_n : t_0)]$ is said to be definable (in the ML^ν-interpretation $\mathcal{J}$) if there exist: (i) a $\mathcal{V} \in Val_{\mathcal{J}}$, (ii) an n-tuple $X = \langle x_1, \ldots, x_n\rangle$ of variables of type $t_1, \ldots, t_n$, respectively, and (iii) a wff p [a term $\Delta \in \mathcal{E}_{t_0}$] such that (3.1) [(3.2)] below holds.

(3.1) $\xi = d(p, X, \mathcal{J}, \mathcal{V}) = \{\langle \xi_1, \ldots, \xi_n, \gamma\rangle : \xi_i \in \mathcal{Q}\mathcal{I}_{t_i} (i = 1, \ldots, n)$ and $\gamma \in des_{\mathcal{J}\mathcal{V}'}(p)$, where $\mathcal{V}' = \mathcal{V}(x_1/\xi_1, \ldots, x_n/\xi_n)\}$.
(3.2) $\zeta = d(\Delta, X, \mathcal{J}, \mathcal{V}) = \{\langle\langle \zeta_1, \ldots, \zeta_n\rangle, des_{\mathcal{J}\mathcal{V}'}(\Delta)\rangle : \zeta_i \in \mathcal{Q}\mathcal{I}_{t_i} (i = 1, \ldots, n)$ and $\mathcal{V}' = \mathcal{V}(x_1/\zeta_1, \ldots, x_n/\zeta_n)\}$.

Here we omit the analogous definition for the $EL^{\nu+1}$-interpretations, since it is quite similar to Definition 3.1; the only relevant difference is in the correspondent of (3.1), which is

(3.3) $d(p, X, I, V) = \{\langle \xi_1, \ldots, \xi_n\rangle \in \Pi_i^n\, Ob_{s_i} : des_{IV'}(p) = 1$, where $V' = V(x_1/\xi_1, \ldots, x_n/\xi_n)\}$.

Definition 3.2 Let $\mathcal{J}[I]$ be an ML^ν-interpretation [$EL^{\nu+1}$-interpretation]. We shall say that $\mathcal{J}[I]$ is general if, for all $t \in \bar{\tau}^\nu[s \in \tau^{\nu+1}]$, every *QI* [object] of type $t[s]$, definable in $\mathcal{J}[I]$, belongs to $\mathcal{Q}\mathcal{I}_t[Ob_s]$.

Theorem 3.1 below is proved (by a Henkin's method) in [8], whereas the proof of Theorem 3.2 can be easily deduced from that of Theorem 2 in [4].

Theorem 3.1 (Completeness for MC^ν) *For every* $p \in \mathcal{E}_0$ *and* $K \subseteq \mathcal{E}_0$, $K \vdash_{MC^\nu} p$ *iff* $K \overset{g}{\models}_{MC^\nu} p$

Theorem 3.2 (Completeness for $EC^{\nu+1}$) *For every* $p \in E_0$ *and* $K \subseteq E_0$, $K \vdash_{EC^{\nu+1}} p$ *iff* $K \overset{g}{\models}_{EL^{\nu+1}} p$.

In this work the expression $d(\Delta, \langle x_1, \ldots, x_n\rangle, \mathcal{J}, \mathcal{V})$ will be often replaced with its equivalent $des_{\mathcal{J}\mathcal{V}}((\lambda x_1, \ldots, x_n)\Delta)$. It is worthwhile remarking, however, that the operator λ is defined in ML^ν by means of the operator ι:

$$(3.4)\quad \begin{cases}(\lambda x_1, \ldots, x_n)p =_d (\imath F)(\forall x_1, \ldots, x_n)(F(x_1, \ldots, x_n) \equiv p)\\ (\lambda x_1, \ldots, x_n)\Delta =_d (\imath f)(\forall x_1, \ldots, x_n) f(x_1, \ldots, x_n) = \Delta,\end{cases}$$

and hence the two expressions are equivalent only if $\mathcal{I}$ is a general ML^ν-interpretation. Analogously we shall write $des_{IV}((\lambda x_1, \ldots, x_n)\Delta)$ instead of $d(\Delta, \langle x_1, \ldots, x_n\rangle, I, V)$.

4 *Isomorphic ML^ν-interpretations and an additional axiom for MC^ν*

Definition 4.1 Let $\mathcal{I}$ and $\mathcal{I}'$ be two ML^ν-interpretations and let $\tilde{w}[w_t(t \in \bar{\tau}^\nu)]$ be a one-to-one correspondence between Γ and Γ' $[\mathcal{L}\mathcal{J}_t$ and $\mathcal{L}\mathcal{J}'_t]$. Then $w = \bigcup_{t\in\bar{\tau}^\nu} w_t \cup \tilde{w}$ is said to be an isomorphism between $\mathcal{I}$ and $\mathcal{I}'$ if the following conditions hold:

(a) $\xi_1, \xi_2 \in \mathcal{L}\mathcal{J}_r (r = 1, \ldots, \nu) \Rightarrow \xi_1(\gamma) = \xi_2(\gamma)$ iff $\xi_1^w(\gamma^w) = \xi_2^w(\gamma^w)$
(b) $\xi \in \mathcal{L}\mathcal{J}_0 \Rightarrow \xi^w = \{\gamma^w: \gamma \in \xi\}$
(c) $\xi \in \mathcal{L}\mathcal{J}_{(t_1,\ldots,t_n)} \Rightarrow \xi^w = \{\langle \xi_1^w, \ldots, \xi_n^w, \gamma^w\rangle: \langle \xi_1, \ldots, \xi_n, \gamma\rangle \in \xi\}$
(d) $\xi \in \mathcal{L}\mathcal{J}_{(t_1,\ldots,t_n:t_0)} \Rightarrow \xi^w(\xi_1^w, \ldots, \xi_n^w) = w(\xi(\xi_1, \ldots, \xi_n))$ for all $\langle \xi_1, \ldots, \xi_n\rangle \in \Pi_i^n \mathcal{L}\mathcal{J}_{t_i}$
(e) $a^{\nu\prime} = w(a^\nu), \mathcal{J}'(c_{tn}) = w(\mathcal{J}(c_{tn}))\ \ (t \in \tau^\nu, n \in Z^+)$.

As usual, two ML^ν-interpretations are said to be *isomorphic* if an isomorphism between them exists. Conditions (b) to (e) above correspond to the ordinary isomorphism conditions between arbitrary structures; whereas the use of (a) depends on the fact that we are dealing with ML^ν-interpretations. The basic entities for these structures are not individuals, but *individual concepts*, i.e., functions, and hence the lowest level relation is not the equality between individuals, but the equality between individual concepts in a given possible case. On the basis of this remark it is clear why no correspondence is assumed between the sets D_r and D'_r $(r = 1, \ldots, \nu)$.

If $\mathcal{I}$ and $\mathcal{I}'$ are isomorphic ML^ν-interpretations and $\mathcal{V} \in Val_\mathcal{I}$, then the $\mathcal{I}'$-valuation defined by $\mathcal{V}'(v_{tn}) = w(\mathcal{V}(v_{tn}))$ will be denoted by $\mathcal{V}^w$. The proof of the following theorem is based on an easy induction on the length of Δ.

Theorem 4.1 *Let w be an isomorphism between the ML^ν-interpretations $\mathcal{I}$ and $\mathcal{I}'$, and let Δ be any wfe of ML^ν. Then*

(4.1) $w(des_{\mathcal{I}\mathcal{V}}(\Delta)) = des_{\mathcal{I}'\mathcal{V}} w(\Delta)$, *for all* $\mathcal{V} \in Val_\mathcal{I}$.

Corollary 4.1 (i) *Any formula p of ML^ν is true [satisfiable] in the ML^ν-interpretation $\mathcal{I}$ iff it is true [satisfiable] in every ML^ν-interpretation isomorphic to $\mathcal{I}$.*
(ii) *If $\mathcal{I}$ is a general ML^ν-interpretation, then every ML^ν-interpretation isomorphic to it is general.*

In the remainder of this section we shall consider an example of isomorphic ML^ν-interpretations to be used later; it refers to those ML^ν-interpretations in which Axiom AS25.1 in [2] holds. Before writing this axiom we need to recall some definitions; in them F is a term of type $t = (t_1, \ldots, t_n)$ and $\bigwedge_i^n p_i$ denotes $p_1 \wedge \ldots \wedge p_n$.

(4.2) $Mconst_t(F) \equiv_d (\forall x_1, \ldots, x_n)(\Diamond F(x_1, \ldots, x_n) \equiv \Box F(x_1, \ldots, x_n))$

(4.3) $Msep_t(F) \equiv_d (\forall x_1, y_1, \ldots, x_n, y_n)\Big(F(x_1, \ldots, x_n) \wedge F(y_1, \ldots, y_n) \wedge \Diamond \bigwedge_i^n x_i = y_i \supset \bigwedge_i^n \Box x_i = y_i\Big)$

(4.4) $Abs_t(F) \equiv_d Mconst_t(F) \wedge Msep_t(F)$

(4.5) $F^{(e)}(x_1, \ldots, x_n) \equiv_d (\exists y_1, \ldots, y_n)\Big(\bigwedge_i^n x_i = y_i \wedge F(y_1, \ldots, y_n)\Big).$

For any given ML^ν-interpretation $\mathcal{J}$ we shall say that the relation $\tilde{R}$ (of type $t = (t_1, \ldots, t_n)$) in $\mathcal{J}$ is *modally constant* or *modally separated* or *absolute* if $des_{\mathcal{J}\mathcal{V}}(Mconst_t(R)) = \Gamma$ or $des_{\mathcal{J}\mathcal{V}}(Msep_t(R)) = \Gamma$ or $des_{\mathcal{J}\mathcal{V}}(Abs_t(R)) = \Gamma$, respectively, when $des_{\mathcal{J}\mathcal{V}}(R) = \tilde{R}$.

In the sequel we shall use the convention of writing "$\langle \xi_1, \ldots, \xi_n \rangle \,\bar{\epsilon}\, \tilde{R}$" for "$\langle \xi_1, \ldots, \xi_n, \gamma \rangle \in \tilde{R}$ for all $\gamma \in \Gamma$". In case $n = 1$, we shall often write $\xi_1 \,\bar{\epsilon}\, \tilde{R}$ for $\langle \xi_1 \rangle \,\bar{\epsilon}\, \tilde{R}$.

Axiom AS25.1 in [2] is

MA4.1 $\quad \Box(\exists F)[Abs_{(r)}(F) \wedge F(a_r^*) \wedge \Box(x)F^{(e)}(x)] \quad (r = 1, \ldots, \nu).$

Let $\mathcal{J}$ be an ML^ν-interpretation in which MA4.1 is true. Then, for every $r \in \{1, \ldots, \nu\}$, there exists a *QI* $\tilde{F}_r$, of type (r), that satisfies the following conditions:

(4.6) $\langle \xi, \gamma \rangle \in \tilde{F}_r$ for some $\gamma \in \Gamma \Rightarrow \langle \xi \rangle \,\bar{\epsilon}\, \tilde{F}_r$

(4.7) $\langle \xi_1 \rangle, \langle \xi_2 \rangle \,\bar{\epsilon}\, \tilde{F}_r$ and $\xi_1 =_\gamma \xi_2$ for some $\gamma \in \Gamma \Rightarrow \xi_1 = \xi_2$

(4.8) $\xi \in \mathcal{Q}\mathcal{I}_r$ and $\gamma \in \Gamma \Rightarrow \xi =_\gamma \xi'$ for some $\langle \xi' \rangle \,\bar{\epsilon}\, \tilde{F}$

(4.9) $\langle a_r^\nu \rangle \,\bar{\epsilon}\, \tilde{F}_r$,

Let us assume $a_r^\nu(\gamma) = a_r (\in D_r)$ for all $\gamma \in \Gamma$; in this case the set $\tilde{C}_r$, defined by

(4.10) $\tilde{C}_r = \{\langle \xi, \gamma \rangle : \xi$ is a constant function in $(\Gamma \to D_r), \gamma \in \Gamma\}$,

fulfills Conditions 4.6-4.9. Since we are dealing with arbitrary (and not standard) ML^ν-interpretations, we cannot know whether $\tilde{C}_r$ is in $\mathcal{Q}\mathcal{I}_{(r)}$ or not; however we can show that the assumption $\tilde{C}_r \in \mathcal{Q}\mathcal{I}_{(r)}$ (in addition to $\langle a_r^\nu \rangle \,\bar{\epsilon}\, \tilde{C}_r$) does not cause loss of generality. Intuitively, we can do this since the basic relation in an ML^ν-interpretation is the equality between individual concepts in a given possible case, and the comparison between the values of individual concepts at different cases has in general no meaning. Thus, for every $\gamma \in \Gamma$ we can change the values at γ of the elements of $\mathcal{Q}\mathcal{I}_r$ (taking care to preserve the equalities) in order to render the *QI*s ξ such that $\langle \xi \rangle \,\bar{\epsilon}\, F_r$, constant functions.

Theorem 4.2 *Let $\mathcal{J}$ be an ML^ν-interpretation in which MA4.1 is true, and let $\tilde{F}_r$ be an element of $\mathcal{Q}\mathcal{I}_{(r)}$ satisfying* (4.6-4.9) *($r \in \{1, \ldots, \nu\}$). Then there exist an ML^ν-interpretation $\mathcal{J}'$ and an isomorphism w between $\mathcal{J}$ and $\mathcal{J}'$, such that $w(\tilde{F}_r) = \tilde{C}_r$.*

Proof: Let $D_r' = D_r$ $(r = 1, \ldots, \nu)$, $\Gamma' = \Gamma$ and $w(\gamma) = \gamma$ for all $\gamma \in \Gamma$. For every $\xi \in \mathcal{Q}\mathcal{I}_r$ and $\gamma \in \Gamma$, let ξ_γ be the unique element of $\mathcal{Q}\mathcal{I}_r$ such that $\langle \xi_\gamma \rangle \,\bar{\epsilon}\, \tilde{F}_r$ and $\xi_\gamma(\gamma) = \xi(\gamma)$ (see (4.8)). Now we fix (once and for all) a $\bar{\gamma} \in \Gamma$ and we set

$\xi^w(\gamma^w) = \xi_\gamma(\bar{\gamma})$ for all γ in Γ. Condition (a) of Definition 4.1 holds for this w, indeed $\xi(\gamma) = \zeta(\gamma) \Longleftrightarrow \xi_\gamma = \zeta_\gamma \Longleftrightarrow \xi_\gamma(\bar{\gamma}) = \zeta_\gamma(\bar{\gamma}) \Longleftrightarrow \xi^w(\gamma^w) = \zeta^w(\gamma^w)$. Hence, if we extend w to QIs of higher type-level, by means of (b) to (e) of Definition 4.1, then it turns out to be an isomorphism between $\mathcal{J}$ and the resulting interpretation $\mathcal{J}'$. Let us consider the QIs $\xi^w(\in \mathcal{Q}\mathcal{I}_r')$ for $\langle\xi\rangle \mathrel{\bar{\in}} \tilde{F}_r$. For these ξ we have $\xi_\gamma = \xi$ and $\xi^w(\gamma^w) = \xi(\bar{\gamma})$ for all $\gamma \in \Gamma$; that is, ξ^w is a constant function from Γ' into D_r'.

Constant functions from possible cases to individuals are called subsistents. By Theorem 4.2, it is natural to extend this notion and call subsistent every QI ξ such that, for some $\tilde{F}_r$ fulfilling (4.6-4.9), $\langle\xi\rangle \mathrel{\bar{\in}} \tilde{F}_r$ (see, e.g., [1]).

5 Representatives of the possible cases in a given general ML^ν-interpretation and a further axiom for MC^ν Let $\mathcal{J}$ be any given general ML^ν-interpretation. In order to make our investigation easier, first note that by the designation rules (δ_1) to (δ_9) (Section 2), for every wfe Δ and every $\mathcal{V} \in Val_{\mathcal{J}}$, $des_{\mathcal{J}\mathcal{V}}(\Delta)$ does not change if we replace $\mathcal{Q}\mathcal{I}_0$ with an arbitrary $QI_0 \supseteq \{des_{\mathcal{J}\mathcal{V}}(p) : p \in \mathcal{E}_0$ and $\mathcal{V} \in Val_{\mathcal{J}}\}$. On the other hand, $des_{\mathcal{J}\mathcal{V}}(p) \in \mathcal{Q}\mathcal{I}_0$ for all p and $\mathcal{V}$. Therefore without any loss of generality we can adopt the following hypothesis.

Hypothesis 5.1 $\mathcal{Q}\mathcal{I}_0 = \{des_{\mathcal{J}\mathcal{V}}(p) : p \in \mathcal{E}_0 \text{ and } \mathcal{V} \in Val_{\mathcal{J}}\}$.

Second, suppose that a subset Γ' of Γ exists such that, for all $p \in \mathcal{E}_0$ and all $\mathcal{V} \in Val_{\mathcal{J}}$, $\Gamma' \subseteq des_{\mathcal{J}\mathcal{V}}(p)$ or $des_{\mathcal{J}\mathcal{V}}(p) \subseteq \Gamma - \Gamma'$. In this case all elements of Γ' behave in the same way with respect to the designata of wffs of ML^ν in $\mathcal{J}$, and hence we can replace Γ' with the singleton $\{\bar{\gamma}\}$ for an arbitrary $\bar{\gamma} \in \Gamma'$ (which means in particular that $\bar{\gamma}$ is substituted for every element of Γ' in all QIs of $\mathcal{J}$). Hence no loss of generality is practically afforded by the following:

Hypothesis 5.2 *If $\gamma_1, \gamma_2 \in \Gamma$ and $\gamma_1 \neq \gamma_2$, then there exist a wff p and a $\mathcal{V} \in Val_{\mathcal{J}}$ such that $\gamma_1 \notin des_{\mathcal{J}\mathcal{V}}(p)$ and $\gamma_2 \in des_{\mathcal{J}\mathcal{V}}(p)$.*

Now we consider a new axiom for MC^ν which allows us to construct in $\mathcal{J}$ a QI $\tilde{\Gamma}$ that will be shown to correspond to the set Γ of possible cases. This axiom is A14''' (II) in [6] and Definitions 5.2 and 5.3 below of "*Modally minimal property*" and "*Actual Elementary Case*" are introduced in that paper too.

(5.1) $F \subseteq H \equiv_d (\forall x_1, \ldots, x_n)(F(x_1, \ldots, x_n) \supset H(x_1, \ldots, x_n))$.
(5.2) $Mmin_t(F) \equiv_d (H)(F \subseteq H \supset \Box F \subseteq H)$, (where $F \in \mathcal{E}_t$).
(5.3) $AEC(c) \equiv_d Mmin_{(1)}(c) \wedge (v_{11})c(v_{11})$.

MA5.1 $\Box(\exists c)AEC(c)$.

The meaning of (5.1-5.3) is easily understood; in particular, for $F \in \mathcal{E}_{(t)}$ and $des_{\mathcal{J}\mathcal{V}}(F) = \tilde{F}$, we have

(5.4) $\gamma \in des_{\mathcal{J}\mathcal{V}}(Mmin_{(t)}(F)) \Longleftrightarrow$ for all $\xi \in \mathcal{Q}\mathcal{I}_{(t)}$, $\tilde{F} \cap (\mathcal{Q}\mathcal{I}_t \times \{\gamma\}) \subseteq \xi \cap (\mathcal{Q}\mathcal{I}_t \times \{\gamma\})$ implies $\tilde{F} \subseteq \xi$.

The extension of (5.4) to other cases (in which F has an arbitrary relation type) is obvious; thus the above restriction on F is also assumed in the following lemma.

Lemma 5.1 *Let us assume:* $\mathcal{V} \in Val_{\mathcal{J}}$, $F \in \mathcal{E}_{(t)}(t \in \tau^{\nu})$, $des_{\mathcal{J}\mathcal{V}}(F) = \tilde{F}$, *and* $\zeta = des_{\mathcal{J}\mathcal{V}}(Mmin_{(t)}(F))$. *Then:* (1) $\tilde{F} = \phi$ *and* $\zeta = \Gamma$, *or* (2) $\tilde{F} \neq \phi$ *and* $\zeta = \phi$, *or* (3) $\tilde{F} \neq \phi$ *and* $\zeta = \{\gamma\}$, *for some* $\gamma \in \Gamma$.

Proof: If $\tilde{F} = \phi$, then (1) holds trivially. Let $\tilde{F} \neq \phi$ and let us suppose $\gamma_1, \gamma_2 \in \zeta$ $(\gamma_1 \neq \gamma_2)$. Remark first that $\tilde{F} \cap (\mathcal{Q}\mathcal{J}_t \times \{\gamma_1\})$ is nonempty: otherwise we could substitute $a^{\nu}_{(t)}(= \phi)$ for ξ and γ_1 for γ in (5.4) and obtain $\tilde{F} = \phi$. By Hypothesis 5.2 there exist a $p \in \mathcal{E}_0$ and a $\mathcal{V}' \in Val_{\mathcal{J}}$ such that $\gamma_1 \notin des_{\mathcal{J}\mathcal{V}'}(p)$ and $\gamma_2 \in des_{\mathcal{J}\mathcal{V}'}(p)$. Let x be any variable of type t not free in p and let ξ be $d(p \wedge x = x, \langle x \rangle, \mathcal{J}, \mathcal{V}')$; by (3.1), $(\mathcal{Q}\mathcal{J}_t \times \{\gamma_2\}) \subseteq \xi$ and $(\mathcal{Q}\mathcal{J}_t \times \{\gamma_1\}) \cap \xi = \phi$. Hence $(\mathcal{Q}\mathcal{J}_t \times \{\gamma_2\}) \cap \tilde{F} \subseteq (\mathcal{Q}\mathcal{J}_t \times \{\gamma_2\}) \cap \xi$ and $(\mathcal{Q}\mathcal{J}_t \times \{\gamma_1\}) \cap \xi \subset (\mathcal{Q}\mathcal{J}_t \times \{\gamma_1\}) \cap \tilde{F}$, which contradicts (5.4).

The proofs of Lemma 5.1 and Definition (5.3) imply the following

Lemma 5.2 *For every* $\mathcal{V} \in Val_{\mathcal{J}}$ *and* $\gamma \in \Gamma$, $des_{\mathcal{J}\mathcal{V}}(AEC(c)) = \{\gamma\}[\phi]$ *iff* $\mathcal{V}(c) = [\neq]\ (\mathcal{Q}\mathcal{J}_1 \times \{\gamma\})$.

Theorem 5.1 *MA5.1 is true in* $\mathcal{J}$ *iff* $\{\gamma\} \in \mathcal{Q}\mathcal{J}_0$ *for all* $\gamma \in \Gamma$.

Proof: Let $des_{\mathcal{J}\mathcal{V}}((\exists c)AEC(c)) = \Gamma$ and let $\gamma \in \Gamma$. Then there is a $\mathcal{V}'$ such that $\gamma \in des_{\mathcal{J}\mathcal{V}'}(AEC(c))$ and, by Lemma 5.2, $\{\gamma\} = des_{\mathcal{J}\mathcal{V}'}(AEC(c))$ which belongs to $\mathcal{Q}\mathcal{J}_0$. Conversely, suppose $\{\gamma\} \in \mathcal{Q}\mathcal{J}_0$ for all $\gamma \in \Gamma$. By Hypothesis 5.1, for every $\gamma \in \Gamma$ there are a $p \in \mathcal{E}_0$ and a $\mathcal{V} \in Val_{\mathcal{J}}$ such that $des_{\mathcal{J}\mathcal{V}}(p) = \{\gamma\}$. Let x be a variable of type 1 not free in p and let $\tilde{c} = d(p \wedge x = x, \langle x \rangle, \mathcal{J}, \mathcal{V})$. Obviously $\tilde{c} = (\mathcal{Q}\mathcal{J}_1 \times \{\gamma\})$, so that $des_{\mathcal{J}\mathcal{V}'}(AEC(c)) = \{\gamma\}$ for $\mathcal{V}'(c) = \tilde{c}$. Since γ was chosen arbitrarily, $des_{\mathcal{J}\mathcal{V}}((\exists c)AEC(c)) = \Gamma$ for all $\mathcal{V}$, and hence MA5.1 is true in $\mathcal{J}$.

Corollary 5.1 *If* $\mathcal{Q}\mathcal{J}_0$ (in $\mathcal{J}$) *is a finite set, then* MA5.1 *is true in* $\mathcal{J}$.

Proof: Let γ be any element of Γ and let $Y = \{\xi \in \mathcal{Q}\mathcal{J}_0 : \gamma \in \xi\}$. By Hypothesis 5.1, $\xi, \zeta \in Y$ implies $\xi \cap \zeta \in Y$ and, by Hypothesis 5.2, $\bigcap Y = \{\gamma\}$. But Y is finite and hence $\{\gamma\} \in Y$.

In what follows $\mathcal{Q}\mathcal{J}_1 \times \{\gamma\}$ will be denoted by $\tilde{\gamma}$. Since $des_{\mathcal{J}\mathcal{V}}(AEC(c)) = \{\gamma\}$ implies $\mathcal{V}(c) = \tilde{\gamma}$, an immediate consequence of MA5.1 is that every $\gamma \in \Gamma$ has a representative $\tilde{\gamma}$ in $\mathcal{Q}\mathcal{J}_{(1)}$. Now, in order to construct a representative (in $\mathcal{Q}\mathcal{J}_{((1))}$) of the set Γ, let us consider the QI $\tilde{\Gamma} = d(\Diamond AEC(c), \langle c \rangle, \mathcal{J}, \mathcal{V})$, which does not depend on $\mathcal{V}$ since c is the only free variable in $AEC(c)$. $\tilde{\Gamma}$ is a modally constant relation (since $\Diamond AEC(c)$ is a modally closed wff) so that, by Lemma 5.2,

$$(5.5) \quad \tilde{\Gamma} = \{\langle (\mathcal{Q}\mathcal{J}_1 \times \{\gamma\}), \gamma' \rangle : \gamma, \gamma' \in \Gamma\} = \{\langle \tilde{\gamma}, \gamma' \rangle : \gamma, \gamma' \in \Gamma\}.$$

In particular, $\langle \tilde{\gamma} \rangle \mathrel{\bar{\epsilon}} \tilde{\Gamma}$ for all $\gamma \in \tilde{\Gamma}$; thus $\tilde{\Gamma}$ can be thought as a representative (in $\mathcal{Q}\mathcal{J}_{((1))}$) of the set Γ. Finally remark that $\tilde{\gamma}_1 =_{\gamma} \tilde{\gamma}_2$ implies $\tilde{\gamma}_1 = \tilde{\gamma}_2$ and hence $\tilde{\Gamma}$ is an absolute relation.

Theorem 5.2 *If the variable* $c(\in \mathcal{E}_{(1)})$ *does not occur free in the wff p, then*

$$(5.6) \quad \{\text{MA5.1}\} \vdash_{MC^{\nu}} \Diamond p \equiv (\exists c)\Diamond(AEC(c) \wedge p)$$

$$(5.7) \quad \{\text{MA5.1}\} \vdash_{MC^{\nu}} \Box p \equiv (c)\Box(AEC(c) \supset p).$$

Proof: Remark first that (5.6) implies {MA5.1} $\vdash_{\overline{MC^\nu}} \sim\Diamond\sim p \equiv \sim(\exists c)\Diamond(AEC(c) \wedge \sim p)$, which is equivalent to (5.7). As for the proof of (5.6) we use Theorem 3.1 and we prove that, for every general ML^ν-interpretation $\mathcal{I}$ in which MA5.1 is true, and every $\mathcal{V} \in Val_{\mathcal{I}}$, $des_{\mathcal{I}\mathcal{V}}(\Diamond p \equiv (\exists c)\Diamond(AEC(c) \wedge p)) = \Gamma$. Let $des_{\mathcal{I}\mathcal{V}}(\Diamond p) = \Gamma$ and let $\gamma \in des_{\mathcal{I}\mathcal{V}}(p)$. Then (by MA5.1) $\gamma \in des_{\mathcal{I}\mathcal{V}'}(AEC(c) \wedge p)$ for $\mathcal{V}' = \mathcal{V}(c/\tilde{\gamma})$, and $des_{\mathcal{I}\mathcal{V}'}(\Diamond(AEC(c) \wedge p)) = \Gamma$; hence $des_{\mathcal{I}\mathcal{V}}((\exists c)\Diamond(AEC(c) \wedge p)) = \Gamma$. Conversely, $des_{\mathcal{I}\mathcal{V}}((\exists c)\Diamond(AEC(c) \wedge p)) = \Gamma$ implies $des_{\mathcal{I}\mathcal{V}'}(AEC(c) \wedge p) \neq \phi$ for a suitable $\mathcal{V}' = \mathcal{V}(c/\xi)$. But c is not free in p and hence $des_{\mathcal{I}\mathcal{V}}(\Diamond p) = \Gamma$.

Theorem 5.2 shows that the elements of $\tilde{\Gamma}$ behave just like case variables. A similar result is achieved in [2] (NN.47-49) by using a concept ($|_u$) analogue to $AEC(c)$. The proof of Theorem 49.2 in [2] (which is the analogue of Theorem 5.2) uses an axiom asserting the existence of a contingent proposition and the strong axiom of "*extensional comprehension*":

(5.8) $\Box(\exists F)(\forall x_1, \ldots, x_n)(F(x_1, \ldots, x_n) \equiv p \,.\wedge.\, Mconst(F))$.

The former (of these axioms) will not be used in this work since we do not exclude the possibility that Γ may have one element. Instead, (5.8) seems to be necessary in order to express the set of possible cases. Indeed in [6] the equivalence (in MC^ν) between (5.8) and the conjunction of MA5.1 with $\Box(\exists F)(\forall x_1, \ldots, x_n)\Box(F(x_1, \ldots, x_n) \equiv p)$ is proved, and this last formula is provable from MA3.1-MA3.18 (see [2], N40); so that the calculus based on MA3.1-MA3.18 and MA5.1 is equivalent to that based on MA3.1-MA3.18 and (5.8).

Part II. An extension of the equivalence theorem

6 *Translation of ML^ν into $EL^{\nu+1}$* In [2] N15 a function η, which translates ML^ν into $EL^{\nu+1}$, is defined on the basis of the semantical rules for ML^ν, substantially by assuming a particular variable $\chi = v_{\nu+1,1}$ of $EL^{\nu+1}$ to represent possible cases; thus, the modal operator $\Box$ shall be translated into the universal quantifier (χ).

In the translation rules (T_1) to (T_9) below the following (metalinguistic) abbreviations (6.1-6.4) are used, in them χ denotes $v_{\nu+1,1}$, $a, b \in E_{r^\eta}(r \in \{1, \ldots, \nu\})$, $F, G \in E_{(t_1,\ldots,t_n)^\eta}$, and $f, g \in E_{(t_1,\ldots,t_n:t_0)^\eta}$, where $t_0, t_1, \ldots, t_n \in \tau^\nu$ (cf. (2.4)).

(6.1) $a =_\chi b \equiv_d a(\chi) = b(\chi)$.

(6.2) $F =_\chi G \equiv_d (\forall x_1, \ldots, x_n)(F(x_1, \ldots, x_n, \chi) \equiv G(x_1, \ldots, x_n, \chi))$.

(6.3) $f =_\chi g \equiv_d (\forall x_1, \ldots, x_n) f(x_1, \ldots, x_n) =_\chi g(x_1, \ldots, x_n)$.

(6.4) $(\exists! x)p \equiv_d (\exists x)(p \wedge (y)(p[x/y] \supset x =_\chi y))$.

Let us remark that the written occurrence of χ in $\Delta_1 =_\chi \Delta_2$ is free, and hence χ has at least one free occurrence in $(\exists! x)p$.

(T_1) $(v_{tn})^\eta = v_{t^\eta n}$, $(c_{tn})^\eta = c_{t^\eta n}$

(T_2) $(\Delta_1 = \Delta_2)^\eta = \Delta_1^\eta =_\chi \Delta_2^\eta$

(T_3) $(R(\Delta_1, \ldots, \Delta_n))^\eta = R^\eta(\Delta_1^\eta, \ldots, \Delta_n^\eta, \chi)$

(T_4) $(f(\Delta_1, \ldots, \Delta_n))^\eta = f^\eta(\Delta_1^\eta, \ldots, \Delta_n^\eta)$

$(T_{5\text{-}6})$ $(\sim p)^\eta = \sim p^\eta$; $(p \wedge q)^\eta = p^\eta \wedge q^\eta$

($T_{7\text{-}8}$) $((x)p)^\eta = (x^\eta)p^\eta; (\Box p)^\eta = (\chi)p^\eta$
(T_9) $((\imath x)p)^\eta = (\imath x^\eta)(\chi)[(\exists!z)p^\eta[x^\eta/z] \wedge (z)(p^\eta[x^\eta/z] \supset x^\eta =_\chi z) .\vee. \sim(\exists!z)p^\eta[x^\eta/z] \wedge x^\eta =_\chi a^*_{t\eta}]$.

These translation rules imply that: (1) if Δ is a term of ML^ν, then Δ^η is closed with respect to the variable χ, and (2) if p is a formula of ML^ν, then p^η is closed with respect to χ iff p is modally closed.

In [2] the definition of "extensional correspondent" of a given ML^ν-interpretation is slightly different from Definition 2.1; in any case the proof of Theorem 16.1 in [2] can be easily turned into a proof of the following theorem.

Theorem 6.1 *Assume that:* (1) $\boldsymbol{I}$ *is an* $EL^{\nu+1}$*-interpretation and* $V \in Val_{\mathcal{I}}$, (2) $\mathcal{I} = \boldsymbol{I}^i$ *and* $\mathcal{V}(\in Val_{\mathcal{I}})$ *satisfies:* $\mathcal{V}(v_{tn}) = V((v_{tn})^\eta)$ *for all* $t \in \tau^\nu$, $n \in Z^+$, *and* (3) p *is a wff of* ML^ν *and* Δ *is a term of* ML^ν. *Then*

(a) $des_{\mathcal{I}\mathcal{V}}(\Delta) = des_{IV}(\Delta^\eta)$,
(b) *for every* $\gamma \in \Gamma$, $\gamma \in des_{\mathcal{I}\mathcal{V}}(p)$ *iff* $des_{IV'}(p^\eta) = 1$ *when* $V' = V(\chi/\gamma)$.

By (3.1-3.3), a corollary of Theorem 6.1 is that, if its assumptions hold, then

(6.5) $d(p, \langle x_1, \ldots, x_n\rangle, \mathcal{I}, \mathcal{V}) = d(p^\eta, \langle x_1^\eta, \ldots, x_n^\eta, \chi\rangle, \boldsymbol{I}, V)$
(6.6) $d(\Delta, \langle x_1, \ldots, x_n\rangle, \mathcal{I}, \mathcal{V}) = d(\Delta^\eta, \langle x_1^\eta, \ldots, x_n^\eta\rangle, \boldsymbol{I}, V)$,

which prove (cf. Definitions 2.1, 3.2):

Theorem 6.2 *If* $\boldsymbol{I}$ *is a general* $EL^{\nu+1}$*-interpretation then* $\boldsymbol{I}^i$ *is a general* ML^ν*-interpretation.*

Corollary 6.1 *If the formula* p *of* ML^ν *is true in every general* ML^ν*-interpretation, then* p^η *is true in every general* $EL^{\nu+1}$*-interpretation.*

Proof: If p^η is not true in the general $EL^{\nu+1}$-interpretation $\boldsymbol{I}$, then (by Theorem 6.1) it is not true in $\boldsymbol{I}^i$, which is general by Theorem 6.2.

*7 **On the equivalence between*** MC^ν ***and*** $EC^{\nu+1}$ From now on we assume the calculus MC^ν to be based on MA3.1-MA3.18, MA4.1, and MA5.1, and by general ML^ν-interpretation (in the second sense) we mean a general ML^ν-interpretation in the first sense in which MA4.1 and MA5.1 are true (that is, a model of the considered version of MC^ν).

Theorem 7.1 *If the formula* p *is a theorem of* MC^ν, *then* p^η *is a theorem of* $EC^{\nu+1}$.

Proof: By the completeness theorems Theorems 3.1, 3.2, and Corollary 6.1, we have only to prove that the translations (by η) of MA4.1 and MA5.1 are true in every general $EL^{\nu+1}$-interpretation. Let $\boldsymbol{I}$ be any such an interpretation and let q be the wff (in $EL^{\nu+1}$) $((\exists x)(w)f(w) = x) \wedge w = w$, where x, f, and w are variables of type $r(\in \{1, \ldots, \nu\})$, $(\nu + 1\colon r)$, and $\nu + 1$, respectively. Obviously $d(q, \langle f, w\rangle, \boldsymbol{I}, V)$ (which does not depend on V) is the set of all ordered pairs $\langle \bar{f}, \gamma\rangle$ such that $\bar{f}$ is a constant function from $Ob_{\nu+1}$ into Ob_r and $\gamma \in Ob_{\nu+1}$, and hence the translation (by η) of MA4.1 holds in $\boldsymbol{I}$ (cf. Section 4). In a similar way the translation of MA5.1 can be proved to hold in $\boldsymbol{I}$, by identifying q with

$x = x \wedge z = z_1$ (where x, z, and z_1 are variables of type 1^{η}, $\nu + 1$, and $\nu + 1$, respectively).

Let us remark that, since MA4.1 is independent of MA3.1-MA3.18, Theorem 7.1 implies that its converse does not hold if we let MC^{ν} be based on MA3.1-MA3.18. As for MA5.1, in this paper we assume that it is independent of MA3.1-MA3.18 and MA4.1 (and actually I think the proof of Gallin (cf., footnote 1) could be extended to prove this), so that this axiom is effectively needed to prove the converse of Theorem 7.1. This fact has an intuitive justification in remarking that in ML^{ν} no expression denotes an individual or a possible case, whereas in $EL^{\nu+1}$ this happens; and hence it is not surprising that the new axioms needed are those which allow us to have expressions (and hence *QI*s) that represent in a certain sense the elements of D_r $(r \in \{1, \ldots, \nu\})$ and Γ (cf. Sections 4, 5).

The proof of the converse of Theorem 7.1 (carried out explicitly in the next sections) is based on the statement that every general ML^{ν}-interpretation has a general extensional correspondent, and this result will be achieved by defining a suitable extensional correspondent $\boldsymbol{I}$ of an arbitrarily fixed general ML^{ν}-interpretation $\mathcal{J}$ and by verifying that $\boldsymbol{I}$ is general. Of course, the crucial point concerns the definability in $\boldsymbol{I}$; indeed, since the sets $Ob_{t^{\eta}}(t \in \bar{\tau}^{\nu})$ are uniquely determined by means of (2.5), the remaining sets Ob_s $(s \neq t^{\eta}$, for all $t \in \bar{\tau}^{\nu})$ must be small [large] enough to fulfill condition (1) [(2)] below: (1) every object of type $t^{\eta}(t \in \bar{\tau}^{\nu})$, definable in $\boldsymbol{I}$, has to belong to $Ob_{t^{\eta}}$ and (2) every Ob_s has to be closed with respect to definability in $\boldsymbol{I}$.

In order to overcome this difficulty, we shall define the sets $Ob_s(s \in \tau^{\nu+1})$ by means of certain representatives of them in $\mathcal{J}$ itself; in this way the problems concerning the definability in $\boldsymbol{I}$ turn out to be reducible to similar problems in $\mathcal{J}$, which are trivial since $\mathcal{J}$ is general.

Unless otherwise stated, from now on $\mathcal{J}$ will denote a general ML^{ν}-interpretation, fixed arbitrarily except that

(7.1) $\tilde{C}_r \in \mathcal{QI}_{(r)}$, $\langle a_r^{\nu}\rangle \,\bar{\in}\, \tilde{C}_r$ (cf. (4.10)) $(r \in \{1, \ldots, \nu\})$,

(7.2) $\mathcal{QI}_0 = \{des_{\mathcal{J}\mathcal{V}}(p)\colon p \in \mathcal{E}_0 \text{ and } \mathcal{V} \in Val_{\mathcal{J}}\}$,

which causes no loss of generality by Theorem 4.2 and Hypothesis 5.1.

8 Construction of an extensional correspondent I of $\mathcal{J}$ For all $s \in \tau^{\nu+1}$, the type $s^*(\in \tau^{\nu})$ is defined recursively by

$$(8.1)\quad \begin{cases} r^* = r\ (r = 1, \ldots, \nu),\ \nu + 1^* = (1) \\ (s_1, \ldots, s_n)^* = (s_1^*, \ldots, s_n^*) \\ (s_1, \ldots, s_n\colon s_0)^* = (s_1^*, \ldots, s_n^*\colon s_0^*). \end{cases}$$

In order to define every set $Ob_s(s \in \tau^{\nu+1})$ for the extensional correspondent $\boldsymbol{I}$ of $\mathcal{J}$ to be constructed, we first consider the representative X_s of it (in $\mathcal{J}$), which turns out to be a subset of $\mathcal{QI}_{s^*}$:

(8.2) $X_r = \{\xi\colon \langle\xi\rangle \,\bar{\in}\, \tilde{C}_r\}$ $(r = 1, \ldots, \nu)$ $X_{\nu+1} = \{\xi\colon \langle\xi\rangle \,\bar{\in}\, \tilde{\Gamma}\}$ (cf. (4.10) and (5.5))

(8.3) $X_{(s_1,\ldots,s_n)} = \{\tilde{R} \in \mathcal{QI}_{(s_1,\ldots,s_n)^*}\colon \tilde{R}$ is modally constant and $\tilde{R} \in \mathcal{P}((\Pi_i^n X_{s_i}) \times \Gamma)\}$

(8.4) $X_{(s_1,\ldots,s_n:s_0)} = \{\tilde{f} \in \mathcal{L}\mathcal{I}_{(s_1,\ldots,s_n:s_0)^*}$: for all $\alpha \in \Pi_i^n \mathcal{L}\mathcal{I}_{s_i}$, $\alpha \in [\notin]\ \Pi_i^n X_{s_i} \Rightarrow \tilde{f}(\alpha) \in X_{s_0}[\tilde{f}(\alpha) = a^{\nu}_{s_0^*}]\}$.

The elements of $X_r (r = 1, \ldots, \nu)$ are constant functions from Γ into D_r, and those of $X_{\nu+1}$ have the form $\{\gamma\} \times \mathcal{L}\mathcal{I}_1$ $(\gamma \in \Gamma)$, so that a natural bijective correspondence σ exists between $X_1, \ldots, X_{\nu+1}$ and the sets $D_1, \ldots, D_{\nu+1} (= \Gamma)$ which the ML^{ν}-interpretation $\mathcal{J}$ is based on: for all ξ in X_r $(r \in \{1, \ldots, \nu\})$ and all $\tilde{\gamma} = \{\gamma\} \times \mathcal{L}\mathcal{I}_1$ in $X_{\nu+1}$, we set

(8.5) $\sigma(\xi)$ = the only element of the range of ξ; $\sigma(\tilde{\gamma}) = \gamma$.

Now the function σ can be extended in a natural way to every X_s, by setting (for arbitrary $\tilde{R} \in X_{(s_1,\ldots,s_n)}$ and $\tilde{f} \in X_{(s_1,\ldots,s_n:s_0)}$)

(8.6) $\sigma(\tilde{R}) = \{\langle\sigma(\xi_1), \ldots, \sigma(\xi_n)\rangle : \langle\xi_1, \ldots, \xi_n\rangle \,\tilde{\in}\, \tilde{R}\}$,
(8.7) $\sigma(\tilde{f}) = \{\langle\langle\sigma(\xi_1), \ldots, \sigma(\xi_n)\rangle, \sigma(\tilde{f}(\xi_1, \ldots, \xi_n))\rangle : \xi_i \in X_{s_i} (i = 1, \ldots, n)\}$.

The sets $\sigma(X_s)(s \in \tau^{\nu+1})$ fulfill conditions (2.1-2.3) for Ob-structures. Thus, if we assume

(8.8) $Ob_s = \sigma(X_s)$ $(s \in \bar{\tau}^{\nu+1})$

(where $\sigma(X_0) = \{0, 1\}$ is obviously understood) then the set $S = \{Ob_s : s \in \bar{\tau}^{\nu+1}\}$ turns out to be an Ob-structure which is sound for the definition of an extensional correspondent I of $\mathcal{J}$, since it is based on the sets $D_1, \ldots, D_\nu$, and Γ. Now we have to prove that $\mathcal{L}\mathcal{I}_t = Ob_{t^\eta}$, for all $t \in \bar{\tau}^{\nu}$ (in addition to defining in a suitable way the function $a^{\nu+1}$ and the valuation I of the constants of $EL^{\nu+1}$).

Lemma 8.1 *The function σ, defined by means of* (8.5-8.7), *is one-to-one.*

Proof: We have already observed that σ is a bijective correspondence between X_r and $D_r (r = 1, \ldots, \nu + 1)$. Let us assume inductively σ to be one-to-one on the sets $X_{s_0}, X_{s_1}, \ldots, X_{s_n}$. Then, by (8.6), $\xi_1, \xi_2 \in X_{(s_1,\ldots,s_n)}$ and $\sigma(\xi_1) = \sigma(\xi_2)$ imply $\xi_1 = \xi_2$ since they are modally constant. Assume now $\xi_1, \xi_2 \in X_{(s_1,\ldots,s_n:s_0)}$ and $\sigma(\xi_1) = \sigma(\xi_2)$. By (8.4), $\xi_1(\alpha) = \xi_2(\alpha) = a^{\nu}_{s_0^*}$ for all α in $(\Pi_i^n \mathcal{L}\mathcal{I}_{s_i^*} - \Pi_i^n X_{s_i})$; hence $\xi_1 = \xi_2$ by (8.7) and the inductive hypothesis.

In order to avoid frequent repetitions, the following conventions are assumed from now on (it is easy to realize that they do not cause any loss of generality): (1) C_r denotes $v_{(r)1} (r = 1, \ldots, \nu)$, (2) c denotes $v_{(1)2}$ and the formula $AEC(c)$ is defined by means of (5.3), (3) for every $\mathcal{V} \in Val_{\mathcal{J}}$, $\mathcal{V}(C_r) = \tilde{C}_r (r = 1, \ldots, \nu)$, (4) no wfe contains free occurrences of $C_1, \ldots, C_\nu$, and c, except those written explicitly.

Propositions 8.1-8.4 below concern the use of the QIs $\tilde{C}_1, \ldots, \tilde{C}_\nu$, and $\tilde{\Gamma}$ as representatives of the sets $D_1, \ldots, D_\nu$, and Γ; in them we assume that (1) $r \in \{1, \ldots, \nu\}$, (2) $x_1, \ldots, x_n, x, y$, and f are variables of type $t_1, \ldots, t_n, r, r$, and $((1): r)$ respectively, (3) $p, q(c) \in \mathcal{E}_0$ and $q(c)$ is modally closed, (4) $\mathcal{V} \in Val_{\mathcal{J}}$, and (5) $\xi_i \in \mathcal{L}\mathcal{I}_{t_i} (i = 1, \ldots, n)$.

Proposition 8.1 *$\xi = des_{\mathcal{J}\mathcal{V}}((\lambda x_1, \ldots, x_n, c)\Diamond(AEC(c) \wedge p))$ is a modally constant relation and $\langle\xi_1, \ldots, \xi_n, \tilde{c}\rangle \,\tilde{\in}\, \xi$ iff there is a $\gamma \in \Gamma$ such that $\tilde{c} = \tilde{\gamma}$ and $\gamma \in des_{\mathcal{J}\mathcal{V}'}(p)$ where $\mathcal{V}' = \mathcal{V}(x_1/\xi_1, \ldots, x_\eta/\xi_n)$.*

Proof: ξ is modally constant since $\Diamond(AEC(c) \wedge p)$ is modally closed. Assume now $\langle \xi_1, \ldots, \xi_n, \tilde{c}\rangle \,\bar{\epsilon}\, \xi$; this is equivalent to the existence of a $\gamma \in \Gamma$ such that $\gamma \in des_{\mathcal{I}\mathcal{V}'}(AEC(c) \wedge p)$ for $\mathcal{V}' = \mathcal{V}(x_1/\xi_1, \ldots, x_n/\xi_n, c/\tilde{c})$ and, by Lemma 5.2, this holds iff $\gamma \in des_{\mathcal{I}\mathcal{V}'}(p)$ and $\tilde{c} = \tilde{\gamma}$.

Proposition 8.2 *If* $\xi = des_{\mathcal{I}\mathcal{V}}((\lambda x_1, \ldots, x_n)(\exists c)(AEC(c) \wedge q(c)))$, *then* $\xi = \{\langle \xi_1, \ldots, \xi_n, \gamma\rangle: des_{\mathcal{I}\mathcal{V}'}(q(c)) = \Gamma$ *for* $\mathcal{V}' = \mathcal{V}(x_1/\xi_1, \ldots, x_n/\xi_n, c/\tilde{\gamma})\}$.

Proof: Since $q(c)$ is modally closed, for every $\mathcal{V} \in Val_{\mathcal{I}}$, $des_{\mathcal{I}\mathcal{V}}(q(c))$ is Γ or ϕ. Assume $\langle \xi_1, \ldots, \xi_n, \gamma\rangle \in \xi$; this holds iff there is a $\tilde{c}$ such that $\gamma \in des_{\mathcal{I}\mathcal{V}'}(AEC(c) \wedge q(c))$ for $\mathcal{V}' = \mathcal{V}(x_1/\xi_1, \ldots, x_n/\xi_n, c/\tilde{c})$, that is, by Lemma 5.2, iff $\tilde{c} = \tilde{\gamma}$ and $des_{\mathcal{I}\mathcal{V}'}(q(c)) = \Gamma$ for $\mathcal{V}'$ as above.

Proposition 8.3 *Let* $\mathcal{V}(x) = \zeta \;(\in \mathcal{L}\mathcal{I}_r)$, $\mathcal{V}(c) = \tilde{c}$, *and let* $\xi = des_{\mathcal{I}\mathcal{V}}((\imath y)\Diamond(AEC(c) \wedge y \in C_r \wedge y = x))$. *Then:*

(i) $\tilde{c} \,\bar{\epsilon}\, \tilde{\Gamma}$ *(say* $\tilde{c} = \tilde{\gamma}$*)* $\Rightarrow \xi$ *is a function with range* $\{\zeta(\gamma)\}$
(ii) $\tilde{c} \,\bar{\notin}\, \tilde{\Gamma} \Rightarrow \xi = a_r^{\nu}$.

Proof: (ii) is trivial since $\tilde{c} \,\bar{\epsilon}\, \tilde{\Gamma}$ iff $des_{\mathcal{I}\mathcal{V}}(\Diamond AEC(c)) = \Gamma$ for $\mathcal{V}(c) = \tilde{c}$ (cf. (5.5) above). Assume now $\tilde{c} = \tilde{\gamma}$. In this case, since $des_{\mathcal{I}\mathcal{V}}(AEC(c)) = \{\gamma\}$, the equality $des_{\mathcal{I}\mathcal{V}'}(\Diamond(AEC(c) \wedge y \in C_r \wedge x = y)) = \Gamma$ where $\mathcal{V}' = \mathcal{V}(y/\zeta')$, implies $\zeta' \,\bar{\epsilon}\, \tilde{C}_r$ and $\zeta'(\gamma) = \zeta(\gamma)$. Such a ζ' is unique and hence $\zeta' = \xi$.

Proposition 8.4 *Let* $\mathcal{V}(f) = \zeta \in X_{(\nu+1:r)}$ *and let* $\xi = des_{\mathcal{I}\mathcal{V}}((\imath x)(\exists c)(AEC(c) \wedge x \in C_r \wedge x = f(c)))$. *Then* $\sigma(\zeta) = \xi$.

Proof: By (8.5 and 8.7) we have to prove that, for all $\gamma \in \Gamma$, $\xi(\gamma)$ is the only element of the range of $\zeta(\tilde{\gamma})$. To this end we first remark that $des_{\mathcal{I}\mathcal{V}}(\exists_1 x)(\exists c)(AEC(c) \wedge x \in C_r \wedge x = f(c)) = \Gamma$, so that the *QI* ξ is determined on the basis of part (b) of designation rule (δ_9) in Section 2. Assume $\gamma \in des_{\mathcal{I}\mathcal{V}'}((\exists c)(AEC(c) \wedge x \in C_r \wedge x = f(c)))$ where $\mathcal{V}' = \mathcal{V}(x/\xi')$. Then $\gamma \in des_{\mathcal{I}\mathcal{V}''}(AEC(c) \wedge x \in C \wedge x = f(c))$ for $\mathcal{V}'' = \mathcal{V}'(c/\tilde{\gamma})$; hence $\xi' =_{\gamma} \zeta(\tilde{\gamma})$. Thus, since by (δ_9) $\xi' =_{\gamma} \xi$, $\xi =_{\gamma} \zeta(\tilde{\gamma})$.

Theorem 8.1 *For all* $t \in \bar{\tau}^{\nu}$, *the function* σ *is a bijection between* $X_{t^{\eta}}$ *and* $\mathcal{L}\mathcal{I}_t$.

Proof: σ is one-to-one (Lemma 8.1), hence we have to prove the equality $\sigma(X_{t^{\eta}}) = \mathcal{L}\mathcal{I}_t (t \in \bar{\tau}^{\nu})$. Let us remark that for every *QI* ξ in $\mathcal{I}$ there exists a wfe Δ and a $\mathcal{V} \in Val_{\mathcal{I}}$ such that $\xi = des_{\mathcal{I}\mathcal{V}}(\Delta)$. Conversely, $des_{\mathcal{I}\mathcal{V}}(\Delta)$ is in $\mathcal{L}\mathcal{I}_t$ for all $\Delta \in \mathcal{E}_t$ and $\mathcal{V}$, since $\mathcal{I}$ is general. Therefore the equality above is a consequence of the following statements:

(1) for every wfe Δ_0 (of type t^{η^*}) and every $\mathcal{V} \in Val_{\mathcal{I}}$ such that $des_{\mathcal{I}\mathcal{V}}(\Delta_0) \in X_{t^{\eta}}$, there is a wfe $\Sigma(\Delta_0)$ (of type t) such that $\sigma(des_{\mathcal{I}\mathcal{V}}(\Delta_0)) = des_{\mathcal{I}\mathcal{V}}(\Sigma(\Delta_0))$;
(2) for every wfe Δ (of type t) and every $\mathcal{V} \in Val_{\mathcal{I}}$, there is a term $\Sigma'(\Delta)$ (of type t^{η^*}) such that $des_{\mathcal{I}\mathcal{V}}(\Sigma'(\Delta)) \in X_{t^{\eta}}$ and $\sigma(des_{\mathcal{I}\mathcal{V}}(\Sigma'(\Delta)) = des_{\mathcal{I}\mathcal{V}}(\Delta)$.

Case 1. $t = 0$. Let $\Sigma(\Delta_0)$ be $(\exists c)(AEC(c) \wedge \Delta_0(c))$. By the hypotheses, $\zeta = des_{\mathcal{I}\mathcal{V}}(\Delta_0)$ is a modally constant QI of type ((1)) in X_{0^η} and, by Proposition 8.2, $\gamma \in \xi = des_{\mathcal{I}\mathcal{V}}(\Sigma(\Delta_0))$ iff $\tilde{\gamma}\ \bar{\epsilon}\ \zeta$; that is, $\sigma(\zeta) = \xi$. Thus (1) holds.

Let $\Sigma'(\Delta)$ be $(\lambda c)\Diamond(AEC(c) \wedge \Delta)$. By Proposition 8.1, $\tilde{c}\ \bar{\epsilon}\ \zeta = des_{\mathcal{I}\mathcal{V}}(\Sigma'(\Delta))$ iff $\tilde{c} = \tilde{\gamma}$ for some γ in $\xi = des_{\mathcal{I}\mathcal{V}}(\Delta)$, which is equivalent to $\sigma(\zeta) = \xi$. Thus (2) holds.

Case 2. $t = r\ (\epsilon\ \{1, \ldots, \nu\})$. (1) follows from Proposition (8.4). Let $\Sigma'(\Delta)$ be $(\lambda c)(\imath y)\Diamond(AEC(c) \wedge y \in C_r \wedge y = \Delta)$. $\zeta = des_{\mathcal{I}\mathcal{V}}(\Sigma'(\Delta))$ is a function in $\mathcal{L}\mathcal{I}_{((1):r)}$ and, by Proposition 8.3, it belongs to X_{r^η} and $\sigma(\zeta) = des_{\mathcal{I}\mathcal{V}}(\Delta)$, which proves (2).

Now the thesis, as well as (1) and (2), are assumed to hold when t is any of $t_0, t_1, \ldots, t_n$.

Case 3. $t = (t_1, \ldots, t_n)$. Let $\Sigma(\Delta_0)$ be $(\lambda x_1, \ldots, x_n)(\exists c)(AEC(c) \wedge \Delta_0(\Sigma'(x_1), \ldots, \Sigma'(x_n), c))$, where $x_1, \ldots, x_n$ are n variables (not free in Δ_0) of type $t_1, \ldots, t_n$ respectively. By Proposition 8.2 and the inductive hypothesis, for all $\langle \xi_1, \ldots, \xi_n\rangle \in \Pi_i^n \mathcal{L}\mathcal{I}_{t_i}$ and $\gamma \in \Gamma$, $\langle \xi_1, \ldots, \xi_n, \gamma\rangle \in \xi = des_{\mathcal{I}\mathcal{V}}(\Sigma(\Delta_0))$ iff $\langle \sigma^{-1}(\xi_1), \ldots, \sigma^{-1}(\xi_n), \tilde{\gamma}\rangle\ \bar{\epsilon}\ \zeta = des_{\mathcal{I}\mathcal{V}}(\Delta_0)$, which proves (1) by (8.6).

Now let $\Sigma'(\Delta)$ be $(\lambda y_1, \ldots, y_n, c)\Diamond(AEC(c) \wedge \Delta(\Sigma(y_1), \ldots, \Sigma(y_n)))$, where $y_1, \ldots, y_n$ are n variables (not free in Δ) of type $t_1^{\eta*}, \ldots, t_n^{\eta*}$, respectively. By Proposition 8.1, for all $\langle \zeta_1, \ldots, \zeta_n\rangle \in \Pi_i^n \mathcal{L}\mathcal{I}_{t_i^{\eta*}}$ and all $\tilde{c} \in \mathcal{L}\mathcal{I}_{(1)}$, ${}_*\langle \zeta_1, \ldots, \zeta_n, \tilde{c}\rangle\bar{\epsilon}$ $\zeta = des_{\mathcal{I}\mathcal{V}}(\Sigma'(\Delta))$ holds iff $\tilde{c} = \tilde{\gamma}$ and $\gamma \in des_{\mathcal{I}\mathcal{V}'}(\Delta(\Sigma(y_1), \ldots, \Sigma(y_n)))$ for $\mathcal{V}' = \mathcal{V}(y_1/\zeta_1, \ldots, y_n/\zeta_n)$. Hence by the inductive hypothesis $(*)$ holds iff $\langle \sigma(\zeta_1), \ldots, \sigma(\zeta_n), \gamma\rangle \in des_{\mathcal{I}\mathcal{V}}(\Delta)$, which in turn occurs iff $\sigma(\zeta) = des_{\mathcal{I}\mathcal{V}}(\Delta)$. Thus (2) holds.

Case 4. $t = (t_1, \ldots, t_n, t_0)$. By the inductive hypothesis and (8.7), $\Sigma(\Delta_0)$ is trivially $(\lambda x_1, \ldots, x_n)\Sigma(\Delta_0(\Sigma'(x_1), \ldots, \Sigma'(x_n)))$ where $x_1, \ldots, x_n$ are as above. Hence (1) holds.

Let $\Sigma'(\Delta)$ be $(\lambda y_1, \ldots, y_n)(\imath y)(\exists x_1, \ldots, x_n)(\Lambda_i^n \Box x_i = \Sigma(y_i) \wedge \Box y = \Sigma(\Delta(x_1, \ldots, x_n)))$, where $y_i, x_i (i = 1, \ldots, n)$ are as above; and let ζ be $des_{\mathcal{I}\mathcal{V}}(\Sigma'(\Delta))$. Hence ζ is a function in $\mathcal{L}\mathcal{I}_{t^{\eta*}}$ and, by the inductive hypothesis, for all $\langle \zeta_1, \ldots, \zeta_n\rangle \in (\Pi_i^n \mathcal{L}\mathcal{I}_{t_i^{\eta*}} - \Pi_i^n X_{t_i^\eta})$, $\zeta(\zeta_1, \ldots, \zeta_n) = a^\nu_{t_0^{\eta*}}$. Conversely, $\langle \zeta_1, \ldots, \zeta_n\rangle \in \Pi_i^n X_{t_i^\eta}$ implies $\zeta(\zeta_1, \ldots, \zeta_n) = \sigma^{-1}(\xi(\sigma(\zeta_1), \ldots, \sigma(\zeta_n)))$, where $\xi = des_{\mathcal{I}\mathcal{V}}(\Delta)$; that is $\zeta \in X_{t^\eta}$ and $\sigma(\zeta) = \xi$. Hence (2) holds.

By (8.8) we have $\mathcal{L}\mathcal{I}_t = Ob_{t^\eta}$ for all $t \in \bar{\tau}^\nu$. In order to define the function $a^{\nu+1}$ (for I), let us remark that a_r^ν (in $\mathcal{I}$) is a constant function from Γ into $D_r(= Ob_r)$ (cf. (7.1)); thus it is natural to assume $a_r^{\nu+1}$ to be the only element of the range of $a_r^\nu (r = 1, \ldots, \nu)$. As for $a_{\nu+1}^{\nu+1}$, we fix a possible case γ^* in Γ and we let $a_{\nu+1}^{\nu+1} = \gamma^*$.

Now the function $a^{\nu+1}$ is determined by the above choice of $a_r^{\nu+1} (r = 1, \ldots, \nu + 1)$ and the semantical conditions corresponding to the axioms on the nonexisting object. It is easy to prove that $a_{t^\eta}^{\nu+1} = a_t^\nu$ for all $t \in \tau^\nu$ and hence, if I is any fixed extensional correspondent of the valuation $\mathcal{I}$ of the constants in $\mathcal{I}$, then the $EL^{\nu+1}$-interpretation I, defined by means of (8.8) and the above choice of $a^{\nu+1}$ and I, is an extensional correspondent of $\mathcal{I}$.

*9 **Proof of the equivalence theorem*** The equivalence theorem between MC^{ν} and $EC^{\nu+1}$ follows from the assertion that the $EL^{\nu+1}$-interpretation $\boldsymbol{I}$, constructed in Section 8, is general. In order to prove this we shall use the above-mentioned possibility (due to the construction of $\boldsymbol{I}$) of turning any problem concerning the existence of definable entities in $\boldsymbol{I}$ into an analogous problem in $\mathcal{J}$.

The following lemma asserts the existence (in $\mathcal{J}$) of certain *QI*s which can represent the sets $Ob_s(s \in \tau^{\nu+1})$ in $\boldsymbol{I}$ (let us remark that, by the intensional semantics for ML^{ν}, the sets X_s defined in Section 8 are not *QI*s in $\mathcal{J}$).

Lemma 9.1 *For all $s \in \tau^{\nu+1}$, the set $\widetilde{X}_s$ defined by*

(9.1) $\widetilde{X}_s = X_s \times \Gamma$

is an absolute QI in $\mathcal{LD}_{(s^*)}$.

Proof: The thesis is trivial for $s \in \{1, \ldots, \nu + 1\}$ (cf. (8.2)) and hence it can be assumed inductively to hold for $s \in \{s_0, \ldots, s_n\}$. Every $\widetilde{X}_s$ is modally constant by (9.1) and, for $s = (s_1, \ldots, s_n)$ the elements of X_s are modally constant by (8.3), so that $\widetilde{X}_s$ is modally separated. Assume now $s = (s_1, \ldots, s_n\colon s_0)$, ξ, $\xi' \in X_s$ and $\xi =_{\gamma} \xi'$; that is, $\xi(\alpha) =_{\gamma} \xi'(\alpha)$ for all $\alpha \in \Pi_i^n \mathcal{LD}_{s_i^*}$. If $\alpha \notin \Pi_i^n X_{s_i}$, then $\xi(\alpha) = \xi'(\alpha) = a_{s_0^*}^{\nu}$; otherwise $\xi(\alpha)$, $\xi'(\alpha)\ \bar{\epsilon}\ \widetilde{X}_{s_0}$, which is modally separated by the inductive hypothesis. Thus $\xi(\alpha) = \xi'(\alpha)$ for all α and hence $\xi = \xi'$.

Let now Y_{s_0}, $Y_{s_1}, \ldots, Y_{s_n}$, R, and f be variables in ML^{ν} of type (s_0^*), $(s_1^*), \ldots, (s_n^*)$, $(s_1, \ldots, s_n)^*$, and $(s_1, \ldots, s_n\colon s_0)^*$, respectively, and let $\mathcal{V}(Y_{s_i}) = \widetilde{X}_{s_i} (i = 0, 1, \ldots, n)$. It is a matter of routine to prove that $des_{\mathcal{J}\mathcal{V}}((\lambda R)(Mconst(R) \wedge (\forall x_1, \ldots, x_n)(R(x_1, \ldots, x_n) \supset \Lambda_i^n \Box x_i \in Y_{s_i})))$ and $des_{\mathcal{J}\mathcal{V}}((\lambda f)(\forall x_1, \ldots, x_n)(\Lambda_i^n x_i \in Y_{s_i} \supset f(x_1, \ldots, x_n) \in Y_{s_0} .\wedge. \sim\Lambda_i^n x_i \in Y_{s_i} \supset f(x_1, \ldots, x_n) = a_{s_0^*}^*))$ are respectively $\widetilde{X}_{(s_1,\ldots,s_n)}$ and $\widetilde{X}_{(s_1,\ldots,s_n:s_0)}$. Hence $\widetilde{X}_s \in \mathcal{LD}_{s^*}$ for all $s \in \tau^{\nu+1}$.

In Lemma 9.2 and Theorem 9.1 below the following conventions are assumed: (1) for every variable v_{sn} of $EL^{\nu+1}$, v_{sn}^* denotes the variable $v_{s^*,n+1}$ of ML^{ν}, (2) for every $s \in \tau^{\nu+1}$, Y_s denotes $v_{(s^*),1}$, and (3) for every $\mathcal{V} \in Val_{\mathcal{J}}$, $\mathcal{V}(Y_s) = \widetilde{X}_s$ (cf. (9.1)).

Lemma 9.2 *Assume that:* (1) $\Delta \in E_s(s \in \bar{\tau}^{\nu+1})$, (2) $V \in Val_{\boldsymbol{I}}$, *and* (3) *no constant occurs in Δ. Then, for every n-tuple $X = \langle x_1, \ldots, x_n\rangle$ of variables in $E_{s_1}, \ldots, E_{s_n}$, respectively, there exist a wfe Δ^* of ML^{ν} and a $\mathcal{V} \in Val_{\mathcal{J}}$, such that, for every $\langle \xi_1, \ldots, \xi_n\rangle \in \Pi_i^n Ob_{s_i}$,*

(9.2) $s \neq 0 \Rightarrow des_{\mathcal{J}\mathcal{V}'}(\Delta^*) = \sigma^{-1}(des_{\boldsymbol{I}V'}(\Delta))$
(9.3) $s = 0 \Rightarrow des_{\mathcal{J}\mathcal{V}'}(\Delta^*) = \Gamma[\phi]$ *iff* $des_{\boldsymbol{I}V'}(\Delta) = 1\ [\phi]$

where $V' = V(x_1/\xi_1, \ldots, x_n/\xi_n)$ and $\mathcal{V}' = \mathcal{V}(x_1^*/\sigma^{-1}(\xi_1), \ldots, x_n^*/\sigma^{-1}(\xi_n))$.

Proof: We use an induction on the number $\imath_{\Delta}$ of occurrences of $\imath$ in Δ and, in correspondence with a given value of $\imath_{\Delta}$, we use an induction on the length of Δ; however, this last part does not depend on $\imath_{\Delta}$ and hence it is carried out explicitly in the initial step only.

Let $\imath_{\Delta} = 0$. If no quantifier occurs in Δ then we let Δ^* be obtained from Δ

by replacing in it every variable x with x^* and we let $\mathscr{V}$ be any $\mathscr{I}$-valuation fulfilling the equality $V(y) = \sigma(\mathscr{V}(y^*))$ for every variable y of $EL^{\nu+1}$. In this case the thesis follows from (8.6 and 8.7) and an induction on the length of Δ. Let now Δ be $(x)p$ $(x \in E_u)$; we assume inductively that the thesis holds for p and the set $X' = X \cup \{x\}$ of variables. We let Δ^* be the wff $(\forall x^* \underset{\sim}{\in} Y_u)p^*$, and let $\mathscr{V}$ be any $\mathscr{I}$-valuation fulfilling the inductive hypothesis. $\tilde{X}_u$ is modally constant and hence, by the inductive hypothesis, $des_{\mathscr{I}\mathscr{V}'}(\Delta^*)$ is Γ or ϕ. Furthermore, $des_{\mathscr{I}\mathscr{V}'}(\Delta^*) = \Gamma$ iff, for all $\zeta \mathrel{\bar{\in}} \tilde{X}_u$ (that is $\zeta \in X_u$), $des_{\mathscr{I}\mathscr{V}''}(p^*) = \Gamma$, where $\mathscr{V}'' = \mathscr{V}'(x^*/\zeta)$; but σ is a bijection between X_u and Ob_u and hence, by the inductive hypothesis, $des_{\mathscr{I}\mathscr{V}'}(\Delta^*) = \Gamma$ iff, for all $\xi \in Ob_u$, $des_{IV''}(p) = 1$ for $V'' = V'(x/\xi)$.

Let now Δ be $(\imath x)p$. We assume inductively that the thesis holds for all wfe Δ' for which $\imath_{\Delta'} < \imath_\Delta$, and we let Δ^* be $(\imath y)((\sim(\exists_1 x)p)^* \supset y = \alpha_s \mathrel{.\wedge.} ((\exists_1 x)p)^* \supset y = (\imath x^* \in Y_s)p^*)$, where y and α_s are distinct variables of type s^*, not free in p^* and different from $x_1, \ldots, x_n$. Furthermore, we assume $\mathscr{V}$ to be any $\mathscr{I}$-valuation fulfilling the inductive hypothesis (relative to p and the set $X' = X \cup \{x\}$), for which $\mathscr{V}(\alpha_s) = \sigma^{-1}(a_s^{\nu+1})$. If $des_{IV'}((\exists_1 x)p) = 0$, then, by the inductive hypothesis, $des_{\mathscr{I}\mathscr{V}'}(((\exists_1 x)p)^*) = \phi$ and $des_{\mathscr{I}\mathscr{V}'}(\Delta^*) = \mathscr{V}'(\alpha_s) = \sigma^{-1}(des_{IV'}(\Delta))$. Assume now $des_{IV'}((\exists_1 x)p) = 1$ and $des_{IV'}(\Delta) = \xi$, which means in particular that $des_{IV''}(p) = 1$ for $V'' = V'(x/\xi)$. Then $des_{\mathscr{I}\mathscr{V}'}(((\exists_1 x)p)^*) = \Gamma$, $des_{\mathscr{I}\mathscr{V}''}(p^*) = \Gamma$ for $\mathscr{V}'' = \mathscr{V}'(x^*/\sigma^{-1}(\xi))$, and $des_{\mathscr{I}\mathscr{V}'}(\Delta^*) = des_{\mathscr{I}\mathscr{V}'}((\imath x^* \in Y_s)p^*)$. Now, the proof of the initial step implies that $((\exists_1 x)p)^*$ is equivalent to $(\exists_1 x^*)(x^* \in Y_s \wedge p^*)$ and hence $des_{\mathscr{I}\mathscr{V}'}(\Delta^*) = \sigma^{-1}(\xi)$.

Theorem 9.1 *The $EL^{\nu+1}$-interpretation I, defined in Section 8, is general.*

Proof: Let ξ be $d(\Delta, \langle x_1, \ldots, x_n\rangle, I, V)$ (cf. (3.2, 3)), where $\Delta \in E_0$, $x_1, \ldots, x_n$ are variables in $E_{s_1}, \ldots, E_{s_n}$, respectively, and $V \in Val_I$. Let us first remark that no loss of generality is afforded by assuming that no constant occurs in Δ, since, otherwise, we can replace every constant a in Δ with a variable x (free for a in Δ and different from the others introduced in the same way), and consider, instead of V, the I-valuation $V' = V(x/I(a))$. Thus Lemma 9.2 can be applied, and a wff Δ^* and a $\mathscr{V} \in Val_{\mathscr{I}}$ exist such that (9.3) holds for all $\langle \xi_1, \ldots, \xi_n\rangle \in \Pi_i^n Ob_{s_i}$. Let Δ_1^* be $(\Lambda_i^n x_i \in Y_{s_i}) \wedge \Delta^*$ and let ζ be $d(\Delta_1^*, \langle x_1^*, \ldots, x_n^*\rangle, \mathscr{I}, \mathscr{V})$. By (8.3), $\zeta \in X_{(s_1,\ldots,s_n)}$ and, by Lemma 9.2, $\langle \xi_1, \ldots, \xi_n\rangle \in \xi$ iff $\langle \sigma^{-1}(\xi_1), \ldots, \sigma^{-1}(\xi_n)\rangle \mathrel{\bar{\in}} \zeta$. Hence, by (8.6) and (8.8) $\xi = \sigma(\zeta)$ and $\xi \in Ob_{(s_1,\ldots,s_n)}$.

In a similar way we can prove that, in case $\Delta \in E_s (s \neq 0)$, $\xi \in Ob_{(s_1,\ldots,s_n:s)}$: if we let Δ_1^* be $(\imath y)(\sim\Lambda_i^n x_i \in Y_{s_i} \supset \Box y = a_{s^*}^* \mathrel{.\wedge.} \Lambda_i^n x_i \in Y_{s_i} \supset \Box y = \Delta^*)$, then it is a matter of routine to verify that $\zeta = d(\Delta_1^*, \langle x_1^*, \ldots, x_n^*\rangle, \mathscr{I}, \mathscr{V})$ is in $X_{(s_1,\ldots,s_n:s)}$ and $\xi = \sigma(\zeta)$.

Now, the converse of Corollary 6.1 is a consequence of the above results. Indeed if a wff p of ML^ν is not true in the general ML^ν-interpretation $\mathscr{I}$, then it is not true in a general ML^ν-interpretation $\mathscr{I}'$ fulfilling (7.1) and (7.2), which has a general extensional correspondent I by Theorem 9.1; and hence, by Theorem 6.1, p^η is not true in I. Thus, by the completeness theorems, Theorems 3.1 and 3.2, the converse of Theorem 7.1 holds, and (1.1) is proved.

10 Uniqueness properties of the general extensional correspondent In this section $\mathcal{J}$ and I still denote the interpretations considered in Sections 8 and 9 and $J = \langle \{Ob'_s: s \in \tau^{\nu+1}\}, b^{\nu+1}, J\rangle$ denotes an arbitrary general extensional correspondent of $\mathcal{J}$. A priori, the only known relation between I and J consists in the equalities $Ob'_{t^\eta} = Ob_{t^\eta}(= \mathcal{L}\mathcal{I}_t)$, for all $t \in \overline{\tau}^{\nu}$; however, if we take into account that I and J are general, then it seems reasonable to ask whether some stronger relation should hold, since in a general interpretation some connections between sets of objects of various types are determined by the required closure of these sets with respect to definability.

In what follows we shall give an exhaustive answer to this problem by proving that $Ob_s = Ob'_s$, for all $s \in \tau^{\nu+1}$. Therefore, since J is chosen arbitrarily, we have that all general extensional correspondents of a given general ML^{ν}-interpretation, are based on the same Ob-structure.

The proof of the uniqueness theorem is substantially based on the possibility of expressing the function σ (cf. (8.5) to (8.7)) inside I and J (let us remark that for every $s \in \tau^{\nu+1}$, X_s (cf. (8.2) to (8.4)) is a subset of $Ob_{s^{*\eta}}(= Ob'_{s^{*\eta}} = \mathcal{L}\mathcal{I}_{s^*})$).

In Lemma 10.1 and Theorem 10.1 below we consider only the cases $s \in \{1, \ldots, \nu + 1\}$ and $s = (s_1, \ldots, s_n)$ because, since I and J are general, every object of type $(s_1, \ldots, s_n: s_0)$ corresponds to exactly one object of type $(s_1, \ldots, s_n, s_0)$.

Lemma 10.1 *For every $s \in \tau^{\nu+1}$, there exists a wfe $\Sigma^s(Z)$ (in which Z is a free variable of type $s^{*\eta}$) such that, for every $V \in Val_I[U \in Val_J]$, $V(Z) \in X_s[U(Z) \in X_s]$ implies $des_{IV}(\Sigma^s(Z)) = \sigma(V(Z))$ $[des_{JU}(\Sigma^s(Z)) = \sigma(U(Z))]$.*

Proof: The proof is quite similar to and simpler than that of Theorem 8.1; therefore we only consider the wfe $\Sigma^s(Z)$. In what follows $x, y, k, x_1, \ldots, x_n$, $Z_1, \ldots, Z_n$ are distinct variables of type r, 1^η, $\nu + 1$, $s_1, \ldots, s_n$, $s_1^{*\eta}, \ldots, s_n^{*\eta}$, respectively.

Case 1. $s = r(\in \{1, \ldots, \nu\})$. $\Sigma^s(Z)$ is $(\imath x)(k)(Z(k) = x)$
Case 2. $s = \nu + 1$. $\Sigma^s(Z)$ is $(\imath k)(y)Z(y, k)$
Case 3. $s = (s_1, \ldots, s_n)$. $\Sigma^s(Z)$ is $(\lambda x_1, \ldots, x_n)(\exists Z_1, \ldots, Z_n)(\Lambda_i^n x_i = \Sigma^{s_i}(Z_i) \wedge (k)Z(Z_1, \ldots, Z_n, k))$.

Theorem 10.1 *For every $s \in \tau^{\nu+1}$, $Ob_s = Ob'_s$.*

Proof: By (8.8) and Lemma 10.1, $Ob_s \subseteq Ob'_s$. If $s \in \{1, \ldots, \nu + 1\}$ then the thesis holds trivially; thus, in considering the case $s = (s_1, \ldots, s_n)$, we can assume inductively $Ob_{s_i} = Ob'_{s_i} (i = 1, \ldots, n)$. Let $\xi \in Ob'_s$ and let ζ be $des_{JV}((\lambda Z_1, \ldots, Z_n, k)(\exists x_1, \ldots, x_n)(\Lambda_i^n x_i = \Sigma^{s_i}(Z_i) \wedge y(x_1, \ldots, x_n)))$, where y and k are variables of type s and $\nu + 1$, respectively, and $U(y) = \xi$. By Lemma 10.1, for every $\langle \zeta_1, \ldots, \zeta_n\rangle \in \Pi_i^n Ob'_{s_i^{*\eta}}$, $\langle \zeta_1, \ldots, \zeta_n\rangle \,\overline{\epsilon}\, \zeta$ iff $\langle \sigma(\zeta_1), \ldots, \sigma(\zeta_n)\rangle \in \xi$, that is, $\xi = \sigma(\zeta)$. Furthermore, $\zeta \in Ob'_{s^{*\eta}} = \mathcal{L}\mathcal{I}_{s^*}$ and, by the inductive hypothesis, $\langle \zeta_1, \ldots, \zeta_n\rangle \,\overline{\epsilon}\, \zeta \Rightarrow \zeta_i \in X_{s_i} (i = 1, \ldots, n)$; by (8.3) this implies $\zeta \in X_s$ and hence $\xi \in Ob_s$.

The above result implies that the only possible differences between I and J, arise from those between the valuations I and J of the constants and those between $a^{\nu+1}$ and $b^{\nu+1}$ (the functions determining the nonexisting objects in $\boldsymbol{I}$ and $\boldsymbol{J}$, respectively). However the latter differences can be disregarded without loss of generality (cf. Theorem 10.2 below).

Let us remark that $a^{\nu+1}$ and $b^{\nu+1}$ are different iff $a^{\nu+1}_{\nu+1} \neq b^{\nu+1}_{\nu+1}$; indeed, for $s \in \{1, \ldots, \nu\}$, $a^{\nu+1}_s$ is the only element of the counterdomain of $a^{\nu+1}_{(\nu+1:s)} (= a^\nu_s)$ and hence it is equal to $b^{\nu+1}_s$, and, for s of a level larger than zero, $a^{\nu+1}_s$ and $b^{\nu+1}_s$ are uniquely determined by means of EA3.15.

Let us assume $a^{\nu+1}_{\nu+1} = \underline{a}$ and $b^{\nu+1}_{\nu+1} = \underline{b}$, and let us consider the function w, of domain $\bigcup_{s \in \tau^{\nu+1}} Ob_s$, which substitutes $\underline{a}$ for $\underline{b}$ in every object in $\boldsymbol{I}$; in other words, w is the identity on $Ob_r (r = 1, \ldots, \nu + 1)$ except that $w(\underline{a}) = \underline{b}$ and $w(\underline{b}) = \underline{a}$, and, for $\overline{R}$ and $\overline{f}$ of type $(s_1, \ldots, s_n)$ and $(s_1, \ldots, s_n : s_0)$ respectively:

(10.1) $\quad w(\overline{R}) = \{\langle w(\xi_1), \ldots, w(\xi_n)\rangle : \langle \xi_1, \ldots, \xi_n\rangle \in \overline{R}\}$

(10.2) $\quad w(\overline{f}) = \{\langle\langle w(\xi_1), \ldots, w(\xi_n)\rangle, w(\overline{f}(\xi_1, \ldots, \xi_n))\rangle : \xi_i \in Ob_{s_i} (i = 1, \ldots, n)\}$.

Lemma 10.2 *For every $s \in \tau^{\nu+1}$ and $\xi \in Ob_s$, $w(\xi) \in Ob_s$.*

Proof: For all $s \in \tau^{\nu+1}$, we define a wfe $W(x) \in E_s$ (in which the variable x, of type s, is free) such that, for a suitable $V \in Val_{\boldsymbol{I}}$, (*) $des_{\boldsymbol{I}V'}(W(x)) = w(\xi)$, where $V' = V(x/\xi)$. For $s \in \{1, \ldots, \nu\}$, $W(x)$ is x. Let now s be $\nu + 1$ and let us fix two distinct variables x_1, x_2 (of type $\nu + 1$) different from x. If we assume V to be any $\boldsymbol{I}$-valuation such that $V(x_1) = \underline{a}$ and $V(x_2) = \underline{b}$, then the definition of $W(x)$ is obvious: $W(x)$ is $(\imath y)(\Lambda^2_i x \neq x_i \supset y = x \,.\!\wedge.\, x = x_1 \supset y = x_2 \,.\!\wedge.\, x = x_2 \supset y = x_1)$. Now we can assume inductively that (*) holds for V as above in case $s \in \{s_0, s_1, \ldots, s_n\}$. If we remark that $w = w^{-1}$ (and hence, by the inductive hypothesis, w is a bijection of Ob_{s_i} onto itself ($i = 1, \ldots, n$)), then the definition of $W(x)$ in the remaining cases turns out to be still straightforward. If s is $(s_1, \ldots, s_n)$ then $W(x)$ is $(\lambda x_1, \ldots, x_n)x(W(x_1), \ldots, W(x_n))$; if s is $(s_1, \ldots, s_n : s_0)$ then $W(x)$ is $(\lambda x_1, \ldots, x_n)W(x(W(x_1), \ldots, W(x_n))$.

By (10.1 and 10.2) and Lemma 10.2, for every $s \in \tau^{\nu+1}$, w is a bijection between Ob_s and Ob'_s, and the equality $w(a^{\nu+1}_s) = b^{\nu+1}_s$ holds. Thus, if the valuations of the constants are disregarded, then w fulfils the obvious definition of isomorphism between $EL^{\nu+1}$-interpretations. In particular, if $V \in Val_{\boldsymbol{I}}$, $U \in Val_{\boldsymbol{I}}$, and $U(x) = w(V(x))$ for every x, then

(10.3) $\quad des_{\boldsymbol{I}V}(p) = des_{\boldsymbol{J}U}(p)$

for every wff p in which no constants occur.

Thus the following theorem holds.

Theorem 10.2 *Let $\boldsymbol{I}$ and $\boldsymbol{J}$ be general $EL^{\nu+1}$-interpretations such that $\boldsymbol{I}^i = \boldsymbol{J}^i$; then, for every closed wff p of $EL^{\nu+1}$, $des_{\boldsymbol{I}}(p) = des_{\boldsymbol{J}}(p)$, whenever no constants occur in p or it is q^η for some q in ML^ν.*

In Theorem 10.3 below (which is the syntactical counterpart of Theorem 10.2) we use the term "*constant-free $EL^{\nu+1}$-extension*" of a set $K \subseteq \mathcal{E}_0$, to denote every set $H \subseteq E_0$ such that

(i) $K^\eta = \{q^\eta : q \in K\} \subseteq H$ and
(ii) if $p \in H - K^\eta$, then no constants occur in p.

Theorem 10.3 *For every maximal consistent set K of totally closed wffs in ML^{ν}, there is exactly one maximal consistent set H, of closed wffs in $EL^{\nu+1}$, which is a constant-free $EL^{\nu+1}$-extension of K.*

Proof: The proof follows from Theorem 10.2 by remarking that every maximal consistent set of closed [totally closed] wffs of $EL^{\nu+1}[ML^{\nu}]$ has a (general) model (Completeness Theorems) and, conversely, every general $EL^{\nu+1}-[ML^{\nu}]-$ interpretation $I[\mathcal{J}]$ determines the (maximal consistent) set of the closed [totally closed] wffs true in $I[\mathcal{J}]$.

NOTES

1. The independence of MA4.1 is proved in [2] (N25). As for MA5.1, it is equivalent to AS12.19 in [2]; Gallin proved the independence of this axiom in a modal calculus having some similarities with Bressan's, but which is simpler (identity between individual expressions is noncontingent and the description operator is not considered) (see [3], Sections 9 and 15).

2. The syntactical counterpart of this procedure is introduced and used by Bressan in [2], within his syntactical proof of (1.1).

3. The description operator $\imath$ can be eliminated from the calculus MC^{ν}, by replacing MA3.14 with an axiom (As.3.17 in [5]) in which $\imath$ does not occur; however, in investigating the problems considered in this work, it seems more convenient to refer to the original version of MC^{ν}.

4. As usual, $\mathcal{P}(A)$ denotes the power set of A, $\Pi_i^n A_i$ denotes $A_1 \times \ldots \times A_n$, and $(A \to B)$ denotes the set of functions from A into B.

5. The concept of QI-structure can also be defined independently (cf., e.g., [8]) by stating the correspondents of (2.1-2.3), which are: $\mathcal{QI}_0 \subseteq \mathcal{P}(\Gamma), \mathcal{QI}_r \subseteq (\Gamma \to D_r), \mathcal{QI}_{(t_1,\ldots,t_n)} \subseteq \mathcal{P}((\Pi_i^n \mathcal{QI}_{t_i}) \times \Gamma), \mathcal{QI}_{(t_1,\ldots,t_n:t_0)} \subseteq ((\Pi_i^n \mathcal{QI}_{t_i}) \to \mathcal{QI}_{t_0})$.

   By (2.4) and (2.5) the two definitions are equivalent since every QI-structure defined in this way can be extended to an Ob-structure.

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