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Probability Theory, Intuitionism, Semantics, and the Dutch Book Argument

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Introduction In a previous paper, [14], we provided a set of constraints for a theory of conditional probability a bit different from that normally seen. We showed that the probability functions satisfying our constraints could be used as valuations on the first-order language of intuitionism, *IL*, and we proved soundness and strong completeness for first-order intuitionistic logic, as well as a number of other results. The reader is referred to our first paper for notation and theoretical particulars. For definiteness, we list our constraints here.

C1 $0 \leqslant Pr^+(A,B) \leqslant 1$
C2 $Pr^+(A,A) = 1$
C3 $Pr^+(A,f) = 1$
C4 $Pr^+(A \supset B, C) = Pr^+(B, A \mathbin{\&} C)$
C5 $Pr^+(A \mathbin{\&} B, C) = Pr^+(A, B \mathbin{\&} C) \times Pr^+(B, C)$
C6 $Pr^+(A \mathbin{\&} B, C) = Pr^+(B \mathbin{\&} A, C)$
C7 $Pr^+(A, B \mathbin{\&} C) = Pr^+(A, C \mathbin{\&} B)$
C8 $Pr^+(A, B \vee C) = Pr^+(A,B) \times Pr^+(A, C \mathbin{\&} (B \supset A))$
C9 $Pr^+((\forall X)A, B) = \underset{i \to \infty}{\text{Limit}}\, Pr^+((\ldots (A'_1 \mathbin{\&} A'_2) \mathbin{\&} \ldots) \mathbin{\&} A'_i, B)$
C10 $Pr^+(A, (\exists X)B) = \underset{i \to \infty}{\text{Limit}}\, Pr^+(A, (\ldots (B'_1 \vee B'_2) \vee \ldots) \vee B'_i)$

For convenience, we will henceforth drop the '+' superscript from our probability functions.

These conditions are obviously not the only ones that could have been employed if our only goal was soundness and completeness. It is well known that one can axiomatize intuitionistic logic by a set of axiom schemes and the rule of modus ponens. (Indeed, we used such an axiomatization in our

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first paper.) Recall that we defined the *logical truths* as those statements A of IL such that for every intuitionistic probability function Pr, $Pr(A,B) = 1$ for every B. In light of this definition, we could have simply taken as our constraints the following:

(1) $Pr(AX, B) = 1$ for all B, where AX is any instance of an axiom scheme.
(2) If $Pr(A \supset B, C) = 1$ and $Pr(A, C) = 1$, then $Pr(B, C) = 1$.

Thus the desire to obtain soundness and completeness results does not impose very stringent requirements. Then why should one be interested in the set of constraints which we proposed? It is the purpose of this paper to answer just that question. Our answer is multi-faceted.

In the completeness proof given in our first paper, we showed that our conditions are consistent by proving that there are functions which satisfy C1-C10. However, all of those functions are limited to the two values 0 and 1. To be properly deemed a theory of probability, there must be functions satisfying our constraints which are not limited to just these two values. So in the first section of this paper, we show that there are functions satisfying our constraints which take on any finite number of values, and further that there are functions satisfying our constraints which are infinitely valued.

But what is the meaning of intuitionistic Pr? In the second section, we consider a very common (and we think very appealing) epistemic interpretation of classical probability. By replacing, in that classical interpretation, talk about truth by talk about proof, we provide a basic interpretation for the meaning of our intuitionistic probability. Note that it is just this shift from talk of truth to talk of proof which motivates intuitionistic logic. As a matter of intellectual fact, it was the basic interpretation described in this section which guided our formulation of constraints C1-C10.

In the third section, we compare our intuitionistic probability theory with classical probability theory and isolate two of our constraints as being nonclassical. Our classical constraints are easily justifiable in terms of our basic interpretation. We go on in the fourth and fifth sections to give detailed justifications for the two nonclassical constraints in terms of our basic interpretation.

Historically, probability theory has been closely connected with betting behavior. One justification offered for the nonquantificational constraints of classical probability has been the so-called Dutch book argument. Roughly speaking, at least part of what such arguments try to establish is that in certain types of highly idealized betting situations, deviation from classical constraints leads to the unwelcome result that one can be put in the position of always suffering a net loss, no matter what the outcome. Now obviously intuitionistic probability will not be appropriate for the same betting situations as classical probability. But in the sixth and last section of this paper, we describe idealized betting situations for which our intuitionistic probability theory is appropriate, and we provide a Dutch book argument for our nonquantificational constraints.

1 Nontriviality Our first task is to show that there are nontrivial functions which satisfy our constraints. In the course of the completeness argument in

[14], we outlined a method of obtaining 0-1-valued functions satisfying our constraints. In this section we will show that our intuitionistic probability functions are not limited to this two-valued case.

A family of three-valued intuitionistic probability functions can be constructed very simply. Let v be a function mapping statements of *IL* into the set of integers {0, 1, 2} subject to the following constraints:

$v1$ $v(f) = 0$
$v2$ If A is atomic, then $v(A)$ is in {0, 1, 2}
$v3$ $v(A \,\&\, B) = min\,\{v(A), v(B)\}$
$v4$ $v(A \vee B) = max\,\{v(A), v(B)\}$
$v5$ $v(A \supset B) = 2$, if $v(A) \leqslant v(B)$
$\qquad\qquad = v(B)$, otherwise
$v6$ Universally quantified expressions are assigned the minimum of the values assigned to their substitution instances.
$v7$ Existentially quantified expressions are assigned the maximum of the values assigned to their substitution instances.

Let r be any real number such that $0 \leqslant r \leqslant 1$. For each choice of r and each v satisfying the above constraints, we may define a function $Pr_3(A,B)$, as follows.

	$Pr_3(A,B)$	$v(B)$ 0	1	2
	0	1	0	0
$v(A)$	1	1	1	r
	2	1	1	1

It is easily verified that each function $Pr_3(A,B)$ so defined meets all of our constraints C1-C10, and hence each is an intuitionistic probability function.

Inspection of C1-C10 will reveal that if Pr' and Pr'' are (not necessarily distinct) intuitionistic probability functions, then the function $Pr(A,B) = Pr'(A,B) \times Pr''(A,B)$ is itself an intuitionistic probability function. In short, the product function formed from any pair of our functions is itself one of our functions. As a result, constructing intuitionistic probability functions with any finite number of values is a simple matter. For example, we may employ n distinct maps v satisfying $v1$-$v7$ above, but all using the same value of r. Each such v will determine a three-valued probability function as we have indicated. The function obtained by taking the n-ary product of these n probability functions will have values in the range $1, r, r^2, \ldots, r^n, 0$. Functions with an infinite number of values can be had by taking limits of infinite products.

2 Basic interpretation There are numerous ways of explicating $Pr(A,B)$, where Pr is a *classical* probability function and statements A and B are accordingly from a classical first-order language. One interpretation which Carnap advocated in his later writings, and which contributors to probabilistic semantics generally favor, is of an epistemic sort.[1] It treats $Pr(A,B)$ as *the degree to which, given B as an assumption,*[2] *one is warranted in believing A to be true* (hence, when A is in the future tense, *in expecting A to be true*). For example, let B be to the effect that in some poker game or other one is dealt five cards and A to the effect that all five cards will be of the same suit. $Pr(A,B)$ would then be the degree to which, being dealt the five cards in question, one is warranted in expecting them to be of the same suit (and hence, if so inclined, in *betting* that one will be dealt a flush). Or let B be the result of interposing '&'s between the (finitely many) postulates of Robinson's arithmetic, and let A be Fermat's last theorem.[3] $Pr(A, B)$ would then be the degree to which, given Robinson's postulates, one is warranted in believing Fermat's last theorem to be true.

For all its merit, the foregoing explication of $Pr(A,B)$ (for classical Pr) gains considerably when an extra factor is brought into play. Consider again the card game example, i.e., the one with B to the effect that in some poker game or other one is dealt five cards and A to the effect that all five cards will be of the same suit. When trying to evaluate $Pr(A,B)$ one makes various "background" assumptions which, unlike B, are left *unstated* but may well affect the value of $Pr(A,B)$. Some of these assumptions will concern the make-up of the deck (i.e., the number of suits in the deck, the number of cards in each suit, etc.); others will concern the "fairness" of the shuffling and dealing; yet others will be snatches of mathematics, physics, etc., relating to the deal (say, to be dealt five cards of the same suit is to be dealt four cards of a given suit plus a fifth highly critical card of that very same suit; cards do not change suit as they are dealt; etc.). All the assumptions we just trotted out are of an obvious sort, and in the course of a poker game would be "in the back of one's mind". Other assumptions might be far less obvious, and a player trying to evaluate $Pr(A,B)$ might even be unconscious of making them: they would then count as assumptions in the sense that the player acts *as though* believing them to be true. But whether obvious or not and whether known or not, it is *given these unstated assumptions* that $Pr(A,B)$ is evaluated. Including the unstated background assumptions makes better sense of requirements for "total evidence" than can be done with the customary account. So adding to the above account, we shall treat $Pr(A,B)$ as *the degree to which, given B as an assumption, plus some set or other S of unstated background assumptions, one is warranted in believing A to be true.*

Our second example (the one concerning the probability of Fermat's last theorem) was not whimsy, but a hint of things to come. As Heyting notes in [8], Brouwer and his followers dealt exclusively with mathematical statements, and to them truth and provability were essentially the same. For intuitionists, the assertion-conditions for the various connectives are proof-theoretic conditions. For example, the assertion-conditions for implication are: "The *implication* $p \rightarrow q$ can be asserted if and only if we possess a construction r, which, joined to any construction proving p (supposing that the latter be effected), would automatically effect a construction proving q. In other words, a proof of p, together with r, would form a proof of q."[4]

Unlike the early intuitionists, we do not require that the statements of *IL* (and its term extensions) be mathematical ones. For us intuitionistic logic is an all-purpose logic, and the probability theory we devised in these pages is an all-purpose one. However, when explicating intuitionistic probabilities, we mimic Heyting and talk of *intuitionistic provability* rather than *truth*. So, where *Pr* is an *intuitionistic* probability function and statements A and B are accordingly from *IL*, we treat $Pr(A,B)$ as *the degree to which, given B as an assumption, plus some set S of unstated background assumptions, one is warranted in believing A to be intuitionistically provable*. Using the notation of our previous paper, we will treat $Pr(A,B)$ as *the degree to which, for some set S of unstated background assumptions, one is warranted in believing that* $S \cup \{B\} \vdash_I A$. Incidentally, since the Deduction Theorem and its converse both hold intuitionistically, '$S \vdash_I B \supset A$' can, and often will, substitute for '$S \cup \{B\} \vdash_I A$' in what follows.

Various points about our explication of $Pr(A,B)$ need underscoring. Note for example, that S may be empty, in which case our explication of $Pr(A,B)$ for classical *Pr* reduces to the more usual one advanced by Carnap and others, as discussed above. (For that matter, note that B may be t, a statement which sheds little light on any A!) As *we* think of it, though, S will frequently have members (and B be other than t). Being more general, our explication of $Pr(A,B)$ for classical probability–and by analogy, for intuitionistic probability–is thus more creditable than the customary one. The S we brought into play also accounts in a rather natural way for the "uncertainty" that is characteristic of probability, a matter we shall touch upon below.

Further points are best underscored by means of an illustration. Suppose a computer programmer (perhaps a PhD student) has set up his machine to determine, given an arbitrary statement of *IL* and an arbitrary set of statements of *IL*, whether the former is intuitionistically provable from the latter. Further, suppose some jealous colleague (perhaps a thesis examiner) challenges him to exhibit his understanding of the machine's behavior by betting on the outcome of several experiments. Each experiment is to consist of feeding the machine two statements A and B of *IL* and asking it whether or not A is intuitionistically derivable from B. The wary programmer may suspect that his colleague has just stored in the machine's memory some undivulged set S of statements of *IL*, in which case what the programmer will be betting on is whether or not A is intuitionistically provable from B *and* the statements in S, rather than from B alone. The betting behavior of an ideally rational programmer would be represented here by an intuitionistic probability function *Pr*, with $Pr(A,B)$ explicated in the belief-theoretic manner we just sketched. Which function our programmer adopts will normally depend on a number of factors, among them his *surmises* about the membership of S. He might suspect his colleague to have fed the machine an inconsistent S, and consequently take $Pr(A,B)$ to equal 1 for any two statements A and B of *IL*. Or he might decide that his best course of action is to act as if S were empty. Note, though, that *a particular actual S may be compatible with many distinct probability functions*. Note further that if A is intuitionistically derivable from B alone, then the programmer should surely take $Pr(A,B)$ to equal 1. Except in this case, though, the value assigned to $Pr(A,B)$ will not depend on whether or not $S \cup \{B\} \vdash_I A$ for the *real* S. However, the value assigned to $Pr(A,B)$ *will* depend on the programmer's *surmises* about S.

In the foregoing illustration we assumed *the membership of S to remain constant* from one experiment to another and hence *through a given sequence* *of bets.* We shall make a similar assumption later when offering a so-called Dutch book argument in behalf of constraints C1-C8. We could relax the assumption a bit, thereby allowing the machine's memory, for example, to occasionally lapse. So long as the changes in membership that S undergoes do not depend upon the sequence of experiments or, more generally, on the sequence of bets, our explication of $Pr(A,B)$ would remain appropriate. Detailed study of this possibility would, however, take us too far afield.

3 Intuitionistic vs classical probability In [16], Popper gave an autonomous characterization of conditional probability theory. His characterization captured the functions included in traditional accounts, while avoiding difficulties previously associated with $Pr(A,B)$ for those cases in which $Pr(B,t) = 0$, for t a tautology, Popper's constraints can be shown to be equivalent to the following ones.[5]

D1 $0 \leqslant Pr(A,B)$

D2 $Pr(A,A) = 1$ (= C2)

D3 If for some C, $Pr(C,B) \neq 1$, then $Pr(\sim A,B) = 1 - Pr(A,B)$.

D4 $Pr(A \,\&\, B, C) = Pr(A, B \,\&\, C) \times Pr(B,C)$ (= C5)

D5 $Pr(A \,\&\, B, C) = Pr(B \,\&\, A, C)$ (= C6)

D6 $Pr(A, B \,\&\, C) = Pr(A, C \,\&\, B)$ (= C7)

Including the following constraint will allow one to handle quantifiers.

D7 $Pr((\forall X)A,B) = \underset{i\to\infty}{\text{Limit}}\, Pr((\ldots(A'_1 \,\&\, A'_2) \,\&\, \ldots) \,\&\, A'_i, B)$ (= C9)

The statements indicated by A'_k are substitution instances of the expression $(\forall X)A$, as in the notation of [14]. With these seven constraints, it is possible to prove soundness and completeness results for classical first-order logic.[6] We shall therefore take D1-D7 to be *characteristic of classical probability theory*.

When talking of either classical logic or classical probability theory, we will take as primitives negation, conjunction, and the universal quantifier. All other connectives, including the existential quantifier, will be assumed to be defined in the usual way. For the classical case, we will assume 'f' to be defined as $A \,\&\, \sim A$, for some predetermined statement A.

Note that five of our constraints for intuitionistic probability theory are included in the constraints for classical probability theory, to wit: C2, C5-C7, and C9. Further, C1, C3, and C10 are all provable from the classical constraints. Our conditions C4 and C8, on the other hand, are independent of D1-D7. These two conditions are thus *nonclassical* probability constraints, indeed *the only nonclassical probability constraints among C1-C10.*

For proof that C4 and C8 are independent of D1-D7, use the four integers 0, 1, 2, and 3 as truth-values. Evaluate conjunctions and negations according to the following table.

	$v(A \,\&\, B)$	$v(B)$ 0	1	2	3	$v(\sim A)$
$v(A)$	0	0	0	0	0	3
	1	0	1	0	1	2
	2	0	0	2	2	1
	3	0	1	2	3	0

Evaluate universally quantified statements as the minimum of the values of their substitution instances. Finally, define a probability function Pr_c as follows.

	$Pr_c(A,B)$	$v(B)$ 0	1	2	3
$v(A)$	0	1	0	0	0
	1	1	1	0	0
	2	1	0	1	1
	3	1	1	1	1

These matrices are due to Popper.[7]

It is easily verified that Pr_c meets constraints D1-D7 and hence is a classical probability function. Yet C4, short here for

$$Pr_c(\sim(A \,\&\, \sim B), C) = Pr_c(B, A \,\&\, C),$$

fails when $v(A)$, $v(B)$, and $v(C)$ equal 1, 2, and 3, respectively. And C8, short here for

$$Pr_c(A, \sim(\sim B \,\&\, \sim C) = Pr_c(A,B) \times Pr_c(A, C \,\&\, \sim(B \,\&\, \sim A)),$$

fails when $v(B)$ equals 1 and both $v(A)$ and $v(C)$ equal 2. So C4 and C8 are, as claimed, independent of D1-D7.

Further, since the intuitionistic probability function used in our completeness proof in [14] does not meet constraint D3, it follows that the classical probability functions are not a subset of ours. Since ours are not a subset of theirs, the two sets of probability functions merely overlap.

Of course it is obvious that the differences between our treatment of disjunction and the conditional and the classical treatment of these connectives will affect the other connectives to which they are related. In particular, our negation will not be classical, since it is defined by $\sim A = A \supset f$. It is possible to define functions satisfying our constraints such that both $Pr(A, t) = 0$ and $Pr(\sim A, t) = 0$, which is impossible in classical probability theory. Similarly, although C10 holds classically, our treatment of the existential quantifier is not classical because disjunction turns up in C10 and our treatment of disjunction is not classical.

Since intuitionistic logic is weaker than classical logic, it admits a smaller set of theorems. Hence, one might expect that intuitionistic probability theory would simply involve a relaxation of the classical probability constraints to admit more functions, thus eliminating more statements as potential logical truths. There is no doubt that one *could* formulate an "intuitionistic semantics" of this sort. Indeed, van Fraassen gives an example in [18]. But it is not clear why one would want to call any particular set of maps from expressions into the closed unit interval an intuitionistic probability theory, unless there is an underlying interpretation concerning the connection of the theory with uncertainty and potential betting behavior.

The problem is that there is an embarrassment of riches. We can show that each probability function determines a set of equivalence classes of expressions, where A and B are in the same equivalence class iff $Pr(A, C) = Pr(B, C)$ for all C. We can then show that for our constraints, the equivalence classes so defined form a Heyting algebra, with meet, join, and pseudo-complement defined in the obvious way. Indeed, van Fraassen proves soundness for his constraints by exactly this technique. Both sets of constraints yield strong completeness results. Both sets of constraints admit nontrivial (in the sense discussed above) functions. But, from the simple example advanced in the introductory section above, we know that there are even weaker sets of constraints about which all of these results hold. Indeed, there are stronger sets of constraints about which all of these results hold.

It is our contention that our constraints are of some interest because they stem from our interpretation of probability in an intuitionistic context. That interpretation parallels a very appealing interpretation of classical probability theory, simply replacing talk about truth by talk about proof in the intuitionistic fashion. Of course we do not claim that ours is the only interpretation possible for our constraints. Nor do we wish to deny that another interpretation might lead one to different constraints. For example, it is quite possible that one may find a quite appealing interpretation leading to van Fraassen's constraints. We do wish to claim that our interpretation forces the constraints we have used. We now turn to an intuitive justification of our nonclassical constraints in terms of the basic interpretation discussed above.

4 Intuitive justification for C4 Condition C4 has a certain slogan appeal: the probability of the conditional is the conditional probability. However, for different purposes, it has been considered and rejected by other authors, e.g., in [17]. And C4 has consistently drawn negative attention in every public presentation of our intuitionistic probability theory. However, given our basic interpretation, C4 is a perfectly natural and obviously correct constraint.

Using the classical probability constraints D1-D7, one can prove both of the following:

(a) $Pr(A \supset B, C) \geqslant Pr(B, A \,\&\, C)$

(b) $Pr(A \supset B, C) = Pr(B, A \,\&\, C)$ iff $Pr(A, C) = 1$ or $Pr(B, A \,\&\, C) = 1$.

In our intuitionistic probability theory, the inequality in (a) is an equality, and, as a result, the condition on the right half of (b) drops out. The functions thus mirror the fact that, *proof-theoretically*, intuitionistic conditionals are stronger than classical ones. Thus probabilistically, intuitionistic conditionals are in general less likely than classical conditionals. The particular modification of (a) and (b) that we chose can be justified in terms of our basic interpretation.

In both the classical and the intuitionistic case, it is possible to prove the following:

$$Pr((A \supset B) \,\&\, A, C) = Pr(B \,\&\, A, C).$$

But the classical conjunction and the intuitionistic conjunction behave alike, provability-wise and probability-wise. So we can apply the product rule for conjunctions and obtain the following:

$$Pr(A \supset B, A \,\&\, C) \times Pr(A, C) = Pr(B, A \,\&\, C) \times Pr(A, C).$$

Hence, as long as $Pr(A, C)$ is not 0, both intuitionistic and classical probability will yield the following result:

(*) $\quad Pr(A \supset B, A \,\&\, C) = Pr(B, A \,\&\, C).$

A major difference between intuitionistic and classical probability theories can now be brought into play. Recall our basic interpretation of intuitionistic probability in terms of provability, and note the following fact about intuitionistic logic:

$$S \cup \{A, C\} \vdash_I A \supset B \text{ iff } S \cup \{C\} \vdash_I A \supset B.$$

But given this fact, assuming A over and above the statements in $S \cup \{C\}$ should not affect one's belief in the (intuitionistic) *provability* of $A \supset B$. Hence, *for intuitionistic probability*, one *should* have the following:

(**) $\quad Pr(A \supset B, A \,\&\, C) = Pr(A \supset B, C).$

But given (*) and (**), we immediately have C4, i.e.,

$$Pr(A \supset B, C) = Pr(B, A \,\&\, C).$$

Note for contrast that assuming A over and above the statements in $S \cup \{C\}$ might well affect, and justifiably affect, one's belief in the *truth* of $A \supset B$. Thus for *classical Pr*, $Pr(A \supset B, C)$ might well exceed $Pr(B, A \,\&\, C)$.

In the course of the foregoing argument, we assumed that $Pr(A, C)$ was not 0. A closely related argument can be used to justify C4, whether or not $Pr(A, C) = 0$. For *classical* probability (though *not* for intuitionistic probability), it is possible to prove the following (see [15]):

$$Pr(D \supset E, F) = 1 - Pr(D, F) + Pr(D \,\&\, E, F).$$

So in particular, the following holds for the classical case:

$$Pr(A \supset B, A \,\&\, C) = 1 - Pr(A, A \,\&\, C) + Pr(A \,\&\, B, A \,\&\, C).$$

But both of the following hold for both classical and intuitionistic probability:

$$Pr(A, A \,\&\, C) = 1$$
$$Pr(A \,\&\, B, A \,\&\, C) = Pr(B, A \,\&\, C).$$

Hence for the *classical* case, we know (*) holds even in those cases when $Pr(A, C) = 0$. The crucial question is the justification for (*) in the intuitionistic case. The following fact about intuitionistic provability settles the question:

$$S \cup \{A \,\&\, C\} \vdash_I A \supset B \text{ iff } S \cup \{A \,\&\, C\} \vdash_I B.$$

This intuitionistic result holds whether or not $S \cup \{C\} \vdash_I A$. Thus given our interpretation of intuitionistic probability, (*) should hold for intuitionistic probability even when $Pr(A,C) = 0$. But we have already argued that (**) should hold intuitionistically, and that argument did not depend on the value of $Pr(A,C)$. Thus C4 should hold intuitionistically regardless of the value of $Pr(A,C)$.

5 ***Intuitive justification for C8*** With $Pr(A,B \vee C)$ understood as the degree of warranted belief that $S \cup \{B \vee C\} \vdash_I A$, C8 corresponds to the inference pattern commonly called *disjunction elimination*. Assume $B \vee C$ is given, and suppose we wish to prove A. First, we would attempt to prove A from $S \cup \{B\}$. Then, given that we can prove A from $S \cup \{B\}$, we would attempt to prove A from $S \cup \{C\}$. If we succeed in both undertakings, then A is provable from $S \cup \{B \vee C\}$. (When matching the foregoing against C8, recall that if A is provable from $S \cup \{B\}$, then exactly the same statements are provable from $S \cup \{B \supset A\}$ as from S alone. So proving A from $S \cup \{C\}$ is tantamount to proving it from $S \cup \{C \,\&\, (B \supset A)\}$.)

For a slightly less direct justification, note that the formula $(B \vee C) \supset A$ is provable from S (and hence from $S \cup \{t\}$) if and only if $(B \supset A) \,\&\, (C \supset A)$ is. So, given our intuitionistic interpretation of Pr, we *should* have:

(C8′) $Pr((B \vee C) \supset A, t) = Pr((B \supset A) \,\&\, (C \supset A), t)$.

Then applying C4 to the left side and C6 to the right side, we obtain:

$$Pr(A, (B \vee C) \,\&\, t) = Pr((C \supset A) \,\&\, (B \supset A), t).$$

Applying the product rule, C5, to the right side and using L2.1(i) of [14] to eliminate some occurrences of 't' gives:

$$Pr(A, B \vee C) = Pr(C \supset A, B \supset A) \times Pr(B \supset A, t).$$

Applying C4 to the conditionals, using L2.1(i) to eliminate 't', and rearrangement of the order of the product gives C8:

(C8) $Pr(A, B \vee C) = Pr(A,B) \times Pr(A, C \,\&\, (B \supset A))$.

Note that the same steps in the reverse order lead from C8 to C8′. So, in the presence of C4, C8 and C8′ are interchangeable. But C8′ holds for classical probability. Thus using C8′ in lieu of C8 would make for a *single* nonclassical constraint on Pr, namely C4.

Constraint C8 leads to an interesting "structural" observation. In classical probability theory, the following constraints (or equivalents thereof) are often imposed to deal with disjunctions:

(c) If $Pr(B \,\&\, C, A) = 0$, then
$Pr(B \vee C, A) = Pr(B,A) + Pr(C,A)$.

(d) If $Pr(B \,\&\, C, A) \neq 0$, then
$Pr(B \vee C, A) = Pr(B,A) + Pr(C,A) - Pr(B \,\&\, C, A)$.

If we interchange the arguments of Pr, replace addition by multiplication, and replace subtraction by division, we obtain:

(e) If $Pr(A,B \,\&\, C) = 0$, then
$Pr(A,B \vee C) = Pr(A,B) \times Pr(A,C)$.

(f) If $Pr(A,B \,\&\, C) \neq 0$, then $Pr(A,B \vee C) = \dfrac{Pr(A,B) \times Pr(A,C)}{Pr(A,B \,\&\, C)}$.

But it can be shown that in the presence of C1-C7, C8 is equivalent to (e)-(f). Hence, C8 may be regarded as a kind of "dual" of (c)-(d). And in the same sense of "dual", C10 is the dual of the usual classical constraint which may be imposed when the existential quantifier is taken as primitive.[8]

*6 **Dutch book argument*** As has been frequently noted before, probabilities often serve as betting odds, and hence $Pr(A,B)$ is often understood (for classical Pr) as the degree to which given $S \cup \{B\}$ one would be warranted in betting that A. So, expectedly enough, betting strategies have been used in classical probability theory to justify certain of the constraints placed on Pr (in particular, the *nonquantificational* constraints). Suppose a gambler were enticed by another (shrewder) gambler into placing such a series of bets that the first gambler would experience a net loss no matter what the outcome. The second gambler would then be said to have made a "Dutch book" against the first. It has been shown, for certain idealized betting situations, that if a gambler, when choosing the odds at which he will bet, uses a probability function that violates constraints akin to D1-D6, it is always possible for another gambler to make a Dutch book against him. As we will now show, the same sort of justification can be offered for the nonquantificational constraints that we used to characterize intuitionistic probability, namely C1-C8.[9]

The betting situations we consider will also involve two gamblers. One of them will be known as the *agent*. It is he who in any series of bets sets the odds, i.e., selects the function Pr (known in the present context as an *odds function*) that takes the pairs (A,B) concerned into reals. The other gambler will be known as the *agent's opponent*, or just the *opponent*, for short. It is he who specifies each pair (A,B) of statements of IL on which a bet is placed,[10] the stakes K, whether the bet is to be unconditional or not (we explain the distinction below), and whether he will bet for or against (A,B). If the opponent bets for (against) (A,B), the agent has to bet against (for) the pair. A *bet* thus consists of: (i) an ordered pair (A,B); (ii) an odds function Pr; (iii) stakes K; (iv) identification of the bet as unconditional or not; and (v) the side taken by the opponent. The odds function is under the control of the agent; all else is determined by his opponent.

New with us is the introduction of a *logical oracle*. The oracle has at its disposal some fixed set S of statements of IL. The provenance of S does not affect any of the results below. Whether the membership of S is known to the agent and his opponent does not matter either. Indeed, leaving them in the dark on this score makes for the (epistemic) uncertainty that is at the heart of probability. New also will be the matters on which bets are placed. Rather than matters of *truth* as in the classical case, they will of course be matters of *intuitionistic provability*; and these matters, the bets once placed, will be submitted to the oracle for adjudication. We assume that the oracle can correctly answer all questions put to it. These questions will be in the form of ordered pairs (A,B), where A and B are statements of IL. The oracle will answer "yes" if $S \cup \{B\} \vdash_I A$, and "no" otherwise. (Recall our basic interpretation of $Pr(A,B)$, for intuitionistic Pr, as the degree to which, for some set S of unstated background assumptions, one is warranted in believing that $S \cup \{B\} \vdash_I A$.)[11]

We will now distinguish unconditional from conditional bets. In an *unconditional* bet, the pair (A,B) on which agent and opponent are betting is submitted to the oracle *as it stands*, with gain and loss determined from the oracle's answer. If the oracle answers "yes", the individual betting "for" (A,B) wins; if the oracle answers "no", the one betting "against" wins. In a *conditional* bet, the pair (B, t) is first submitted to the oracle. If the oracle answers "no", the bet is terminated without gain or loss to either bettor.[12] If the oracle answers "yes", the pair (A, t) is then submitted to it with gain and loss determined from its answer to this second question. In both cases it should be noted that to bet against (A,B) is not the same as betting for $({\sim}A,B)$.

Gain and loss are calculated exactly as in the classical case. We assume that the stakes K are a sum of money greater than 0. (If K equaled 0, no bet would be placed; if K were smaller than 0, the positions of winner and loser would simply be reversed.) We also assume that all betting situations are zero sum. Gain and loss for the agent are then as in the following table, where K represents the stakes and r represents the odds:

		oracle's answer: yes	oracle's answer: no
agent	for	$(1-r)K$	$-rK$
	against	$-(1-r)K$	rK

The entries on a corresponding table for the agent's opponent would be the negatives of those for the agent.

We shall say that an agent's odds function is *rational* just in case it is impossible for his opponent to make a Dutch book against him, i.e., just in case it is impossible for the opponent to place such a series of bets that, no matter what the outcome, the agent experiences a net loss.[13]

We first establish that if a function Pr is to be rational, it must meet C1-C2 plus two constraints involving intuitionistic provability. On the strength of this result, we then establish that if Pr is to be rational, it must meet each of C1-C8. Whether it must also meet C9-C10 is, as of this writing, an open question.[14]

Theorem 1 *Let Pr take pairs of statements of IL into reals. If Pr is to constitute a rational odds function, then Pr must meet the following constraints.*

(a) $0 \leqslant Pr(A,B) \leqslant 1$
(b) $Pr(A,A) = 1$
(c) *If for every* S, $S \vdash_I B \supset A$ *iff* $S \vdash_I D \supset C$, *then* $Pr(A,B) = Pr(C,D)$.

(d) *If for every* S, $S \vdash_I B \supset A$ *iff both* $S \vdash_I D \supset C$ *and* $S \vdash_I F \supset E$, *then* $Pr(A,C) = Pr(C,D) \times Pr(F \supset E, D \supset C)$.

Proof: *Ad* (a): Let $Pr(A,B) = r$. (i) Suppose for reductio that $r > 1$. Then the agent's opponent can bet unconditionally against (A,B) for any stakes K he pleases, in which case the agent must bet for (A,B). If the oracle answers "yes", the return to the agent is $(1 - r)K$, which will be negative for $r > 1$. If, on the other hand, the oracle answers "no", the return to the agent is $-rK$, which will also be negative. In either case, therefore, the agent is sure to experience a loss, and his odds function Pr is irrational. (ii) Suppose next that $r < 0$. Then the agent's opponent can bet unconditionally for (A,B) for stakes K again, so the agent is forced to bet against (A,B). If the oracle answers "yes", the return to the agent is $-(1 - r)K$, which is negative for $r < 0$. On the other hand, if the oracle answers "no", the return to the agent is rK, which will also be negative for $r < 0$. Again, in either case the agent is sure to experience a loss, and his odds function is irrational. Hence (a) is established.

Ad (b): Let $Pr(A,A) = r$. By (a), we know $r \leqslant 1$, so suppose for reductio that $r < 1$. Then the agent's opponent can bet unconditionally for (A,A) for stakes K, forcing the agent to bet against. But since A is intuitionistically derivable from A, no matter what S is, the oracle has to answer "yes". So the return to the agent is sure to be $-(1 - r)K$, which is negative for $r < 1$. Thus the agent is sure to experience a loss, so Pr is irrational. Hence (b) is established.

Ad (c): Let $Pr(A,B)$ and $Pr(C,D)$ respectively equal r and r'. By virtue of (a), we know both must lie somewhere in the closed interval from 0 to 1. (i) Suppose for reductio that $r' < r$. Then the agent's opponent can bet unconditionally against (A,B) and for (C,D), for stakes K in both cases. As a result, the agent must bet for (A,B) and against (C,D). Now, by the hypothesis of (c), the oracle is sure to give the same answer to both pairs. If the answer is "yes" to both, the net return to the agent is $(1 - r)K - (1 - r')K$, which is just $(r' - r)K$. But this amount is negative for $r' < r$. On the other hand, if the oracle answers "no" to both, the net return to the agent is $-rK + r'K$, which again is just $(r' - r)K$, a negative amount. (ii) Suppose next that $r < r'$. If the agent's opponent bets unconditionally for (A,B) and against (C,D) for stakes K in both cases, the net return to the agent can be shown by a symmetric argument to be negative, no matter what the oracle answers. Thus, so long as r and r' are different, the agent is sure to experience a loss, so Pr is irrational. Hence (c) is established.

Ad (d): Let $Pr(A,D)$, $Pr(C,D)$, and $Pr(F \supset E, D \supset C)$ respectively equal r, r', and r''. By (a), all must be reals in the closed interval from 0 to 1. (i) Suppose for reductio that $r > r'r''$. Then the agent's opponent can bet unconditionally against (A,B) for stakes K, bet unconditionally for (C,D) for stakes $r''K$, and bet conditionally for $(F \supset E, D \supset C)$ for stakes K. Three outcomes are possible: *Outcome one*: The oracle answers "yes" to (A,B). By the hypothesis of (d), it follows that the oracle must then answer "yes" to all of (C,D), $(D \supset C, t)$, and $(F \supset E, t)$. The net return to the agent will then be $(r'r'' - r)K$, which will be negative for $r > r'r''$. *Outcome two*: The oracle answers "no" to (A,B) and to (C,D). Hence the oracle will answer "no" to $(D \supset C, t)$, and the conditional bet on $(F \supset E, D \supset C)$ will be terminated. The net return to the agent will again be $(r'r'' - r)K$, which is negative. *Outcome three*: The oracle answers "no" to (A,B) and "yes" to (C,D). By the hypothesis of (d), the oracle must then answer "yes" to $(D \supset C, t)$ and "no" to $(F \supset E, t)$. In this case again, the net return to the agent will be $(r'r'' - r)K$, a negative amount. (ii) Suppose next that $r < r'r''$. If the agent's opponent bets unconditionally against (A,B), unconditionally for (C,D), and conditionally for $(F \supset E, D \supset C)$, for stakes K, $r''K$, and K, respectively, the net return to the agent can be shown by a symmetric argument to be negative, no matter what the oracle answers. Thus so long as r is different from $r'r''$, the agent is sure to experience a loss, so *Pr* is irrational. Hence (d) is established, and that completes the proof of Theorem 1.

Theorem 2 *Let Pr take pairs of statements of IL into reals. If Pr is to constitute a rational odds function, then Pr must meet constraints C1-C8.*

Proof: Theorem 1(a) and (b) assure C1 and C2, respectively. Condition C3 follows from L3.1(a) and (b) of [14], in conjunction with Theorem 1(b) and (c). Conditions C4, C6, and C7 follow respectively from L3.1(c), (e), and (f) of [14], in conjunction with Theorem 1(c). Conditions C5 and C8 follow respectively from L3.1(d) and (g) of [14], in conjunction with Theorem 1(d). Thus Theorem 2 is established.

As remarked earlier, whether *Pr* must also meet C9-C10 in order to be rational is an open question. We suspect the answer to this question is yes. And whether *Pr* is sure to be rational providing it meets C1-C10 is also an open question. We suspect the answer to this latter question is negative, and hence that C1-C10 are not sufficient to guarantee that *Pr* constitutes a rational odds function. Note indeed that the characteristic differences between classical and intuitionistic logic played no role in our Dutch book argument. In fact, when answering the questions posed to her, our oracle could just as well have called on her knowledge of classical logic. Since the characteristic differences between the two logics are bound to be felt somewhere, perhaps they will be felt when matters of sufficiency are addressed.

NOTES

1. As regards Carnap, see in particular [2], pp. 7-16, where 'credence' is to be understood as 'rational credibility'. Among contributors to probabilistic semantics, Bendall, Ellis, Field, Harper, etc., think of *Pr*(A,B) epistemically, and indeed they refer to probabilistic semantics as *belief-theoretic* semantics. See [1], [3]-[6], etc.
2. Or, to use a less prosaic phrase, '*in light of B*'.
3. Concerning Robinson's arithmetic, see [10], *passim*.
4. The quotation is from pp. 98-99 of [8].
5. Plus an extra constraint to the effect that $(\exists C)(\exists D)(Pr(A,B) \neq Pr(C,D))$, which, as shown in [7], is equivalent to $(\exists A)(\exists B)(Pr(A,B) \neq 1)$. The constraint is dispensable in probabilistic semantics, as the soundness and completeness results of [7], [11], etc., attest. And, given our account of *Pr*(A,B), it is unwelcome as well. Indeed, were *S*

inconsistent, a possibility we do not rule out, the believability of A given $S \cup \{B\}$ should equal 1 for *any* A and B. Hence, for *some* probability function Pr, we should have $Pr(A,B) = 1$ for *any* A and B. Proof that the constraints we list are equivalent to Popper's will be found in [7].

6. Proof will be found in several places, among them [6] and [11].

7. See [16], p. 338.

8. C10 follows by strictly classical means from its "dual". Whether the "dual" follows by like means from C10 is an open question.

9. So-called "Dutch book arguments" for classical probability theory were proposed by B. de Finetti and F. P. Ramsey in independent papers dated 1931; by A. Shimony, R. S. Lehman, and J. G. Kemeny in independent papers dated 1955; and by E. Adams in a 1959 paper. For a review of the results obtained and pertinent references, see [2], pp. 105-116.

10. In the terminology of Dutch book arguments, to place a bet on a pair (A,B) is of course to bet on A (i.e., for or against A) in light of B.

11. Our oracle enables us to couch in *the indicative* matters which otherwise *ought* to be couched in the *subjunctive*, a nicety to which the writers in Note 9 are all too often insensitive. To illustrate the point, turn to the table of gains and losses and suppose that the agent bets for (A,B). To say that the agent wins $(1 - r)K$ if the oracle answers "yes" is simply to say that the agent would win $(1 - r)K$ if A were provable in IL from B and the assumptions in S. Since intuitionistic logic is undecidable, *we* may never know whether or not A is so provable, and hence *we* may never know whether or not the agent wins $(1 - r)K$. However, our oracle does know, thereby enabling us as we prove Theorem 1 below to report the *actual* return to the agent. Or take the matter of terminated bets. To say that the bet on (A,B) is terminated if the oracle answers "no" to the question (B, t) is simply to say that the bet would be terminated if B were not provable in IL from t and the assumptions in S. *We* may never know whether or not B is so provable, and hence *we* may never know whether or not the bet is terminated. Our oracle, however, does know, thereby enabling us as before to report the *actual* return to the agent. Objections have recently been raised against Dutch book arguments for classical probability theory. In particular, see [9]. At least some of these objections are met by employing our oracle.

12. We mimic here (but, as claimed in the preceding note, may be improving upon) instructions in the literature whereby a conditional bet on (A,B) "is" terminated if B "is" false.

13. We borrow the appelation 'rational' from [13], one of the 1955 papers mentioned in Note 9. Other epithets used are 'coherent' (de Finetti), 'consistent' (Ramsey), 'fair' (Kemeny), etc. But, as agents are commonly referred to in action theory as rational, calling the rates at which they bet rational seemed to us particularly appropriate.

14. In [13] Lehman suggests that in classical probability theory a function Pr is rational if and only if it meets constraints akin to D1-D7. However, to simplify matters, he limits his treatment to nonquantificational constraints, a step we deeply regret.

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