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# Material Equivalence and
# Tautological Entailment

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> The introduction of any new device into the symbolism of logic is neces-
> sarily a momentous event. In logic a new device should not be introduced in
> brackets or in a footnote with what one might call a completely innocent air.
> (Tractatus, 5.452)

. . .

It is my purpose in this paper to argue that Anderson and Belnap's
([1], Chap. 3) system of tautological entailment is, at best, a mere fragment of
a full theory of entailment for truth-functions and that no guide has been given
by Anderson and Belnap as to how to complete it. A secondary purpose is to
show that two recently published accounts of entailment are not coextensional
with tautological entailment, as their proponents claim, nor with each other. I
shall assume familiarity with the theory of tautological entailment throughout
this paper.

In Lewy's *Meaning and Modality* there is a long and absorbing discussion
of the Lewis paradoxes, concluding with a definition of "strictly entails" that
Lewy conjectured would be satisfied by tautological entailment, ([11], p. 150).
Since that definition and the conjecture were made, however, Clark and Dunn
have separately shown the definition to be defective. Both have proposed ways
in which they believe Lewy's definition may be repaired, and these repairs
reveal definitions that are coextensional with those of tautological entailment
(see [3], [2], and [7]). If these results were correct then some support would
be given to the thesis that entailment was to be identified with tautological
entailment in the same manner that Church's thesis gained support from the
coextensiveness of Turing computability, $\lambda$-definability, etc. I shall argue that
any such clustering is extremely limited in the case of the recent entailment
definitions.

*1*      I shall begin by examining two claims that Clark makes about the relationship between the definitions of entailment given by Geach, Smiley, and Anderson and Belnap. The definition of Smiley's with which Clark is concerned is

$A \overset{S}{\rightarrow} B$ iff $A \supset B$ is an *SI* of some tautology $C \supset D$ with contingent $C, D$ (*S*-necessitation). ([13], p. 240)

Clark claims this definition is equivalent to one of Geach's ([3], p. 10); he also claims that Anderson and Belnap have shown that if $A \overset{t}{\rightarrow} B$ (*A* tautologically entails *B*) then $A \overset{S}{\rightarrow} B$. It is time to put the record straight. Firstly, Geach's later papers on entailment ([8] and [9]) have not given a definition of "*A* entails *B*" *simpliciter*: what Geach does give is a definition of the *entailment extension* of a logical system. Thus whether "$(p \supset p) \rightarrow (p \equiv p)$" belongs to the entailment extension of a logical system will depend on the system. Geach made this very clear in the exchange of papers between us concerning this very example ([9] and [4]). It is therefore quite incorrect to see Geach's definition as equivalent to Smiley's: Geach gives no definition of "$A \rightarrow B$" at all.

Smiley's account of *S*-necessitation, on the other hand, is given entirely in terms of tautology and contingency. There is no mention of different logical systems in Smiley's definition: it presumes nothing more than the notion of a tautology. Such an approach is not without its difficulties, however. Smiley says he is restricting his discussion to "the elementary formal logic of propositions" but, unfortunately, does not clarify what he takes this to be. The occurrence of the word "formal" would suggest that a *formal system* treatment of propositional logic is intended. This is belied by the indication of uniqueness: "*the* elementary formal logic of propositions". Now there are many different formal systems of elementary (i.e., classical, two-valued) logic. Although there is a sense in which these formal systems are equivalent, it is nevertheless the case that they are different *formal* systems. (One system may contain only the connectives for negation and disjunction, another may contain the connectives of negation and conjunction, one system may contain the rule of *modus tellens*, another may not.) Smiley must have been aware of such difficulties since he says that certain statements involving *S*-necessitation have their truth-values dependent on whether or not "$\equiv$" is taken as primitive. Although he does not consider the matter further, it is clear that the definition cannot avoid some move in the direction of Geach's system dependence.

For both Smiley and Geach it is the case that their definitions have complete generality. Given any two sentences of a language of propositional logic it can be determined when one 'entails' the other or when a sentence belongs to the entailment extension of a system of propositional logic. Whatever other demerits the Smiley definition may have it has the virtue that it may be applied to any sentences of any system of propositional calculus. Indeed, a theory of entailment should be capable of fulfilling at least this requirement.

Despite Smiley's worries over "$\equiv$" he himself uses a definition of tautology which relies on the familiar truth-table rules for the connectives, including "$\equiv$" ([13], p. 235). I shall assume that these familiar rules for the connectives provide *a* correct way of introducing the truth-functional connectives (I shall

not assume that it is the only correct way or that it has any logical primacy) so that there can be no objection to defining material equivalence, "≡", by the table "(TFFT)($p,q$)" (in the notation of the *Tractatus*). Since this is the apparatus Smiley uses I shall consider that any occurrence of "≡" in a formula of the form $A \xrightarrow{s} B$ is not just an abbreviation for some longer expression.

After this discussion it will be illuminating to look at the claim that if $A \xrightarrow{t} B$ then $A \xrightarrow{s} B$. There are many apparent counterexamples to this claim:

$$\sim p \rightarrow (p \equiv p), \, p \rightarrow (p \equiv p), \, (p \equiv \sim p) \rightarrow (p \equiv q), \, (p \equiv \sim p) \rightarrow (p \equiv p),$$
$$(p \, . \, \sim p) \rightarrow (p \equiv q), \, (p \, . \, \sim p) \rightarrow (p \equiv p).$$

Clearly none of these is an example of $S$-necessitation when "≡" is defined by means of its truth table. They are, however, tautological entailments as may be seen from the fact that "$A \equiv B$" is, in the system of Anderson and Belnap, no more than an abbreviation of "$(\sim A \vee B) \, . \, (\sim B \vee A)$". If the proof ([1], p. 219) of the claim that if $A \xrightarrow{t} B$ then $A \xrightarrow{s} B$ is examined it may be seen that it ignores the occurrence of "≡" in either the antecedent $A$ or the consequent $B$. Certainly the proof shows that if only the connectives for negation, conjunction, and disjunction occur in $A$ and $B$ then $A \xrightarrow{t} B$ only if $A \xrightarrow{s} B$. Since Anderson and Belnap do not consider "≡" in their proof it must be because they take it that it has no real occurrence in their system, although they have introduced a 'convention' that "$A \equiv B$" is short for "$(A \supset B) \, . \, (B \supset A)$" where "$A \supset B$" in turn is short for "$\sim A \vee B$". Since "≡" occurs by 'convention' it is a sensible question to ask whether it has the same logical powers as "≡" as defined by means of the truth table. Now in the two-valued standard logic "$A \equiv B$" and "$(\sim A \vee B) \, . \, (\sim B \vee A)$" coincide in their truth values (TFFT) and so for most purposes it is unimportant if "≡" is used as a primitive in the system or as an abbreviatory device: there is no loss in the powers of expression in a language which employs only conjunction, negation, and disjunction. However, when we leave the well-trodden grounds of ordinary two-valued logic we leave our intuitions (if not our senses) behind. In the two-valued case "$(A \, . \, B) \vee (\sim A \, . \, \sim B)$" also coincides in its truth-values with "$A \equiv B$". There is of course nothing novel here: every truth-function has a conjunctive normal form as well as a disjunctive normal form. Why did not Anderson and Belnap use "$A \equiv B$" as a conventional shortening of the disjunctive normal form? As Prior [12] says in another, similar context it is just a pretense that there is any arbitrariness in giving such abbreviations. "≡" is given this definition because its presence is necessary to give any semblance of completeness to their theory of entailment. Arbitrariness enters only in that some choice between the disjunctive normal form and the conjunctive normal form had to be made. In this context, however, this is a matter of more than minor importance, for in Anderson and Belnap's theory of tautological entailment "$(A \, . \, B) \vee (\sim A \, . \, \sim B)$" and "$(\sim A \vee B) \, . \, (\sim B \vee A)$" have different entailments since the latter does not tautologically entail the former.

Since tautological entailment was introduced to deal with the problem of entailment between truth-functions we should expect that it would answer such questions as whether "$A \equiv B$", considered as defined by its truth-table, entails "$(A \, . \, B) \vee (\sim A \, . \, \sim B)$". Just because Anderson and Belnap have introduced the symbol "≡" as an abbreviatory device it does not follow that material equiva-

lence occurs in their system. The apparent occurrence of material equivalence, through the introduction of "≡", is the merest trick of the light. Of course it may be that Anderson and Belnap believe that their convention embodies certain truths about material equivalence, but this would imply that the introduction of "≡" represents far more than an abbreviation. It would also need argument to establish that the conjunctive normal form rather than the disjunctive normal form should be taken as the definiens.

It is clear, then, that Anderson and Belnap's theory of tautological entailment is restricted to the truth-functions for negation, disjunction, and conjunction. What we are presented with is a fragment that awaits extension to cover the other truth-functions. I am not sure that Anderson and Belnap intended their notion of tautological entailment to be of such small scope. The encyclopaedic nature of their *Entailment* volume would suggest that a complete treatment of entailment for truth-functional sentences at least was intended and nowhere in the work is it specified that only a fragment had been given. I can only assume that the authors believed they had given a full treatment: the appearance of all the standard connectives may fool us (and them) into the belief that there is no more to be done. To test this belief, however, the reader should investigate whether the exclusive disjunction of $A$ and $B$ is or is not tautologically entailed by $A . \sim A$. Again in the two-valued case it does not matter if we define exclusive disjunction as a conjunctive normal form "$(A \lor B) . (\sim A \lor \sim B)$" or as a disjunctive normal form "$(A . \sim B) \lor (\sim A . B)$", but which one should be chosen for tautological entailment? Only the disjunctive normal form is tautologically entailed by $A . \sim A$.

What is needed here is a semantics that would motivate one choice or another (or perhaps neither). Unfortunately, Anderson and Belnap have little to offer on this topic. There are general exhortations to be relevant but such appeals do not help in the case of material equivalence or exclusive disjunction. Semantics have been provided by others to fill this gap and I shall return to them later. Without semantics Anderson and Belnap have provided no more than a fragment of what we might expect a definition of entailment to achieve.

*2*    I shall now turn to Clark's definition which he claims Lewy accepts as capturing his notion and which he claims is equivalent to tautological entailment. The definition is as follows:

> $A \rightarrowtail B$ iff $A \supset B$ is an *SI* of some tautology $C \supset D$ with contingent $C, D$ such that, for every primitive tautology $T$ and every primitive inconsistent wff $I$,
>      (1) if $D \xrightarrow{S} T$ then $A \xrightarrow{S} T$
>      (2) if $I \xrightarrow{S} C$ then $I \xrightarrow{S} D$
>      (3) if $I \xrightarrow{S} C$ and $D \xrightarrow{S} T$ then $I \xrightarrow{S} T$. ([3], p. 10)

A primitive tautology is a tautologous disjunction of propositional variables or their negations; a primitive inconsistent wff is an inconsistent conjunction of the same.

There is no mention in the above definition that it should be restricted to sentences involving only conjunction, disjunction, and negation. Indeed, if Lewy has agreed that the definition has captured his notion then it would seem likely that he interprets the definition in the *general* way I have shown that

Smiley's definition should be interpreted. Lewy himself makes much use of material equivalence throughout his discussion, though he appears to leave it undefined. I shall assume that he had some such definition as the truth-table definition in mind rather than as an abbreviatory device. With this interpretation, the following are not Clark entailments

$$(p \equiv q) . (q \equiv r) \twoheadrightarrow (p \equiv r)$$
$$(\sim p \vee q) . (\sim q \vee p) \twoheadrightarrow (p \equiv q)$$
$$(p \equiv q) \twoheadrightarrow (p . q) \vee (\sim p . \sim q)$$

since, respectively, $(p . q . \sim q . \sim r) \overset{S}{\twoheadrightarrow} (p \equiv q) . (q \equiv r)$; $q . \sim q \overset{S}{\twoheadrightarrow} (\sim p \vee q) . (\sim q \vee p)$; $(p . q) \vee (\sim p . \sim q) \overset{S}{\twoheadrightarrow} q \vee \sim q$. The first shows that transitivity of material equivalence fails (a result that also holds for tautological entailment when the appropriate expansion is made). The second and third examples show that material equivalence will not coentail either its conjunctive normal form or its disjunctive normal form. I do not know whether Lewy finds these results acceptable since his suspicion that his original definition was satisfied by Anderson and Belnap's entailment relation complicates matters. For not only are there my original counterexamples to the thesis that if $A \overset{b}{\twoheadrightarrow} B$ then $A \overset{S}{\twoheadrightarrow} B$, which *a fortiori* are counterexamples to the thesis that if $A \overset{b}{\twoheadrightarrow} B$ then $A \twoheadrightarrow B$ but also there are new additions: the second example above is an $S$-necessitation and a tautological entailment but not a Clark entailment. There are also Clark entailments that are $S$-necessitations and yet are not tautological entailments: $(p . q) \vee (\sim p . \sim q) \twoheadrightarrow (p \equiv q)$, for example.

This last example raises new issues. The proof Clark gives of the theorem $A \overset{b}{\twoheadrightarrow} B$ iff $A \twoheadrightarrow B$ does not mention material equivalence at all. Now, as I have pointed out, Anderson and Belnap's language contains only negation, disjunction, and conjunction so the claim that $A \overset{S}{\twoheadrightarrow} B$ holds if $A \overset{b}{\twoheadrightarrow} B$ holds may be maintained despite the apparent counterexamples. With this qualification Clark's claim that $A \overset{b}{\twoheadrightarrow} B$ only if $A \twoheadrightarrow B$ may be shown. The converse, if $A \twoheadrightarrow B$ then $A \overset{b}{\twoheadrightarrow} B$, holds only if "$A \equiv B$" is treated as an abbreviation for "$(\sim A \vee B)$ . $(\sim B \vee A)$" in the unspecified language in which he frames his definition, for it is only thus that he will achieve the coextensiveness of the two definitions. From Lewy's discussion, however, there is no hint that he would take such a view of material equivalence. The relationship between the two definitions is better stated as being coextensional provided that the languages for which both definitions are given are restricted in logical vocabulary to the connectives of disjunction, conjunction, and negation. I see no reason why Lewy should so restrict his definition or any repaired version of it: at least it would require philosophical argument to maintain that for the purposes of a theory of entailment all connectives other than those for negation, conjunction, and disjunction have to be defined in terms of those three.

What Clark's definition gives is a possible way of *extending* tautological entailment to include other logical connectives such as material equivalence. This extension may result in the divergence of material equivalence from the definition given by Anderson and Belnap of the sign "$\equiv$", as I have shown.

To sum up this section: in the sense in which "$A \twoheadrightarrow B$" can be shown to be coextensive with "$A \overset{b}{\twoheadrightarrow} B$" it is surely not the case that it captures Lewy's original notion; in the sense in which it is an extension of tautological entail-

ment then there is no identity between material equivalence and Anderson and Belnap's abbreviatory "≡".

*3*     I shall now turn to the definition given by Dunn in terms of truth trees and falsity trees. I shall assume some familiarity with these concepts throughout this section. Dunn's definition is as follows:

$A \overset{d}{\twoheadrightarrow} B$ iff   (i)  $A \supset B$ is an *SI* of a tautology $C \supset D$ such that
            (ii)  the truth tree of $C$ has no closed branch and
            (iii)  the falsity tree of $D$ has no closed branch.[1]

He then proves that $A$ entails $B$ in his sense ($A \overset{d}{\twoheadrightarrow} B$) iff $A \overset{t}{\twoheadrightarrow} B$. As in Clark's proof there is no mention of material equivalence or of the sign "≡". Indeed, in his rules for the construction of truth trees Dunn gives no rule for the connective "≡". It would appear then that Dunn is restricting himself to the language of negation, conjunction, and disjunction. If this is the case then there can be no reason to suppose that it captures Lewy's intention since it is severely limited in its scope. Could the definition be extended to a language in which rules for the construction of truth trees and falsity trees for such connectives as "≡" are given? There is an immediate difficulty that presents itself for there are different rules that may be given in the case of such connectives as "≡" which would lead, according to the above definition, to different entailments holding depending on the choice of rules. Also, whichever definition is chosen it will inevitably lead to entailments that differ from Anderson and Belnap's tautological entailments. Suppose then Jeffrey's rules ([10], p. 72) for "≡" are followed in the construction of truth trees:
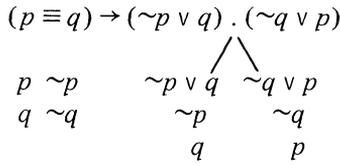
$$\begin{array}{cc} \underline{A \equiv B} & \underline{\sim(A \equiv B)} \\[4pt] A \quad \sim A \;, & \sim A \quad A \\ B \quad \sim B & B \quad \sim B \end{array}$$

Then the following tree demonstrates that $(\sim p \vee q) . (\sim q \vee p) \overset{d}{\twoheadrightarrow} (p \equiv q)$ does not hold

$$(\sim p \vee q) . (\sim q \vee p) \rightarrow (p \equiv q)$$
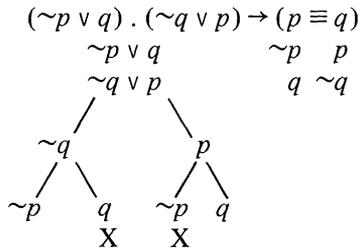


The corresponding tautological entailment with "≡" read as an abbreviation is trivially true. As in the case of Clark's definition, if the tree definition is extended to cover other connectives including "≡" then there is no coincidence between material equivalence and Anderson and Belnap's use of "≡".

    Nor, if this extension is made, will the definition be equivalent to Clark's since the following tree shows a formula which is a true Dunn entailment but not a true Clark entailment:

$$(p \equiv q) \rightarrow (\sim p \vee q) \, . \, (\sim q \vee p)$$

$$
\begin{array}{ccc}
 & & \diagup\diagdown \\
p \;\; \sim p & \;\; \sim p \vee q & \sim q \vee p \\
q \;\; \sim q & \sim p & \sim q \\
 & q & p
\end{array}
$$

(It does not represent a true Clark entailment since $(\sim p \vee q) \, . \, (\sim q \vee p) \xrightarrow{\;\;}$ $(q \vee \sim q)$.)

In sum: either Dunn intends to offer his definition as limited to conjunction, disjunction, and negation in which case it does not capture Lewy's notion or it is intended to be supplemented by rules for other connectives in which case there is a divergence from Anderson and Belnap for some formulas including "≡". In addition there is a divergence between this extension and the general interpretation of Clark's definition.

*4*    In the previous three sections I have shown that if Anderson and Belnap's, Clark's, and Dunn's definitions are limited in scope to conjunction, disjunction, and negation then they are indeed equivalent, but they do not remain equivalent if Clark's and Dunn's definitions are extended to cover other connectives. It surely behooves a theory of entailment for truth-functions to cover *all* truth-functions including material equivalence. This Anderson and Belnap's theory does not do. More importantly, it does not tell us how to extend their theory to fit the other truth-functions. Clark's definition does lend itself to this extension (perhaps it was even intended to have this generality originally) and is at least satisfying in that respect. Dunn's definition may be extended by giving rules for the other connectives in terms of truth tree construction, though here there may be some difficulty over which rules to use. Anderson and Belnap's theory is a rococo extravagance: it is decorative, it is elegant, it is unsatisfying.

How might Anderson and Belnap's definition be extended to the other connectives? Semantics have been given by Dunn and others for tautological entailment together with explanations of the motivation behind them. I do not wish in this paper to examine whether there is good reason to adopt any of these semantics but I want to see what happens if we take the semantics and apply it to "≡", considered as defined by the truth table. In conformity with Dunn's semantics and using his notation the following should be added to his recursive definition of *relevance valuation*:

$T \, \epsilon \, V(A \equiv B)$ iff either $T \, \epsilon \, V(A)$ and $T \, \epsilon \, V(B)$ or $F \, \epsilon \, V(A)$ and $F \, \epsilon \, V(B)$ and
$F \, \epsilon \, V(A \equiv B)$ iff either $T \, \epsilon \, V(A)$ and $F \, \epsilon \, V(B)$ or $F \, \epsilon \, V(A)$ and $T \, \epsilon \, V(B).$[2]

This seems to be the most appropriate addition, bearing in mind the motivation behind the definition for the other connectives. However, with this addition *none* of the above definitions will fit the Dunn semantics. To see this, Anderson and Belnap's theorem $(q \, . \, \sim q) \xrightarrow{\;\;} (p \equiv q)$ will not be relevantly valid, for it takes the value $F$ when $T \, \epsilon \, V(q)$ and $F \, \epsilon \, V(q)$ and neither truth value belongs to $V(p)$. For Clark and Dunn the formula $p \rightarrow (p \equiv p)$ is not true on either extension, yet it will be relevantly valid. Dunn's semantics provide another possible extension to the theory of tautological entailment.

That there are so many different ways of extending Anderson and Belnap's theory shows, I believe, that their theory was not grounded on anything firmer than a dislike of the disjunctive syllogism. Their cavalier treatment of material equivalence shows how shallow that theory is.

## NOTES

1. See [3], p. 43. I have combined a number of Dunn's definitions here for brevity.

2. [6]; There are other proposed semantics but these turn out to be isomorphic with Dunn's, as his discussion in his paper shows. Each of these semantics is formally equivalent to a four-valued logic characterized by Smiley's matrices as I have shown in [5].

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