Notre Dame Journal of Formal Logic
Volume 23, Number 4, October 1982

A Note on Initial Segment Constructions in Recursively Saturated Models of Arithmetic

C. SMORYŃSKI*

This note is something of an addendum to our earlier paper [5], in which, with the aid of a simple method of constructing elementary initial segments of recursively saturated models of arithmetic, we constructed a continuum of elementarily inequivalent structures $(\mathfrak{N}; \mathfrak{M})$, where $\mathfrak{N}$ was a fixed countable recursively saturated model of *PA* and $\mathfrak{M}$ a recursively saturated elementary initial segment of $\mathfrak{N}$. Herein we take a close look at this method, recounting past glories, offering new facts, and citing a few minor open problems.

The basic construction we are referring to was originally performed in joint work with Jonathan Stavi. In Section 1, we review this construction and the original application from [7], and follow it up with a few related observations. Section 2 reviews the application cited above and, again, follows it up with a few related observations. This application produces a large variety of pairs, $(\mathfrak{N}; \mathfrak{M})$, with $\mathfrak{M}$ a recursively saturated elementary initial segment of a given countable recursively saturated model $\mathfrak{N}$. In Section 3, we ask the question of obtaining variety without the basic construction. This question has two senses: First, can we construct a large number of decidedly distinct such pairs, $(\mathfrak{N}; \mathfrak{M})$, without using the basic construction? And second, can we construct such a large number of such pairs, $(\mathfrak{N}; \mathfrak{M})$, where $\mathfrak{M}$ is not of the form provided by the basic construction? We have only partial, but positive, solutions to

*The new results presented herein were obtained during the author's stay in Warsaw. He would like to thank Cecylia Rauszer for the invitation and the organisors of the Foundations Seminar for the necessity of preparing the lecture which resulted in the present note. Thanks are also due to Roman Kossak and Henryk Kotlarski for observations noted within.

Received December 16, 1980

offer at present. Finally, there is a small appendix containing some observations of Henryk Kotlarski made after the rest of the paper had been written.

The notation used is reasonably near-standard. Anything unfamiliar is explained in our earlier papers [4] and [5]. (Although it is probably not necessary, it would be helpful if the reader had these papers at hand.) An important local notational convention we should stress is this: Except for the first half of Section 1, the letters $\mathfrak{N}$ and $\mathfrak{M}$ will denote countable recursively saturated models of arithmetic. Moreover, we will usually regard $\mathfrak{M}$ as an elementary initial segment of $\mathfrak{N}$—although which of the two models is fixed and which is variable or to be constructed will depend on the context. Finally, we only consider countable models—even in the first part of Section 1, where the assumption is not needed.

1 Introduction to the basic construction For the time being, let $\mathfrak{N}$ be a fixed countable nonstandard model of *PA*. We do not yet assume $\mathfrak{N}$ to be recursively saturated.

We begin with some preliminary definitions and explanatory remarks.

1.1 Definition Let $a,b \in |\mathfrak{N}|$. We say a and b *belong to the same sky*, written $a \approx b$, if there are parameter-free Skolem functions F,G such that $b \leqslant Fa$ and $a \leqslant Gb$. The equivalence class of an element $a \in |\mathfrak{N}|$ with respect to this relation is called its *sky*, and is written

$$Sk(a) = \{b \in |\mathfrak{N}| : a \approx b\}.$$

The skies of $\mathfrak{N}$ inherit the ordering of the model,

$$Sk(a) < Sk(b) \text{ iff } a < b \text{ and } Sk(a) \neq Sk(b).$$

We also write $a \ll b$ for $Sk(a) < Sk(b)$.

In general the only restriction on the ordering of the skies of $\mathfrak{N}$ is that there be a least sky; if, however, $\mathfrak{N}$ is recursively saturated, the remaining skies are ordered in the type of the rationals.

Concerning our next definition, recall that every natural number codes a finite sequence of natural numbers. If x codes $(x_0, \ldots, x_{n-1})$, we write $n = lh(x)$ and $x_i = (x)_i$. This notation extends to nonstandard codes.

1.2 Definition Let $b \in |\mathfrak{N}|$. We say *b codes an ascending sequence of skies* in $\mathfrak{N}$, written $b \in ASS(\mathfrak{N})$, if both

i. $lh(b)$ is nonstandard
ii. for $c < d < lh(b)$, we have $(b)_c \ll (b)_d$.

In general, $ASS(\mathfrak{N})$ can be empty. If, however, $\mathfrak{N}$ is recursively saturated, elements of $ASS(\mathfrak{N})$ exist in abundance. Indeed, it was proven in our joint work with Stavi [7] that this is characteristic of recursively saturated models:

1.3 Theorem *The following are equivalent*:

i. $\mathfrak{N}$ *is recursively saturated*
ii. $\forall a \in |\mathfrak{N}|\ \exists b \in ASS(\mathfrak{N})[(b)_0 \geqslant a]$
iii. $\forall a \in |\mathfrak{N}|\ \exists b \in ASS(\mathfrak{N})[lh(b) \geqslant a]$.

As we said, the proof appears in [7], the title of which paper states an immediate corollary.

It is because of this theorem and our evident interest in elements coding ascending sequences of skies that we will assume $\mathfrak{N}$ to be recursively saturated. But we do not make this assumption quite yet. We first wish to introduce the simple initial segment construction cited in the introduction and remark on its rôle in the proof of Theorem 1.3.

1.4 Definition Let $b \in ASS(\mathfrak{N})$ and let $I \subseteq_e \mathfrak{N}$ be a proper initial segment of $\mathfrak{N}$ closed under successor, with $I < lh(b)$. We define $\mathfrak{M}(I,b) \prec_e \mathfrak{N}$ by

$$|\mathfrak{M}(I,b)| = \bigcup_{i \in I} [0,(b)_i],$$

where $[0,c]$ is the initial segment $\{d \in |\mathfrak{N}|: d \leqslant c\}$ of $\mathfrak{N}$.

[That $\mathfrak{M}(I,b)$ is an elementary substructure of $\mathfrak{N}$ follows from the closure of $|\mathfrak{M}(I,b)|$ under Skolem functions–which follows from the fact that there is no last sky represented by the $(b)_i$'s for $i \in I$.]

This operation bears on the proof of Theorem 1.3 via the following result of [7].

1.5 Theorem *Let $b \in ASS(\mathfrak{N})$ and let $I \subseteq_e \mathfrak{N}$ be a proper initial segment of $\mathfrak{N}$ closed under successor, with $I < lh(b)$. Then: $\mathfrak{M}(I,b)$ is recursively saturated.*

If $\mathfrak{N}$ is already recursively saturated, the recursive saturation of $\mathfrak{M}(I,b)$ is not particularly remarkable, If, however, $\mathfrak{N}$ is not assumed to be recursively saturated, an argument is required. [Briefly: Because $\mathfrak{M}(I,b)$ has no greatest sky, we can assume any recursive type τv to contain a bound, say: $v < (\bar{b})_{\bar{i}_0}$. Moreover, since the Skolem functions cannot reach from one $(b)_i$ to the next, the quantifiers in all formulae of τv can be bounded by projections of b. Thus τv is reduced from being a recursive type over $\mathfrak{M}(I,b)$ to being a recursive type of Δ_0-formulae over $\mathfrak{N}$ (over $\mathfrak{N}$ since the parameter b is not in $|\mathfrak{M}(I,b)|$). But a Σ_1-truth definition and Overspill promptly realise the new type and, from the bound $v < (\bar{b})_{\bar{i}_0}$, it follows that the realisation actually takes place in $\mathfrak{M}(I,b)$. For further details, consult [7].]

We can now assume that $\mathfrak{N}$, as well as any other model of arithmetic we happen to consider, is recursively saturated. If we do this, then, as we have already said, Theorem 1.5 becomes quite unremarkable–except, of course, as a means of producing recursively saturated elementary initial segments.

The importance of the construction $\mathfrak{M}(I,b)$ as a producer of initial segments will be demonstrated in the next section and discussed in that following it. For the moment, we wish simply to discuss the converse to Theorem 1.5. That this theorem possesses converses was pointed out to us in conversation by Roman Kossak. Indeed, it has a very strong converse:

1.6 Theorem *Let $\mathfrak{M}$ be recursively saturated and $I \subseteq_e \mathfrak{M}$ an initial segment of $\mathfrak{M}$ closed under successor. There is a recursively saturated elementary end extension $\mathfrak{N} \succ_e \mathfrak{M}$ and an element $b \in ASS(\mathfrak{N})$, with $lh(b) > I$, such that $\mathfrak{M} = \mathfrak{M}(I,b)$ in $\mathfrak{N}$.*

The easy case of this is that in which $I = \omega$: Let $\mathfrak{N}$ be an isomorphic copy of $\mathfrak{M}$ and note that, for any $b \in ASS(\mathfrak{N})$, $\mathfrak{M}(\omega,b)$ is isomorphic to $\mathfrak{M}$ (since it possesses the same isomorphism invariants—cf. e.g. [4]). The isomorphism preserves ω.

A slightly less obvious case is that in which $I = |\mathfrak{M}|$. This is actually quite simple via the following result of [5]:

1.7 Theorem *Let $(\mathfrak{N};\mathfrak{M})$ be recursively saturated with $\mathfrak{M} \prec_e \mathfrak{N}$. Then: $\mathfrak{M} = \mathfrak{M}(I,b)$ in $\mathfrak{N}$ for some $b \in ASS(\mathfrak{N})$ and $I = |\mathfrak{M}|$.*

The proof of Theorem 1.7 is a simple matter of describing the recursive type over the pair $(\mathfrak{N};\mathfrak{M})$ that b must realise for $\mathfrak{M}$ to be $\mathfrak{M}(|\mathfrak{M}|,b)$ in $\mathfrak{N}$.

The use of Theorem 1.7 in settling the problem at hand is simple: Let $\mathfrak{N}$ be an isomorphic copy of $\mathfrak{M}$ and appeal to resplendence to find $\mathfrak{H} \prec_e \mathfrak{N}$ such that $(\mathfrak{N};\mathfrak{H})$ is recursively saturated. Evidently $\mathfrak{M}$ is isomorphic to $\mathfrak{H}$ and $I = |\mathfrak{M}|$ is preserved under the corresponding identification.

Roman Kossak and Henryk Kotlarski both noted the relevance of the following result to the general case of Theorem 1.6 in which $I \subseteq_e \mathfrak{M}$ is a proper initial segment other than ω:

1.8 Theorem *Let $I \subseteq_e \mathfrak{H} \prec_e \mathfrak{M}$, where $\mathfrak{H}$, $\mathfrak{M}$ are recursively saturated and both extensions are proper. Then:*

i. $(\mathfrak{H};I) \cong (\mathfrak{M};I)$
ii. $(\mathfrak{H};I) \prec (\mathfrak{M};I)$.

The proof of this theorem lies beyond the scope of the present note. It is a back-and-forth argument with special care taken to preserve a nonfinite amount of information. A detailed proof can be found in [6].

To apply Theorem 1.8 to the remaining case of Theorem 1.6, let $I \subseteq_e \mathfrak{M}$ be a proper initial segment of $\mathfrak{M}$ closed under successor. Let $\mathfrak{H} = \mathfrak{M}(I,b)$ in $\mathfrak{M}$ for some $b \in ASS(\mathfrak{M})$ and note that, since the isomorphism $(\mathfrak{H};I) \cong (\mathfrak{M};I)$ preserves I, it makes $\mathfrak{M} = \mathfrak{M}(I,b)$ inside the large copy of $\mathfrak{M}$.

2 On the use of the basic construction in providing variety One of the basic facts about recursively saturated models of arithmetic is that a countable recursively saturated model $\mathfrak{N}$ has a continuum of recursively saturated elementary initial segments $\mathfrak{M} \prec_e \mathfrak{N}$ and that, in fact, all these segments are isomorphic to $\mathfrak{N}$. However, these segments cease to be automatically isomorphic to each other if we take into account not only them, but also the manners in which they fit into $\mathfrak{N}$. There are nonisomorphic pairs $(\mathfrak{N};\mathfrak{M}_0)$ and $(\mathfrak{N};\mathfrak{M}_1)$ with $\mathfrak{M}_0, \mathfrak{M}_1$ isomorphic recursively saturated proper elementary initial segments of $\mathfrak{N}$. Indeed, there are some obvious possible differences between such pairs:

i. $(\mathfrak{N};\mathfrak{M}_0)$ is recursively saturated and $(\mathfrak{N};\mathfrak{M}_1)$ is not
ii. $\mathfrak{M}_0$ is of the form $\mathfrak{M}(I,b)$ in $\mathfrak{N}$ and $\mathfrak{M}_1$ is not
iii. $\mathfrak{M}_0$ is semiregular (regular) [strong] {etc.} in $\mathfrak{N}$ and $\mathfrak{M}_1$ is not (not) [not] {etc.}.

Aside from the individual questions of which of these possibilities is realised, there is the general question: How great is the variety among such pairs $(\mathfrak{N};\mathfrak{M})$? In [5], we exploited the basic construction to give the following answer:

2.1 Theorem *Let $\mathfrak{N}$ be recursively saturated. There are continuum many pairwise nonisomorphic structures of the form $(\mathfrak{N};\mathfrak{M})$, where $\mathfrak{M}$ is a recursively saturated elementary initial segment of $\mathfrak{N}$.*

The idea of the proof is simple: By work of Jensen and Ehrenfeucht [1], there are continuum many elementarily inequivalent initial segments $I \subseteq_e \mathfrak{N}$ modeling PA. For each such I, we pick $b \in ASS(\mathfrak{N})$ with $I < lh(b)$ and note that I is parametrically definable in $(\mathfrak{N};\mathfrak{M}(I,b))$ via the parameter b to get a continuum of distinct types—which require a continuum of nonisomorphic models in which to realise all of them.

This theorem is not the last word—assuming each $I \subseteq_e \mathfrak{N}$ used in the above proof to be semiregular in $\mathfrak{N}$ (which is always possible by a remark in [2]), we can avoid the use of the parameter in defining I in $(\mathfrak{N};\mathfrak{M}(I,b))$:

2.2 Theorem *Let $\mathfrak{N}$ be recursively saturated. There are continuum many pairwise elementarily inequivalent structures of the form $(\mathfrak{N};\mathfrak{M})$, where $\mathfrak{M}$ is a recursively saturated elementary initial segment of $\mathfrak{N}$.*

For detailed proofs of these two theorems, we refer the reader to [5]. The relevant fact about the proof for our present discussion is this: If we allow $b \in ASS(\mathfrak{N})$ and $I < lh(b)$ to vary, we obtain a continuum of decidedly distinct pairs $(\mathfrak{N};\mathfrak{M}(I,b))$.

One of the most simple-minded questions that one can ask when presented with a construction depending on two parameters is: What happens if we fix one of the parameters? And here the answer is simple: If we fix $b \in ASS(\mathfrak{N})$, we still get our continua; if we fix $I \subseteq_e \mathfrak{N}$, we do not.

That $b \in ASS(\mathfrak{N})$ can be held fixed and we can still get a continuum of elementarily inequivalent structures of the form $(\mathfrak{N};\mathfrak{M}(I,b))$ is not difficult to see. As observed by Lipshitz [3], any Diophantine correct model of PA possessing the same standard system as $\mathfrak{N}$ can be initially embedded below $lh(b)$ in $\mathfrak{N}$. The methods of Jensen and Ehrenfeucht [1] still yield a continuum of elementarily inequivalent such models and, again, they can be assumed semiregularly embedded.

That we do not similarly get a continuum by fixing I and varying $b \in ASS(\mathfrak{N})$ is trivial—there are only countably many b's in $ASS(\mathfrak{N})$. But we can ask if varying the b's as far as they can be varied does give any variety to the pairs $(\mathfrak{N};\mathfrak{M}(I,b))$: For fixed I, is there a countable infinity of pairwise nonisomorphic (or: elementarily inequivalent) such pairs? For semiregular I, the answer is no. There are at most two such pairs.

The following example will first explain why there can be two pairs instead of just one:

2.3 Example Let $\mathfrak{N}$ be recursively saturated. There are semiregular $I \subseteq_e \mathfrak{N}$ and $a,b \in ASS(\mathfrak{N})$ with $I < lh(a), lh(b)$ such that

$$(\mathfrak{N};\mathfrak{M}(I,a)) \not\equiv (\mathfrak{N};\mathfrak{M}(I,b)).$$

Proof: Let $(\mathfrak{N};\mathfrak{M})$ be recursively saturated with $\mathfrak{M} \prec_e \mathfrak{N}$ and $I = |\mathfrak{M}|$ semiregular in $\mathfrak{N}$. Choose $a \in ASS(\mathfrak{N})$ by Theorem 1.7 so that $\mathfrak{M} = \mathfrak{M}(I,a)$ in $\mathfrak{N}$; and choose $b \in ASS(\mathfrak{N})$ with $I < lh(b)$ and $I < (b)_0$. The two models $(\mathfrak{N};\mathfrak{M}(I,a))$ and $(\mathfrak{N};\mathfrak{M}(I,b))$ are elementarily inequivalent because: (i) $|\mathfrak{M}(I,a)| = I$ is semiregular in $\mathfrak{N}$, while $|\mathfrak{M}(I,b)|$ is not, and (ii) the semiregularity of an initial segment J in $\mathfrak{N}$ is a first-order property of the pair $(\mathfrak{N};J)$. QED

We claim that this is essentially the only variety available to us: If, for a fixed semiregular $I \subseteq_e \mathfrak{N}$, we have $a,b \in ASS(\mathfrak{N})$ of sufficient length which do not map I into itself, then $(\mathfrak{N};\mathfrak{M}(I,a)) \cong (\mathfrak{N};\mathfrak{M}(I,b))$. We first prove the following special case of this result:

2.4 Theorem *Let $\mathfrak{N}$ be recursively saturated and $a,b \in ASS(\mathfrak{N})$. Then*:

$$(\mathfrak{N};\mathfrak{M}(\omega,a)) \cong (\mathfrak{N};\mathfrak{M}(\omega,b)).$$

Proof: We will construct a number $d \in ASS(\mathfrak{N})$ such that both

i. for some $c > \omega$, $\tau_{a\restriction c} = \tau_{d\restriction c}$
ii. for all $x \in \omega$, $(b)_x < (d)_x < (b)_{x+1}$,

where, for any element $e \in |\mathfrak{N}|$, τ_e denotes the type of e over $\mathfrak{N}$ and $e\restriction c$ denotes the restriction of the sequence coded by e to coordinates $i < c$. This will prove the theorem since any automorphism of $\mathfrak{N}$ mapping $a\restriction c$ to $d\restriction c$ will make

$$(\mathfrak{N};\mathfrak{M}(\omega,a)) \cong (\mathfrak{N};\mathfrak{M}(\omega,d))$$

since $|\mathfrak{M}(\omega,a)|$ will be mapped onto $|\mathfrak{M}(\omega,d)|$, while

$$(\mathfrak{N};\mathfrak{M}(\omega,d)) = (\mathfrak{N},\mathfrak{M}(\omega,b))$$

by ii: $|\mathfrak{M}(\omega,d)| = |\mathfrak{M}(\omega,b)|$.

The constructions of d and c are by recursive saturation. To verify the consistency of the necessary type, we construct an auxiliary sequence $d_0,d_1,\ldots$ such that, for $d^x = (d_0,\ldots,d_{x-1})$, we have

i. $\tau_{d^x} = \tau_{a\restriction x}$
ii. $(b)_i < d_i < (b)_{i+1}$ for $i < x - 1$
iii. $d_0 \ll \ldots \ll d_{x-1}$.

This sequence is constructed by induction.

Basis. Without loss of generality, we can assume $(a)_0,(b)_0 \gg 0$. We need a small lemma:

2.5 Lemma *Let $\mathfrak{H}$ be recursively saturated and let $e \gg 0$ in $\mathfrak{H}$. Then $\tau_e v$ is realised arbitrarily highly in $\mathfrak{H}$.*

Proof: Suppose d bounds all elements realising $\tau_e v$. Then

$$Th(\mathfrak{H}) + \tau_e v + \tau_d v_0 \vdash v < v_0,$$

i.e., for some $\phi_0,\ldots,\phi_{m-1} \in \tau_e$ and $\psi_0,\ldots,\psi_{k-1} \in \tau_d$,

$$\mathfrak{H} \models \forall v v_0[\bigwedge\!\!\!\bigwedge \phi_i v \wedge \bigwedge\!\!\!\bigwedge \psi_j v_0 \rightarrow v < v_0].$$

Letting $d^* = \mu v_0[\bigwedge\!\!\!\bigwedge \psi_j v_0]$ we see $e < d^*$, contrary to our assumption that $e \gg 0$. QED

We can now complete the proof of the basis step: Let $\mathfrak{H} \prec_e \mathfrak{N}$ be recursively saturated such that

$$(b)_0 \in |\mathfrak{H}| < (b)_1$$

and let $(d_0) > ((b)_0)$ realise $\tau_{a\restriction 1}$ in $\mathfrak{H}$. [We note that, since $(b)_0 \gg 0$, $\tau_{a\restriction 1}$ can be realised below $(b)_0$ in $\mathfrak{N}$—whence in $\mathfrak{H}$.] Conditions i and ii are automatically satisfied and iii is vacuous.

Induction step. We do the same sort of thing. First, we relativise the lemma to some parameters:

2.6 Lemma *Let $\mathfrak{H}$ be recursively saturated and $e \gg f \gg 0$ in $\mathfrak{H}$. Then $\tau_{ef}v\bar{g}$ is realised arbitrarily highly in $\mathfrak{H}$ for any g realising τ_f.*

We omit the proof as it consists merely in sticking parameters into the previous proof.

To finish the proof of the induction step, let $\mathfrak{H} \prec_e \mathfrak{N}$ be recursively saturated such that

$$(b)_x \in |\mathfrak{H}| < (b)_{x+1}$$

and choose $d_x \in |\mathfrak{H}|$ realising $\tau_{(a)_x, a\restriction x}(v, \overline{d^x})$ with $d_x > (b)_x$. Conditions i-iii are again satisfied.

Having the sequence $d_0, d_1, \ldots$, we now proceed to use only its existence. Let $\tau v_0 v_1 \overline{ab}$ be the set of formulae:

i. $\phi(\bar{a}\restriction v_1) \leftrightarrow \phi(v_0\restriction v_1)$, all ϕ with only one free variable
ii. $\forall v_2 < v_1 \dot{-} \bar{2}[F((v_0)_{v_2}) < (v_0)_{v_2 + \bar{1}}]$, all parameter-free Skolem functions F
iii. $\forall v_2 < v_1 \dot{-} \bar{2}[(\bar{b})_{v_2} < (v_0)_{v_2} < (\bar{b})_{v_2+\bar{1}}]$
iv. $v_1 < lh(\bar{a}) \wedge v_1 < lh(\bar{b})$
v. $v_1 > \bar{0}, v_1 > \bar{1}, \ldots$.

By the existence of the d_x's, $\tau v_0 v_1 \overline{ab}$ is a type. It is also recursive and so realised by some d,c. But, by v, $c > \omega$ and, by i, $\tau_{a\restriction c} = \tau_{d\restriction c}$. QED

As we said, Theorem 2.4 is but a special case of a more general result on the isomorphism of all pairs $(\mathfrak{N}; \mathfrak{M}(I,b))$ where I is semiregular in $\mathfrak{N}$ and $b \in ASS(\mathfrak{N})$ is of proper length and maps I beyond itself. By truncating $a,b \in ASS(\mathfrak{N})$ if necessary, we can state this result in the following form more closely analogous to that of Theorem 2.4:

2.7 Theorem *Let $\mathfrak{N}$ be recursively saturated and let $I \subseteq_e \mathfrak{N}$ be semiregular in $\mathfrak{N}$. For any $a,b \in ASS(\mathfrak{N})$ with $I < lh(a), lh(b)$, if $I < (a)_0, (b)_0$, then*:

$$(\mathfrak{N}; \mathfrak{M}(I,a)) \cong (\mathfrak{N}; \mathfrak{M}(I,b)).$$

We shall not give the proof here. It requires the same special back-and-forth technique used to prove Theorem 1.8 and, like the proof of this latter theorem, it fits more naturally in [6] (in rather a strong sense: There it will be (modulo a small trick supplied by Kotlarski to handle a key difficulty) just another of several applications of a simple, but moderately powerful, tool. Here, without a reasonably leisurely digression explaining the method, it would give an undue impression of depth and/or cleverness.).

One quick thing we can say about the proof of Theorem 2.7 is that it does follow the main line of the proof of Theorem 2.4: To get the isomorphism, it suffices to construct an element $d \in ASS(\mathfrak{N})$ onto a large initial segment of which one can map a corresponding segment of a by some automorphism preserving I. Both the construction of the element d and the subsequent construction of the automorphism require special care.

3 Variety without the basic construction We hope that, through the results of Sections 1 and 2, we have established the interest of the basic construction. If so, our new countergoal might appear perverse: We wish in this section to avoid the basic construction. We do so on general principle: To understand a phenomenon, it often helps to understand its nonoccurrence as well as its occurrence, and, additionally, the occurrence of other related phenomena.

Stated less grandiloquently: We wish to consider the possibilities of: (i) constructing recursively saturated elementary initial segments $\mathfrak{M} \prec_e \mathfrak{N}$ *not* of the form $\mathfrak{M}(I,b)$ in $\mathfrak{N}$; (ii) obtaining a variety of such segments; and (iii) barring (ii), at least obtaining a variety of initial segments without the *use* of the basic construction. One may view such consideration as the modest beginning of a search for further constructions of utility and interest comparable to that of the basic construction. We confess to (and complain of) only partial success in all these endeavours.

It is a pleasure to acknowledge the fact that the results of the present section were obtained in response to a question of Roman Kossak: Are all recursively saturated $\mathfrak{M} \prec_e \mathfrak{N}$ of the form $\mathfrak{M} = \mathfrak{M}(I,b)$ in $\mathfrak{N}$? We hope that the results and problems offered below will attest to the fruitfulness of this question. Belying such fruitfulness is an immediate negative answer to this question:

3.1 Theorem *Let $\mathfrak{M} \prec_e \mathfrak{N}$ be recursively saturated and suppose $|\mathfrak{N}| - |\mathfrak{M}|$ has a minimum sky. Then: $\mathfrak{M}$ is not of the form $\mathfrak{M}(I,b)$ in $\mathfrak{N}$ for any I,b.*

Proof: Suppose $\mathfrak{M} = \mathfrak{M}(I,b)$ for some $b \in ASS(\mathfrak{N})$ and some $I \subseteq_e \mathfrak{N}$ closed under successor, with $I < lh(b)$. Let c be an element of the minimum sky of $|\mathfrak{N}| - |\mathfrak{M}|$. Observe that I is parametrically definable in $\mathfrak{N}$: For all $i \in |\mathfrak{N}|$,

$$i \in I \text{ iff } \mathfrak{N} \models (\bar{b})_{\bar{i}} < \bar{c}.$$

But, an initial segment closed under successor cannot be defined in any model of PA, and we have a contradiction. QED

Given $\mathfrak{N}$, there are only countably many $\mathfrak{M} \prec_e \mathfrak{N}$ for which $|\mathfrak{N}| - |\mathfrak{M}|$ has a minimum sky. A plethora of counterexamples to the basic construction is provided by the following result:

3.2 Theorem *Let $\mathfrak{N}$ be recursively saturated. There are continuum many recursively saturated $\mathfrak{M} \prec_e \mathfrak{N}$ which are not of the form $\mathfrak{M} = \mathfrak{M}(I,b)$ in $\mathfrak{N}$.*

Proof: The proof is a simple ω-step construction. We alternate between diagonalising on $b \in ASS(\mathfrak{N})$ (even-numbered stages) to guarantee the outcome not to be of the form $\mathfrak{M}(I,b)$ and making a sequence of binary choices (odd-numbered stages) to guarantee a continuum of possible outcomes.

Let $b_0, b_1, \ldots$ enumerate all elements $b_n \in ASS(\mathfrak{N})$ for which $(b_n)_0 = 0$. (This restriction is inessential.) Toward the diagonalisation, we will construct successively smaller intervals $[a_n, c_n]$, setting $J_n = [0, a_n]$. This will require $a_n \ll c_n$. Moreover, to guarantee that $J = \cup_n J_n$ is the domain of a recursively saturated elementary initial segment $\mathfrak{M} \prec_e \mathfrak{N}$, we require $a_0 \ll a_1 \ll \ldots$.

On to the construction:

Stage 0. Set $a_0 = (b_0)_0$, $c_0 = (b_0)_1$, $J_0 = [0, a_0]$. Note that b_0 will not code any cofinal map $I \to J$ when we are finished.

Stage 2n + 1. We are given

$$a_0 \ll \ldots \ll a_{2n} \ll c_{2n} \leqslant \ldots \leqslant c_0,$$

and $J_{2n} = [0, a_{2n}]$. Choose c^* such that

$$a_{2n} \ll c^* \ll c_{2n}$$

and either set

$$a_{2n+1} = c^*, \; c_{2n+1} = c_{2n}, \; J_{2n+1} = [0, a_{2n+1}]$$

or set $c_{2n+1} = c^*$ and choose a_{2n+1} such that

$$a_{2n} \ll a_{2n+1} \ll c^*$$

and set $J_{2n+1} = [0, a_{2n+1}]$.

Stage 2n + 2. We are given

$$a_0 \ll \ldots \ll a_{2n+1} \ll c_{2n+1} \leqslant \ldots \leqslant c_0$$

and $J_{2n+1} = [0, a_{2n+1}]$. Consider b_{n+1}.

If, for all $i < lh(b_{n+1})$, $(b_{n+1})_i \leqslant Sk(a_{2n+1})$, then we simply let a_{2n+2} be such that $a_{2n+1} \ll a_{2n+2} \ll c_{2n+1}$ and set $c_{2n+2} = c_{2n+1}$, and $J_{2n+2} = [0, a_{2n+2}]$. Note that b_{n+1} will not code any cofinal map $I \to J$ when we are finished.

If there is an $i < lh(b_{n+1})$ such that $(b_{n+1})_i \leqslant Sk(a_{2n+1})$ and $(b_{n+1})_{i+1} \gg a_{2n+1}$, we choose c_{2n+2} such that

$$a_{2n+1} \ll c_{2n+2} \ll min\{(b_{n+1})_{i+1}, c_{2n+1}\},$$

and further choose a_{2n+2} such that $a_{2n+1} \ll a_{2n+2} \ll c_{2n+2}$, and set $J_{2n+2} = [0, a_{2n+2}]$. Again, b_{n+1} will not code any cofinal map $I \to J$ when we are finished.

Finally, suppose there is an $I < lh(b_{n+1})$ closed under successor such that, for all $i \in I$, $(b_{n+1})_i \ll a_{2n+1}$ and, for all $i \notin I$, if $i < lh(b_{n+1})$ we have $a_{2n+1} \ll (b_{n+1})_i$. [Note: Since I is closed under successor and $lh(b_{n+1}) - I$ is closed under predecessor, we cannot have $(b_{n+1})_i \in Sk(a_{2n+1})$ for any $i < lh(b_{n+1})$.] If, for $I < i < lh(b_{n+1})$, one always has $(b_{n+1})_i \geqslant Sk(c_{2n+1})$, then we simply choose a_{2n+2} such that $a_{2n+1} \ll a_{2n+2} \ll c_{2n+1}$ and let $c_{2n+2} = c_{2n+1}$ and $J_{2n+2} = [0, a_{2n+2}]$. If, however, for some $I < i < lh(b_{n+1})$, it happens that $a_{2n+1} \ll (b_{n+1})_i \ll c_{2n+1}$, then choose $a_{2n+2} = (b_{n+1})_i$ and $c_{2n+2} = min\{(b_{n+1})_{i+1}, c_{2n+1}\}$

and $J_{2n+2} = [0,a_{2n+2}]$. Either way, neither I nor any other initial segment can ever again threaten to be mapped cofinally into J by b_{n+1}.

We finish the proof by letting $J = \cup_n J_n = \cup_n [0,a_n]$. From the inequalities, $a_0 \ll a_1 \ll \ldots$, it follows that J is closed under Skolem functions and constitutes the domain of a recursively saturated elementary initial submodel $\mathfrak{M} \prec_e \mathfrak{N}$. Varying our choices in the odd-numbered stages, we obtain continuum many distinct such segments J, hence continuum many distinct such models $\mathfrak{M}$. Moreover, for each b_n, either, for some $i < lh(b_n)$, we have $(b_n)_i \in |\mathfrak{M}|$ and $(b_n)_{i+1} \notin |\mathfrak{M}|$, or, for all $i < lh(b_n)$, we have $(b_n)_i \in |\mathfrak{M}|$ (and so all $(b_n)_i$ bounded in I by $(b_n)_{lh(b_n)-1}$). Hence, for no $b \in ASS(\mathfrak{N})$ and $I \subseteq_e \mathfrak{N}$ do we have $\mathfrak{M} = \mathfrak{M}(I,b)$ in $\mathfrak{N}$. QED

By Theorems 3.1 and 3.2 there are both natural and many examples of initial segments $\mathfrak{M} \prec_e \mathfrak{N}$ with $\mathfrak{M}$ not of the form $\mathfrak{M}(I,b)$ in $\mathfrak{N}$. But is there any variety to such segments? The models of Theorem 3.1, though only countable in number, do offer as much variety as possible: As shown in [5], there is a countable infinity of pairwise elementarily inequivalent structures of the form $(\mathfrak{N};\mathfrak{M})$, where $\mathfrak{M} \prec_e \mathfrak{N}$ are recursively saturated and $|\mathfrak{N}| - |\mathfrak{M}|$ possesses a minimum sky. While this is as much as we can say at present, it does suggest an eventual positive solution to the following problem:

3.3 Open Problem Let $\mathfrak{N}$ be recursively saturated. Do there exist continuum many pairwise elementarily inequivalent (or, at least, nonisomorphic) structures of the form $(\mathfrak{N};\mathfrak{M})$, where $\mathfrak{M}$ is a recursively saturated elementary initial segment of $\mathfrak{N}$ *not* of the form $\mathfrak{M}(I,b)$ in $\mathfrak{N}$?

As indicated, we do not know how to construct a continuum of pairwise nonisomorphic pairs $(\mathfrak{N};\mathfrak{M})$ with $\mathfrak{N}$ not of the form $\mathfrak{M}(I,b)$ in $\mathfrak{N}$. However, we can construct a continuum for which we do not know $\mathfrak{M}$ to be of the form $\mathfrak{M}(I,b)$ in $\mathfrak{N}$—i.e. we can reprove Theorem 2.1 without using the basic construction. To do so, we introduce a new construction very similar to, but definitely different from, our basic construction.

3.4 Definition Let $<_r$ be a primitive recursive ordering of the natural numbers in the type of the rationals. Let $b \in |\mathfrak{N}|$. We say *b codes a rational sequence of skies* in $\mathfrak{N}$, written $b \in RSS(\mathfrak{N})$, if

i. $lh(b)$ is nonstandard
ii. for $c,d < lh(b)$, $(b)_c \ll (b)_d$ iff $\mathfrak{N} \models \bar{c} <_r \bar{d}$.

If $\mathfrak{N}$ is recursively saturated, then $RSS(\mathfrak{N})$ is nonempty.

3.5 Definition Let $b \in RSS(\mathfrak{N})$ and let C be a Dedekind cut in $(\omega,<_r)$. We define $\mathfrak{M}[C,b] <_e \mathfrak{N}$ by

$$|\mathfrak{M}[C,b]| = \bigcup_{x <_r C} [0,(b)_x],$$

where x ranges over ω.

[That $\mathfrak{M}[C,b]$ is a recursively saturated elementary substructure of $\mathfrak{N}$ follows, as usual, from the fact that $|\mathfrak{M}[C,b]|$ has no highest sky. (This, incidentally, is the reason both for using the strict inequality ("$x <_r C$") in the definition and for restricting x in the definition to ω. In the case of a rational cut, the strict inequality keeps us from providing $|\mathfrak{M}[C,b]|$ with a maximum element. Similarly, restricting x to ω guarantees us to be choosing a cut in a dense linear order–funny things can happen in the nonstandard part. (Another reason for this restriction will emerge in the proof of the next theorem.))]

We shall now use this new construction to give a new proof of Theorem 2.1. We should stress that this new proof is more elementary than the old insofar as it replaces the appeal to the results of Jensen and Ehrenfeucht [1] by a simple reference to the existence of a continuum of Dedekind cuts in the rationals.

3.6 Theorem *Let $\mathfrak{N}$ be recursively saturated. There are continuum many pairwise nonisomorphic structures of the form $(\mathfrak{N};\mathfrak{M}[C,b])$, where $b \in RSS(\mathfrak{N})$ and C is a Dedekind cut.*

Proof: Fix $b \in RSS(\mathfrak{N})$ and, for C any Dedekind cut, define

$$C^* = \{a < lh(b): \exists x \in \omega(a <_r x <_r C)\}.$$

For $C_1 \neq C_2$, we clearly have $C_1^* \cap \omega \neq C_2^* \cap \omega$.

We get our continuum by noting C^* to be uniformly definable in $(\mathfrak{N};\mathfrak{M}[C,b];b)$: For $a \in |\mathfrak{N}|$,

$$a \in C^* \text{ iff } (\mathfrak{N};\mathfrak{M}[C,b];b) \models \bar{a} < lh(\bar{b}) \wedge (\bar{b})_{\bar{a}} \in |\mathfrak{M}[C,b]|.$$

The continuum of distinct C^*'s defined in the pairs $(\mathfrak{N};\mathfrak{M}[C,b])$ by a single formula with parameter b gives us a continuum of distinct types realised by b's in these models and require, as before, a continuum of nonisomorphic models in which to realise them. QED

This new construction raises all sorts of questions. An obvious first one is whether or not Theorem 2.2 can be obtained by means of the new construction.

3.7 Open Problem Let $\mathfrak{N}$ be recursively saturated. Is there a continuum of pairwise elementarily inequivalent structures of the form $(\mathfrak{N};\mathfrak{M}[C,b])$?

There is no reason to believe that C, under any special assumptions, can be recovered without b from $(\mathfrak{N};\mathfrak{M}[C,b])$ in analogy to the recovery of I, under assumption of semiregularity, from $(\mathfrak{N};\mathfrak{M}(I,b))$. However, the situation is not hopeless. The expressive power of the structure $(\mathfrak{N};\mathfrak{M}[C,b])$ is very great and it is likely that the differences between many such pairs can be expressed without parameters.

Of course, both Theorem 3.6 and an affirmative solution to Problem 3.7 might cease to be interesting should the new models $\mathfrak{M}[C,b]$ coincide with the old models $\mathfrak{M}(I,b)$. As to the relations between these two types of models, we can find examples of $\mathfrak{M} \prec_e \mathfrak{N}$ such that

i. $\mathfrak{M}$ has neither the form $\mathfrak{M}(I,b)$ nor $\mathfrak{M}[C,b]$ in $\mathfrak{N}$
ii. $\mathfrak{M}$ has the form $\mathfrak{M}(I,b)$, but not the form $\mathfrak{M}[C,b]$, in $\mathfrak{N}$
iii. $\mathfrak{M}$ has both forms in $\mathfrak{N}$.

Specifically, we have: (i) a continuum of examples of the first sort, with at least a countable infinity of pairwise elementarily inequivalent pairs $(\mathfrak{N};\mathfrak{M})$; (ii) a continuum of examples of the second sort, comprised of a countable infinity of elementary equivalence classes; and (iii) a countably infinite single isomorphism class of examples of the third sort.

The first continuum is given by modifying the construction of the proof of Theorem 3.2: While we see no way of obtaining models of the form $\mathfrak{M}[C,b]$ by means of this construction, we can easily modify it so as to avoid models of such a form: We simply change it to a congruence-mod-3 construction in which every third step diagonalises against the new construction $\mathfrak{M}[C,b]$.

The countable families of elementary equivalence classes of structures $(\mathfrak{N};\mathfrak{M})$ with $\mathfrak{M}$ obtainable by neither of our constructions or only by the basic construction are given by Theorems 3.1 and 1.7, respectively:

3.8 Theorem *Let $\mathfrak{M} \prec_e \mathfrak{N}$ be recursively saturated.*

i. *If $|\mathfrak{N}| - |\mathfrak{M}|$ has a minimum sky, then $\mathfrak{M}$ is neither of the form $\mathfrak{M}(I,b)$ nor of the form $\mathfrak{M}[C,b]$ in $\mathfrak{N}$.*
ii. *If $(\mathfrak{N};\mathfrak{M})$ is recursively saturated, then $\mathfrak{M}$ is of the form $\mathfrak{M}(I,b)$, but not of the form $\mathfrak{M}[C,b]$, in $\mathfrak{N}$.*

Proof: The information regarding the form $\mathfrak{M}(I,b)$ has already been established. It thus suffices to show $\mathfrak{M}$ not to be of the form $\mathfrak{M}[C,b]$ in each of these two cases:

i. Let a belong to the minimum sky of $|\mathfrak{N}| - |\mathfrak{M}|$ and suppose $\mathfrak{M} = \mathfrak{M}[C,b]$ in $\mathfrak{N}$. We will construct $c \in ASS(\mathfrak{N})$ and show $\mathfrak{M} = \mathfrak{M}(\omega,c)$ in $\mathfrak{N}$, which we know not to be the case.

Fix $i_0 \in \omega$ such that $(b)_{i_0} < a$ and define F by primitive recursion:

$$F0 = i_0$$
$$F(i+1) = \begin{cases} \mu j[(b)_i < (b)_j < a], & \text{if such exists} \\ b, & \text{otherwise} \end{cases}$$

Set $d = \mu i[Fi = b]$ and simply let

$$c = ((b)_{F0}, \ldots, (b)_{F(d-1)}).$$

Obviously $c \in ASS(\mathfrak{N})$.

It remains to be seen that $|\mathfrak{M}(\omega,c)| = |\mathfrak{M}[C,b]|$. Clearly

$$|\mathfrak{M}(\omega,c)| \subseteq |\mathfrak{M}[C,b]|.$$

To see the converse inclusion, let $(b)_x <_r C$ for some $x \in \omega$. The only way x cannot occur in the list $F0, \ldots, Fx$ is for $(b)_x < (b)_{Fy}$ for some $y < x$. Thus, the set $\{(c)_x : x \in \omega\}$ is cofinal in $|\mathfrak{M}[C,b]|$.

We have thus proven the contradictory claim that $\mathfrak{M} = \mathfrak{M}(\omega,c)$ in $\mathfrak{N}$ for some $c \in ASS(\mathfrak{N})$ and must discard our assumption $\mathfrak{M} = \mathfrak{M}[C,b]$.

ii. Let $(\mathfrak{N};\mathfrak{M})$ be recursively saturated. To see that $\mathfrak{M}$ is not of the form $\mathfrak{M}[C,b]$ in $\mathfrak{N}$, we note that ω cannot be parametrically defined in any recursively saturated model $(\mathfrak{N};\mathfrak{M})$ and show that ω is parametrically definable in each structure $(\mathfrak{N};\mathfrak{M}[C,b])$.

To define ω in $(\mathfrak{N}; \mathfrak{M}[C,b])$, recall from the proof of Theorem 3.6 the definability of

$$C^* = \{a < lh(b)\colon (b)_a \in |\mathfrak{M}[C,b]|\}.$$

Using this definition, define

$$Iv\colon \exists v_0 \in |\mathfrak{M}[C,b]|\, \forall v_1 < v[v_1 \in C^* \to (\bar{b})_{v_1} < v_0].$$

Iv defines the initial segment of those $a < lh(b)$ such that the intersection of $|\mathfrak{M}[C,b]|$ with the range of $b \restriction a$ is not cofinal in $|\mathfrak{M}[C,b]|$. A moment's thought reveals that Iv defines ω. QED

The existence of a countable infinity of pairwise elementarily inequivalent pairs $(\mathfrak{N}; \mathfrak{M})$ of each of the two given forms of the theorem was established in [5]. There are only countably many such pairs in which $|\mathfrak{N}| - |\mathfrak{M}|$ has a minimum sky (indeed, they are indexed by the minimum skies), but a continuum of recursively saturated pairs–forming a countable family of isomorphism (= elementary equivalence) classes.

Our models of the third sort are very simple:

3.9 Theorem *Let $\mathfrak{N}$ be recursively saturated and $a \in ASS(\mathfrak{N})$. Then: $\mathfrak{M}(\omega,a)$ is of the form $\mathfrak{M}[C,b]$ in $\mathfrak{N}$.*

Proof: Let $a \in ASS(\mathfrak{N})$.

Let $x_y \in \omega$ correspond, for each $y \in \omega$, to $1 - 1/y$ in $(\omega,<_r)$ and define the recursive type $\tau v\bar{a}$ to consist of all formulae

$$(v)_{\overline{x_y}} = (\bar{a})_{\bar{y}}, \qquad (*)$$

and all those formulae which together assert $v \in RSS(\mathfrak{N})$. Thus, if $b \in |\mathfrak{N}|$ realises $\tau v\bar{a}$, then $b \in RSS(\mathfrak{N})$. Moreover, if C is the cut determining 1 in $(\omega,<_r)$, then by $(*)$ and the fact that $\lim_{y\to\infty} x_y = 1$ we have $|\mathfrak{M}[C,b]| = |\mathfrak{M}(\omega,a)|$. QED

As we proved in Section 2, the models of the form $\mathfrak{M}(\omega,a)$ in $\mathfrak{N}$ all result in isomorphic pairs $(\mathfrak{N}; \mathfrak{M}(\omega,a))$.

Let us add to this short unstructured list of facts one obvious open problem.

3.10 Open Problem Let $\mathfrak{N}$ be recursively saturated. Is any $\mathfrak{M} \prec_e \mathfrak{N}$ of both the form $\mathfrak{M}[C,b]$ and the form $\mathfrak{M}(I,a)$ in $\mathfrak{N}$, where, in the latter case, I is a semiregular initial segment of $\mathfrak{N}$ other than ω?

A negative answer to this problem would be quite nice.

Finally, let us remark that our new construction $\mathfrak{M}[C,b]$ seems to be of some minor interest in its own right, irrespective of any relation it might or might not have with our earlier construction $\mathfrak{M}(I,b)$. We list a few interesting facts about $\mathfrak{M}[C,b]$:

3.11 Facts *Let $\mathfrak{N}$ be recursively saturated.*

i. *ω is uniformly (*non*parametrically) definable in the structures $(\mathfrak{N}; \mathfrak{M}[C,b])$.*
ii. *Satisfaction for $\mathfrak{M}[C,b]$ is uniformly (nonparametrically) definable in the structures $(\mathfrak{N}; \mathfrak{M}[C,b])$.*

iii. *A model* $\mathfrak{M}[C,b]$ *has cofinality* ω *in* $\mathfrak{N}$ *iff it is also of the form* $\mathfrak{M}(\omega,a)$ *in* $\mathfrak{N}$.

The first two facts are established exactly as the analogous facts were established for $(\mathfrak{N};\mathfrak{M})$, with $|\mathfrak{N}| - |\mathfrak{M}|$ possessing a minimum sky, in [5]: We start with our parametric definition of ω (from the proof of 3.8.ii) and quantify it out. Using ω, we get a parametric definition of satisfaction for $\mathfrak{M}[C,b]$; but, again, we can quantify out the parameter (cf. [5] for details).

The third point is more interesting. For, at first glance, all the models $\mathfrak{M}[C,b]$ look like they should have cofinality ω.

First, we explain that an initial segment $I \subseteq_e \mathfrak{N}$ is said to have cofinality ω in $\mathfrak{N}$ if there is an element $a \in |\mathfrak{N}|$ such that

$$I = \bigcup_{x \in \omega} [0,(a)_x],$$

where the sequence coded by a is assumed (without loss of generality) to be strictly increasing.

Proof of 3.11.iii: Clearly, if $\mathfrak{M} = \mathfrak{M}(\omega,a)$ in $\mathfrak{N}$, then $\mathfrak{M}$ has cofinality ω in $\mathfrak{N}$. Thus, assume $\mathfrak{M} = \mathfrak{M}[C,b]$ has cofinality ω in $\mathfrak{N}$ and let the sequence coded by a witness this cofinality. First, fix $i_0, j_0 \in \omega$ so that

$$(b)_{i_0} < (a)_{j_0}.$$

[Note: Since $i_0, j_0 \in \omega$, it follows that $(b)_{i_0} \in |\mathfrak{M}[C,b]|$.] Now define functions F,G by simultaneous recursion:

$$F0 = i_0 \qquad G0 = j_0$$
$$G(i+1) = \begin{cases} \mu j \exists v_0[(b)_{Fi} < (b)_{v_0} < (a)_j], & \text{if such exists} \\ b, & \text{otherwise,} \end{cases}$$
$$F(i+1) = \begin{cases} \mu j[(b)_{Fi} < (b)_j < (a)_{G(i+1)}], & \text{if such exists} \\ b, & \text{otherwise.} \end{cases}$$

Letting $d = \mu i[Fi = b]$ and defining

$$c = ((b)_{F0}, \ldots, (b)_{F(d-1)}),$$

as before, we quickly see $|\mathfrak{M}[C,b]| = |\mathfrak{M}(\omega,c)|$. QED

One last pair of facts of interest is the following:

3.12 Facts Let $\mathfrak{N}$ be nonstandard, not necessarily recursively saturated.

i. The following are equivalent:
   a. $\mathfrak{N}$ is recursively saturated
   b. $\forall a \in |\mathfrak{N}| \exists b \in RSS(\mathfrak{N})[(b)_0 = a]$.

ii. For any $b \in RSS(\mathfrak{N})$ and any cut C, $\mathfrak{M}[C,b]$ is recursively saturated.

Proof: These reduce quickly to Theorem 1.5. For, though it is difficult to make $\mathfrak{M}[C,b]$ assume the form $\mathfrak{M}(\omega,c)$, we can easily construct $c \in ASS(\mathfrak{N})$ such that $\mathfrak{M}[C,b] \subseteq \mathfrak{M}(\omega,c)$.

Fix $x \in \omega$ such that $|\mathfrak{M}[C,b]| < (b)_x$. Define the usual sort of function F by primitive recursion:

$$F0 = x$$
$$F(i+1) = \begin{cases} \mu j[(b)_{Fi} < (b)_j], & \text{if such exists} \\ b, & \text{otherwise.} \end{cases}$$

Set $d = \mu i[Fi = b]$ and $c = ((b)_{F0}, \ldots, (b)_{F(d-1)})$ and note that $c \in ASS(\mathfrak{N})$ and $\mathfrak{M}[C,b] \subseteq \mathfrak{M}(\omega,c)$. QED

Appendix After we wrote the preceding, Henryk Kotlarski pointed out the utility of topological tools for an analysis of recursively saturated elementary initial segments of a given recursively saturated model of arithmetic.

For a fixed recursively saturated $\mathfrak{N}$, define

$$R = \{\mathfrak{M} \prec_e \mathfrak{N} : \mathfrak{M} \text{ is recursively saturated}\},$$

where we assume $\prec_e$ to indicate a proper inclusion.

A.1 Fact *R is ordered in the type of the reals by* $\prec_e$.

The quickest proof of this fact is given by noting the skies of $\mathfrak{N}$ to be ordered in the type of the rationals and the obvious correspondence between cuts (with rational cuts putting the rationals on the right) in this ordering and elements of R. (This was apparently first noticed by Kotlarski.)

Another proof can be found in [7], where it is shown that the set of models of the form $\mathfrak{M}(\omega,b)$ for $b \in ASS(\mathfrak{N})$ is dense in this ordering. Despite this density, Kotlarski noted the following:

A.2 Theorem *The following sets are of the first category in* $(R,\prec_e)$:

i. $\{\mathfrak{M}(I,b): I < lh(b) \text{ and } b \in ASS(\mathfrak{N})\}$
ii. $\{\mathfrak{M}[C,b]: C \text{ is a cut in } (\omega,<_r) \text{ and } b \in RSS(\mathfrak{N})\}$.

Proof: i. For each fixed $b \in ASS(\mathfrak{N})$ we show the set of models of the form $\mathfrak{M}(I,b)$ to be nowhere dense in $(R,\prec_e)$.

Note that any pair $a \ll c$ in $\mathfrak{N}$ determines a nonempty open interval $(a,c)_R = \{\mathfrak{M} \in R: a \in |\mathfrak{M}| < c\}$ in R. We must find a nonempty open subinterval containing no model $\mathfrak{M}(I,b)$. But this is easy: If there is no or only one $(b)_i$, with $i < lh(b)$, such that $a < (b)_i < c$, then $(a,c)_R$ will do; otherwise there are $(b)_i,(b)_{i+1}$, with $i+1 < lh(b)$, such that $a < (b)_i < (b)_{i+1} < c$ and the interval $((b)_i,(b)_{i+1})_R$ will do.

ii. Again fix $b \in RSS(\mathfrak{N})$. We can quickly reduce ourselves to considering intervals $((b)_x,(b)_y)_R$ for $x,y \in \omega$ with $x <_r y$. We must find an open subinterval of this interval containing no models $\mathfrak{M}[C,b]$.

By the density of $<_r$ and Overspill, there is an infinite $i < lh(b)$ such that $(b)_x < (b)_i < (b)_y$. Let

$$C_i = \{z \in \omega: (b)_z < (b)_i\}$$

be the cut determined by $(b)_i$ and let $\mathfrak{M}_i$ be the maximum element of R below $(b)_i$:

$$|\mathfrak{M}_i| = \{a \in |\mathfrak{N}|: a \ll (b)_i\}.$$

Since $|\mathfrak{N}| - |\mathfrak{M}_i|$ has a minimum sky, Theorem 3.8.i tells us that the inclusion,

$$\mathfrak{M}[C_i,b] \prec_e \mathfrak{M}_i,$$

is indeed proper. Looking at skies of elements $a \in |\mathfrak{M}_i| - |\mathfrak{M}[C_i,b]|$, one quickly sees that the open interval

$$I = \{\mathfrak{M} \in R\colon \mathfrak{M}[C_i,b] \prec_e \mathfrak{M} \prec_e \mathfrak{M}_i\}$$

is nonempty. Moreover, for any cut $C \neq C_i$, we have either

$$\mathfrak{M}[C,b] \prec_e \mathfrak{M}[C_i,b] \text{ or } \mathfrak{M}_i \prec_e \mathfrak{M}[C,b],$$

whence no model $\mathfrak{M}[C,b]$ is in the interval I. QED

By the Baire Category Theorem, A2 yields the existence of uncountably many models $\mathfrak{M} \in R$ of neither the form $\mathfrak{M}(I,b)$ nor the form $\mathfrak{M}[C,b]$ in $\mathfrak{N}$. Unfortunately, sets of the second category need not have the power of the continuum unless the continuum hypothesis is assumed, and, since most logicians fail to agree on the obvious truth of this hypothesis, we are stuck with the more complicated proof of Theorem 3.2 and the later remark generalising this if we wish to know the exact cardinalities of the sets of nonexamples of our constructions. Nonetheless, Kotlarski's topological observation does seem to shed some additional light on the situation.

REFERENCES

[1] Jensen, D. and A. Ehrenfeucht, "Some problem in elementary arithmetics," *Fundamenta Mathematicae*, vol. 92 (1976), pp. 223-245.

[2] Kirby, L., *Initial Segments of Models of Arithmetic*, Unpublished Ph.D. Dissertation, University of Manchester.

[3] Lipshitz, L., "Diophantine correct models of arithmetic," *Proceedings of the American Mathematical Society*, vol. 73 (1979), pp. 107-108.

[4] Smoryński, C., "Recursively saturated nonstandard models of arithmetic," *The Journal of Symbolic Logic*, forthcoming.

[5] Smoryński, C., "Elementary extensions of recursively saturated models of arithmetic," *Notre Dame Journal of Formal Logic*, vol. 22, No. 3 (1981), pp. 193-203.

[6] Smoryński, C., "Back-and-forth inside a recursively saturated model of arithmetic," in preparation.

[7] Smoryński, C. and J. Stavi, "Cofinal extension preserves recursive saturation," in *The Model Theory of Algebra and Arithmetic*, eds., L. Pacholski and A. Wilkie, Springer-Verlag, Heidelberg, forthcoming.

Department of Mathematics
Ohio State University