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An Actualistic Semantics for Quantified Modal Logic

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In the past two decades considerable energy and effort have been expended in the study of quantified modal logic. The first formal semantics and completeness theorem were published by Kripke in 1959 [2]. Since then, many different systems have been developed including those given by Kripke in [3] and Hughes and Cresswell in [1]. In each case the central informal idea underlying the formal semantics is the notion of *possible world*, a total way things could have been. Formally, a *model structure* for a first-order language L is a quadruple (D, W, ψ, ϕ) such that:

a. W and D are nonempty sets (of possible worlds and objects, respectively),
b. ψ is a function which assigns to each world w a nonempty subset of D ($\psi(w)$ is the set of objects existing in w),
c. ϕ is a function which assigns to each pair (F, w), where F is a predicate symbol and w a world, a set of tuples of objects in D.[1]

From each model structure *models* are formed by singling out individual worlds to play the role of the actual world. Finally, valuation rules are provided under which models assign truth values to formulas relative to an assignment of the individual variables of L to objects in D. Different systems arise from variations in the valuation rules and from added conditions on the components of a model structure.

Formally, all of the systems referred to above work equally well and all are interesting. However, it does not follow from a system's formal success that it is philosophically significant or useful. To determine whether a formal semantics is philosophically successful, we must consider the *applied* semantics which accompanies it. If we are to understand the formal semantics as being about possible worlds and we take possible worlds seriously, then not just any

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formal semantics will do. We will want a formal semantics that fits the facts, one that reflects what really is the case about possible worlds. Formally it may be interesting to know that there are some semantics in which

(1) $\forall x \Box Fx \supset \Box \forall x Fx$

is valid and some in which it is not. Philosophically, however, we must ask whether all propositions whose form is (1) are true, for if the answer is *no* any semantics in which (1) is valid will fail to be a good model for the modal notion of necessity.

Are these systems philosophically successful? Alvin Plantinga has argued in *The Nature of Necessity* and 'Actualism and Possible Worlds' that the applied semantics given by Kripke fails because it requires that there be objects that do not exist and assumes that objects can have properties in worlds in which they do not exist. For the former, Plantinga presents the following argument. Consider the proposition

(2) Possibly, there is an object distinct from each object which exists in α

where 'α' is a name of the actual world. If (2) is true,

> then (on the Canonical Scheme) there is a possible world W in which there exists an object distinct from each of the things that exists in α. $\psi(W)$, therefore, contains an object that is not a member of $\psi(\alpha)$; hence the same can be said for U. Accordingly, U contains an object that does not exist in α; this object, then, does not exist in the actual world and hence does not exist. We are committed to the view that there are some things that do not exist, therefore, if we accept the Canonical Conception and consider that there could have been a thing distinct from each thing that does in fact exist. ([5], p. 138)

In fact, it seems in most cases that the motivation for the particular formal semantics is technical, that the semantics is chosen to fit a previously given set of axioms.[2] If the formal system is to be philosophically revealing, it would seem that the following procedure ought to be followed. First, an applied semantics must be found which does justice to the modal notions of necessity and possibility and avoids the infelicities of Kripke's applied semantics. Then, this applied semantics must be modeled in a formal semantics for first-order modal logic. Finally, an axiomatic system must be found which has as its theorems exactly those formulas which are valid in the semantics. It is this procedure which is followed in this paper.

*1 **Applied semantics*** Let us call the position that, necessarily, there are no objects which do not exist *actualism*; and the position that, necessarily, whatever has a property in a world exists in that world *serious actualism*. What would be an acceptable applied semantics for a (serious) actualist? How, for example, can we understand (2) so that we are not committed to there being nonexistent possible objects and still keep its modal force? Plantinga suggests that we interpret (2) to be about special properties called *essences*.

Let F be a property. We say that *F is exemplified in a possible world w* if the proposition

(3) Something has F

would have been true had w been actual. We say that an object x has the property of being F in a world w if *x has F* would have been true had w been actual. We say that an object *x has F essentially* if x has F in every world in which x exists. We can then define *individual essence* (or *essence* for short) as follows:

(4) An essence is a property E which is exemplified in some possible world and is such that, in every possible world, for every x, if x has E then: (a) x has E essentially and (b) in no world does anything distinct from x have E.

In a given world w, an essence may or may not be exemplified. For example, an essence of Socrates–Socrateity we might say–is exemplified only in worlds in which Socrates exists. Some essences, presumably, will not be exemplified in the actual world. There is a natural connection between essences which are actually exemplified and objects, for essences by definition have unique exemplifications in a world if any at all. Yet, since essences are properties and properties are necessary beings, all essences exist necessarily. The essence Socrateity, for example, exists in every possible world even though Socrates himself fails to exist in some of them. Which essences are exemplified varies from world to world, but the set of existing essences is the same in each world. Thus, although ordinary mortals like Socrates have properties in only some worlds, essences like Socrateity have properties in all worlds.

If F and G are properties, we say that *F and G are coexemplified in w* if

(5) Something has both F and G

would have been true had w been actual. If F is an essence, then to say that F and G are coexemplified in w is to say that, had w been actual, what would have been the unique exemplification of F would have had G. Unlike Kripke we will interpret quantifiers in modal contexts as ranging over essences rather than possible objects. In general, we interpret propositions of the form

(6) $\forall xFx$

as being true in a world w if every essence which is exemplified in w is coexemplified in w with the property of being F. Propositions of the form

(7) $\exists xFx$

are true in w if some essence is coexemplified in w with the property of being F. Let us now consider (2). It is true if

(8) $(\exists x)$ (x is distinct from every object existing in α)

is true in some world w; i.e., if in some world w some essence is coexemplified with the property of being distinct from every object existing in α. Thus, (2) is true if for some world w some essence is exemplified in w but not in α. Since, by definition, every essence is possibly exemplified, the acceptance of (2) only commits us to the existence of an essence not actually exemplified. Essences, then, provide a way of understanding quantification in modal contexts without referring to possible objects.

Singular propositions can be treated in similar fashion. Consider the propositions:

(9) Socrates is wise
(10) Socrates is not wise.

(9) is true in a world *w* if the essence Socrateity is coexemplified with wisdom in *w*. (10) is ambiguous and may be understood as meaning either

(10′) Socrates is nonwise

or

(10″) *Socrates is wise* is false.

In the actual world (10′) and (10″) have the same truth value, but this is not the case in all worlds. Suppose, for example, that *w* is a world in which Socrates fails to exist. In *w*, then, Socrates has no properties, neither wisdom nor nonwisdom. Thus, in *w*, both (9) and (10′) are false so that (10′) and (10″) have opposite truth values. (10′) is true in a world *w* if Socrateity is coexemplified with the property of being nonwise in *w*, whereas (10″) is true in *w* if either Socrateity is not exemplified in *w* or Socrateity is coexemplified with nonwisdom. Thus (10′) is the stronger claim. Because (10′) asserts that Socrates has a property (being unwise) while (10″) asserts that the proposition (9) has a property (falsehood), we will call (10′) the *de re* interpretation of (10) and (10″) the *de dicto* interpretation of (10). Finally, let us note that for necessary beings *de dicto* and *de re* denial are equivalent.

How are statements containing ‘necessarily’ to be understood? Consider

(11) Necessarily, Socrates is a person.

There are two traditional ways of understanding (11). We can interpret it as meaning

(11′) *Socrates is a person* is a necessary truth.

In this case to assert (11) is to assert that Socrateity is coexemplified with personhood in every possible world. Since Socrates is not a necessary being, (11′) is false. We could also interpret (11) as meaning

(11″) Socrates has personhood essentially;

i.e., Socrates could not have existed without being a person, or *Socrates is a person* is true in every world in which Socrates exists. In this case (11) is true in a world *w* if Socrateity is exemplified in *w* and is coexemplified with personhood in every world in which Socrateity is exemplified. Like denial, then, necessity has two senses: a *de dicto* sense and a *de re* sense. Again, for necessary beings these two senses coincide.

Consider the proposition

(12) Possibly, Socrates is wise.

The usual procedure is to equate ‘possibly’ with ‘not necessarily not’. There are two ways of understanding ‘not’ and ‘necessarily’, so it would seem that there are eight ways of interpreting (12). However, because the two senses of denial and necessity coincide when applied to necessary beings, only four cases actually arise. If, for example, we treat the right-hand instance of ‘not’ as *de dicto* denial, (12) becomes

(13) Not necessarily, *Socrates is wise* is false.

But, presumably propositions like properties are necessary beings, so that '*Socrates is wise* is false' asserts that the necessary being *Socrates is wise* has the property of falsehood. Thus, however we interpret 'not' and 'necessarily' in (13) the result is the same. The four distinct renderings of (12) are

(12a) Socrates has the property of not being essentially unwise
(12b) It is false that *Socrates is wise* is necessarily false
(12c) It is not the case that *Socrates is nonwise* is necessarily true.
(12d) It is false that Socrates is essentially nonwise.

(12a) and (12b) might be called, respectively, the pure *de re* and *de dicto* interpretations of (12). (12a) predicates of Socrates the property of being possibly wise; it is true in a world w if Socrates exists in w and in some world Socrates is wise. (12b) predicates of the proposition *Socrates is wise* the property of being possibly true; it is true in a world w if *Socrates is wise* is true in some world. The hybrid versions (12c) and (12d) provide strange and, as we will see, unacceptable ways of understanding (12). Consider, for example, (12c). It is true if *Socrates is nonwise* is false in some world w. But this will be the case if either Socrates does not exist in w, or Socrates does exist and is wise in w.[3] That this way of understanding (12) is unacceptable can readily be seen by noting that

(14) Possibly, Socrates is a round square

similarly understood would be true just because there are worlds in which Socrates does not exist. Similarly, (12d) is unacceptable as an interpretation of (12), for if we were to read (14) in similar fashion it would be true in any world in which Socrates fails to exist.

2 ***The formal semantics A*** Let L be a first-order modal language. We now turn to the problem of formulating a pure semantics for L which models the applied semantics developed above. A *model structure* for L is a quadruple (D, W, ψ, ϕ) such that

a. D and W are nonempty sets
b. ψ is a function which assigns to each $w \in W$ a nonempty subset D_w of D such that everything in D belongs to some D_w
c. ϕ is a function which assigns to every pair (F, w), where F is an n-ary predicate symbol and $w \in W$, a set of n-tuples of objects from D_w.

The components of the quadruple are to be interpreted as follows: D is a set of essences, W a set of possible worlds, D_w the set of essences exemplified in w, and $\phi(F, w)$ the set of n-tuples of essences which are coexemplified with F in w. That every element of D belongs to some D_w corresponds to the requirement that every essence be possibly exemplified, and that $\phi(F, w)$ only contain tuples whose components are in D_w corresponds to the requirement of serious actualism that only objects existing in a world can have properties or stand in relations in that world. If M is a model structure for L and $w \in W$, then the pair

(M, w) is a *model* for L with *domain* D_w. The model (M, w) will be represented by 'M_w'.

Let $M = (D, W, \psi, \phi)$ be a model structure for L. A function θ which assigns to each individual variable of L an object in D is an *essence assignment.* We wish to define for each model M_w, essence assignment θ, and wff α the notion that M_w *satisfies* α *relative to* θ. (This will be the formal counterpart to a proposition's being true in a world.) For atomic wffs

(a) $M_w \models_\theta Fx_1 \ldots x_n$ iff $(\theta(x_1), \ldots, \theta(x_n)) \in \phi(F, w)$.

In particular, if M_w satisfies '$Fx_1 \ldots x_n$' relative to θ, $\theta(x_1), \ldots, \theta(x_n)$ must be in the domain of M_w. For conjunctions

(b) $M_w \models_\theta \alpha \wedge \beta$ iff $M_w \models_\theta \alpha$ and $M_w \models_\theta \beta$.

If α is a wff and x an individual variable, we follow our interpretation of (7) and define

(c) $M_w \models_\theta \forall x\alpha$ iff $M_w \models_{\theta'} \alpha$ for every θ' such that $\theta'(x) \in D_w$ and θ' has the same values as θ for all variables other than x.

Before we define '$M_w \models_\theta \Box\alpha$' we must decide whether to treat '$\Box$' as a *de re* or a *de dicto* operator. If '$\Box$' is to be a *de dicto* operator, then

$$M_w \models_\theta \Box Fx \text{ iff } M_{w'} \models_\theta Fx \text{ for all } w' \in W;$$

whereas, if '$\Box$' is to be a *de re* operator, then

$$M_w \models_\theta \Box Fx \text{ iff } \theta(x) \in D_w \text{ and } M_{w'} \models_\theta Fx \text{ for all } w' \in W \text{ such that } \theta(x) \in D_{w'}.$$

The applied semantics seems to favor neither the one nor the other. However, as Plantinga has pointed out, *de dicto* necessity is a special case of *de re* necessity. Consider

(15) Necessarily, Socrates is wise.

Interpreted as *de dicto* necessity, (15) means that

(9) Socrates is wise

is true in every world. But, (9) is a proposition and presumably exists in every possible world. Thus, we can understand (15) as asserting that (9) has truth essentially. Thus, we take *de re* necessity as the more basic sense of necessity and define

(d) $M_w \models_\theta \Box\alpha$ iff $\theta(x) \in D_w$ for all x free in α and $M_{w'} \models_\theta \alpha$ for all $w' \in W$ such that $\theta(x) \in D_{w'}$ for all x free in α.

We note that if α is a closed wff then (d) treats '$\Box$' as *de dicto* necessity. Our applied semantics also allows for two ways of treating '$\sim$'. If '$\sim$' is *de dicto* then

$$M_w \models_\theta {\sim}Fx \text{ iff it is not the case that } M_w \models_\theta Fx;$$

whereas, if '$\sim$' is *de re* then

$$M_w \models_\theta {\sim}Fx \text{ iff } \theta(x) \in D_w \text{ and it is not the case that } M_w \models_\theta Fx.$$

But, we have seen the unpleasant consequences of mixing *de re* and *de dicto* operators, so our choice of *de re* necessity requires us to choose *de re* denial. Thus

(e) $M_w \models_\theta \sim\alpha$ iff $\theta(x) \in D_w$ for every variable x free in α and it is not the case that $M_w \models_\theta \alpha$.

(a)-(e) define uniquely the notion of satisfaction for all M_w, θ, and α.

If M is a model structure, then a wff α is *M-valid* if $M_w \models_\theta \alpha$ for every model M_w and essence assignment θ. A wff α is *valid* if it is M-valid for every model structure M. Using induction on the formation rules, it is easy to show that if $M_w \models_\theta \alpha$ then $\theta(x) \in D_w$ for every x free in α. Suppose α has a free variable x. Let $M = (\{a, b\}, \{w_1, w_2\}, \psi, \phi)$ where $D_{w_1} = \{a\}$ and $D_{w_2} = \{b\}$. If $\theta(x) = b$, then it cannot be the case that $M_{w_1} \models_\theta \alpha$. Thus, only closed formulas are valid.

In our definition of *model structure,* we required each element of D to belong to some D_w. This was dictated by our applied semantics since each essence is possibly exemplified. However, because of the formal limitations of the first-order modal language L, this requirement is irrelevant in the sense that the same formulas would have been valid had we not required each element of D to belong to some D_w. Let A' be the semantics which results from deleting this requirement. We will argue that A' is equivalent to A in the sense given above. For the sake of clarity, we will let '$\dashv$' denote satisfaction in A'. If α is a wff, the *closure* of α is the wff which results from prefixing α with the universal quantifiers of all the variables free in α.[4] The closure of α will be denoted by '$C\alpha$'.

If $M = (D, W, \psi, \phi)$ is an A-model structure and $M' = (D', W', \psi', \phi')$ is an A'-model structure, then we say that $M \simeq M'$ if $W = W'$, $\psi = \psi'$, $\phi = \phi'$. We let 'V' denote the set of individual variables of L. For each subset U of V and essence assignment θ, we let $\theta(U) = \{\theta(x): x \in U\}$.

Lemma 2.1 *If M is an A-model structure, M' is an A'-model structure, and $M \simeq M'$, then for any wff α, $w \in W$ and θ such that $\theta(V) \subseteq D$,*

$$M_w \models_\theta \alpha \textit{ if and only if } M'_w \dashv_\theta \alpha.$$

Proof: The proof is a standard induction argument on the formation rules for wffs.

Theorem 2.2 *If α is any wff, then $C\alpha$ is A-valid if and only if $C\alpha$ is A'-valid.*

Proof: Since every A-model structure is an A'-model structure, every A'-valid wff is A-valid.

Suppose $C\alpha$ is A-valid. Let $M' = (D', W, \psi, \phi)$ be any A'-model structure. Let $M = \left(\bigcup_{w \in W} \psi(w), W, \psi, \phi\right)$. Then, M is an A-model structure and $M \simeq M'$. Let $w \in W$ and θ be any M'-assignment. θ may not be an M-assignment, for $\theta(V) \subseteq D'$ but perhaps not $\theta(V) \subseteq D$. However, since $C\alpha$ is a closed wff $C\alpha$ is M'-valid if and only if

$$M'_w \dashv_{\theta^*} C\alpha$$

for every $w \in W$ and θ^* such that $\theta^*(V) \subseteq \bigcup_{w \in W} \psi(w)$. Thus, we may assume

$\theta(V) \subseteq D$. Hence,

$$M'_w \dashv_\theta C\alpha \text{ iff } M_w \models_\theta C\alpha.$$

Since α is A-valid, it is A'-valid.

Because of Theorem 2.2, the requirement that $D = \bigcup_{w \in W} D_w$ is technically irrelevant. It will be convenient in some of the following sections to dispense with this requirement.

3 ***Kripke's system revisited*** Plantinga has argued that the applied semantics used by Kripke to interpret his system is unacceptable from the point of view of a serious actualist. Can we associate with his formal system a different, acceptable applied semantics? Suppose in formulating a pure semantics we had chosen *de dicto* necessity and denial as a basis. The corresponding satisfaction rules would be

(d′) $M_w \models_\theta \Box\alpha$ iff $M_{w'} \models_\theta \alpha$ for all $w' \in W$
(e′) $M_w \models_\theta \sim\alpha$ iff it is not the case that $M_w \models_\theta \alpha$.

Let us call the semantics which results A^* and Kripke's 1963 (*S5*) semantics K. A^* is an acceptable formal semantics and the satisfaction rules for A^* are equivalent to the valuation rules for K. A^* and K still differ, however. In K a world can assign to a predicate letter tuples whose components are not all from the domain of that world. In terms of our applied semantics this corresponds to the possibility of an object's having a property or standing in a relation in a world in which it does not exist. Clearly, this is anathema to the serious actualist. But we might hope that this feature of K is somehow nonessential, that for K and A^* the same formulas might be valid. It is easy to show, for example, that every K-valid formula is also A^*-valid. Also, since quantifiers range only over the domain of a world, that objects can have properties in worlds in which they don't exist may not affect the validity of closed formulas. Unfortunately, this is not the case.

Consider the formula

(16) $(\exists x)\Box Fx \supset \Box\exists xFx.$

If '$\Box$' is taken as *de dicto* necessity, (16) ought to be valid. If something has F in every possible world, then it must exist and have F in every possible world so that in every world something has F. Formally, it can easily be shown that (16) is valid in A^*. In K, however, (16) is invalid. If $D = \{a, b\}$, $W = \{w_1, w_2\}$, $D_{w_1} = \{a\}$, $D_{w_2} = \{b\}$, $\phi(F, w_1) = \{a\}$, and $\phi(F, w_2) = \{a\}$, then

$$M_{w_1} \models_\theta \exists x\Box Fx$$

but not

$$M_{w_1} \models_\theta \Box\exists xFx.$$

Thus, not only is Kripke's applied semantics unacceptable for the serious actualist, the formal semantics K contains an irremovable defect.

4 ***Axiomatics for A*** If α is wff of the first-order modal language L, we will write '$\vdash\alpha$' if the *closure* of α is a theorem.

We have the following axiom formation rules.

(Q1) If α is a substitution instance of a truth-functional tautology, then $\vdash\alpha$.

(Q2) If α and β are wffs and x is an individual variable, then $\vdash(\forall x)(\alpha \supset \beta) \supset (\forall x\alpha \supset \forall x\beta)$.

(Q3) If x is a variable which is not free in α, then $\vdash\alpha \supset \forall x\alpha$.

(Q4) If α' is like α except for containing free occurrences of x' wherever α contains free occurrences of x, then $\vdash\forall x\alpha \supset \alpha'$.

(M1) If α is any wff, then $\vdash\Box\alpha \supset \alpha$.

(M2) If α and β are wffs and every variable free in α is free in β, then $\vdash\Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta)$.

(M3) If α is any wff, then $\vdash\Diamond\alpha \supset \Box\Diamond\alpha$.[5]

In addition, there are two inference rules: modus ponens and necessitation. Formally,

MP If $\vdash\alpha \supset \beta$ and $\vdash\alpha$, then $\vdash\beta$.
N If $\vdash\alpha$, then $\vdash\Box\alpha$.[6]

In essence (Q1)-(Q4) and MP is the system for quantification given by Quine in *Mathematical Logic*. (M1)-(M3) and N is a variation of the modal component of the system *MPC* + *S5* of Hughes and Cresswell. There are two significant differences: the meaning of '$\vdash$' and the weakened formation rule (M2). Since every *MPC* + *S5* model is an A-model, it is not surprising that an axiom system for A is a weakening of *MPC* + *S5*. (If it were the case that all essences were exemplified in all possible worlds, *MPC* + *S5* would have been the right system of quantified modal logic.)

It is easy to show that every theorem in this system is A-valid. The remainder of this paper is concerned with showing that every A-valid wff is a theorem.

5 ***Completeness*** We present a Henkin-style completeness proof, following in outline the completeness proofs given by Hughes and Cresswell. We continue to assume that L is a first-order modal language and V is the set of individual variables of L. If α is a wff, the set of variables free in α will be denoted by '$f(\alpha)$'.

A *Henkin system* for A is a set Ω of pairs (H, V_H) such that

(a) H is a set of wffs of L
(b) V_H is a set of variables
(c) if $\alpha \in H$, then $f(\alpha) \subseteq V_H$
(d) if $f(\alpha) \subseteq V_H$, then exactly one of α and $\sim\alpha$ is in H
(e) if $\alpha, \alpha \supset \beta \in H$, then $\beta \in H$
(f) if $f(\alpha) \subseteq V_H$ and $\vdash\alpha$, then $\alpha \in H$
(g) if $\sim\Box\alpha \in H$, then for some H', $\sim\alpha \in H'$
(h) if $\Box\alpha \in H$, then $\alpha \in H'$ for each H' such that $f(\alpha) \subseteq V_{H'}$

(i) if $\sim\forall x\alpha \in H$, then there is a variable $y \in V_H$ such that y does not occur in $\forall x\alpha$ and $\sim\alpha' \in H$ where α' is the result of replacing all free occurrences of x in α with y.

Let Ω be a Henkin system for A. Let D be the set of variables of L, $W = \Omega$, $\psi(H, V_H) = V_H$, and $\phi(F, (H, V_H)) = \{(x_1, \ldots, x_n) | Fx_1 \ldots x_n \in H\}$. The model structure $M_\Omega = (D, W, \psi, \phi)$ is the *A-system associated with* Ω.[7] Let θ be the function which assigns to each variable in V itself as an element of D.

Theorem 5.1 *If Ω is a Henkin system for A and M_Ω the associated A-system, then for every wff α,*

$$M_{(H, V_H)} \models_\theta \alpha \text{ iff } \alpha \in H.$$

Proof: The argument proceeds by induction on the formation rules of L.

A wff α is *A-consistent* if $C{\sim}\alpha$ is not a theorem of A. Equivalently, α is A-consistent if and only if not $\vdash\sim\alpha$. A finite set of wffs is A-consistent if the conjunction of all its wffs is A-consistent. An arbitrary set of wffs S is A-consistent if every finite subset of S is A-consistent. A set of wffs S is *maximally A-consistent with respect to a set of variables* V^* if

(a) S is A-consistent,
(b) if $\alpha \in S$, then $f(\alpha) \subseteq V^*$, and
(c) if $f(\alpha) \subseteq V^*$, then either $\alpha \in S$ or $S \cup \{\alpha\}$ is A-inconsistent.

The following four lemmas can be proved easily:

Lemma 5.2 *If S is A-consistent and α is any wff, then it cannot be the case that α and $\sim\alpha$ are both in S.*

Lemma 5.3 *If S is maximally consistent with respect to V^* and $f(\alpha) \subseteq V^*$, then either $\alpha \in S$ or $\sim\alpha \in S$.*

Lemma 5.4 *If S is maximally A-consistent with respect to V^*, $f(\alpha) \subseteq V^*$, and $\vdash\alpha$, then $\alpha \in S$.*

Lemma 5.5 *If S is maximally A-consistent with respect to V^* and $\alpha \supset \beta$, $\alpha \in S$, then $\beta \in S$.*

Lemma 5.6 *If $\beta, \gamma_1, \ldots, \gamma_n, \tau_1, \ldots, \tau_l$ are wffs such that for all i, $f(\gamma_i) \subseteq f(\beta)$ and $f(\tau_i) \subseteq f(\beta)$, and $\{\sim\Box\beta, \Box\gamma_1, \ldots, \Box\gamma_n, \sim\tau_1, \ldots, \sim\tau_l\}$ is A-consistent, then $\{\sim\beta, \Box\gamma_1, \ldots, \Box\gamma_n, \sim\Box\tau_1, \ldots, \sim\Box\tau_l\}$ is A-consistent.*

Proof: Suppose $\{\sim\beta, \Box\gamma_1, \ldots, \Box\gamma_n, \sim\Box\tau_1, \ldots, \sim\Box\tau_l\}$ is A-inconsistent. Then

$$\vdash\sim(\sim\beta \wedge \Box\gamma_1 \wedge \ldots \wedge \Box\gamma_n \wedge \sim\Box\tau_1 \wedge \ldots \wedge \sim\Box\tau_l).$$

By (Q1) and modus ponens,

(5.1) $\quad \vdash(\Box\gamma_1 \wedge \ldots \wedge \Box\gamma_n) \supset (\beta \vee \Box\tau_1 \vee \ldots \vee \Box\tau_l)$

We next show that

(5.2) $\quad \vdash\Box(\beta \vee \Box\tau_1 \vee \ldots \vee \Box\tau_l) \supset (\Box\beta \vee \tau_1 \vee \ldots \vee \tau_l)$

By (Q1),

$$\vdash(\beta \vee \Box\tau_1 \vee \ldots \vee \Box\tau_l) \supset [\sim\Box\tau_1 \supset (\beta \vee \Box\tau_2 \vee \ldots \vee \Box\tau_l)].$$

Thus, by necessitation and (M_2),

$$\vdash\Box(\beta \vee \Box\tau_1 \vee \ldots \vee \Box\tau_l) \supset \Box[\sim\Box\tau_1 \supset (\beta \vee \Box\tau_2 \vee \ldots \vee \Box\tau_l)]$$
$$\text{(by M2)} \qquad \supset [\Box\sim\Box\tau_1 \supset \Box(\beta \vee \Box\tau_2 \vee \ldots \vee \Box\tau_l)]$$
$$\text{(by M3)} \qquad \supset [\sim\Box\tau_1 \supset \Box(\beta \vee \Box\tau_2 \vee \ldots \vee \Box\tau_l)]$$
$$\text{(by Q1)} \qquad \supset [\Box\tau_1 \vee \Box(\beta \vee \Box\tau_2 \vee \ldots \vee \Box\tau_l)]$$

Thus, by (Q1) and modus ponens,

$$\vdash\Box(\beta \vee \Box\tau_1 \vee \ldots \vee \Box\tau_l) \supset [\Box\tau_1 \vee \Box(\beta \vee \Box\tau_2 \vee \ldots \vee \Box\tau_l)].$$

By repeated use of this argument, we have

$$\vdash\Box(\beta \vee \Box\tau_1 \vee \ldots \vee \Box\tau_l) \supset [\Box\beta \vee \Box\tau_1 \vee \ldots \vee \Box\tau_l].$$

Using (M1) together with (Q1) and modus ponens gives (5.2).

Applying necessitation, (M2), and modus ponens, from (5.1) we deduce

(5.3) $\qquad \vdash\Box(\Box\gamma_1 \wedge \ldots \wedge \Box\gamma_n) \supset \Box(\beta \vee \Box\tau_1 \vee \ldots \vee \Box\tau_l).$

Combining (5.3) and (5.2) gives

(5.4) $\qquad \vdash\Box(\Box\gamma_1 \wedge \ldots \wedge \Box\gamma_n) \supset \Box\beta \vee \tau_1 \vee \ldots \vee \tau_l.$

Now, by (Q1),

$$\vdash\Box\gamma_1 \supset (\Box\gamma_2 \supset \ldots \supset (\Box\gamma_n \supset (\Box\gamma_1 \wedge \ldots \wedge \Box\gamma_n) \ldots)).$$

Applying necessitation and (M2) repeatedly gives

$$\vdash\Box\Box\gamma_1 \supset (\Box\Box\gamma_2 \supset \ldots \supset (\Box\Box\gamma_n \supset \Box(\Box\gamma_1 \wedge \ldots \wedge \Box\gamma_n) \ldots).$$

By (Q1),

$$\vdash(\Box\Box\gamma_1 \wedge \ldots \wedge \Box\Box\gamma_n) \supset \Box(\Box\gamma_1 \wedge \ldots \wedge \Box\gamma_n).$$

But, from (M3) it follows in the usual way that

$$\vdash\Box\gamma \supset \Box\Box\gamma$$

for any γ. Thus

(5.5) $\qquad \vdash(\Box\gamma_1 \wedge \ldots \wedge \Box\gamma_n) \supset \Box(\Box\gamma_1 \wedge \ldots \wedge \Box\gamma_n).$

Combining (5.5) and (5.4) gives

$$\vdash(\Box\gamma_1 \wedge \ldots \wedge \Box\gamma_n) \supset (\Box\beta \vee \tau_1 \vee \ldots \vee \tau_l).$$

Thus,

$$\vdash\sim(\sim\Box\beta \wedge \Box\gamma_1 \wedge \ldots \wedge \Box\gamma_n \wedge \sim\tau_1 \wedge \ldots \wedge \sim\tau_l).$$

Hence, $\{\sim\Box\beta, \Box\gamma_1, \ldots, \Box\gamma_n, \sim\tau_1, \ldots, \sim\tau_l\}$ is A-inconsistent.

Lemma 5.7 *If S is A-consistent, y is a variable which has no free occurrences in any wff of S and no occurrences at all in $\forall x\beta$, and β' is like β except for containing occurrences of y wherever β has free occurrences of x, then*

$$S \cup \{\sim\forall x\beta \supset \sim\beta'\}$$

is A-consistent.

Proof: The proof is standard.

Let y be a variable. An *E-formula with respect to y* is a formula of form

$$\sim\forall x\beta \supset \sim\beta'$$

where β' is like β except for containing free occurrences of y wherever β has free occurrences of x, and y does not occur in β. An *E-form* is a set consisting of all E-formulas with a given antecedent. If S is a set of wffs, '$f(S)$' will denote the set of variables free in some wff of S.

The proofs of the following two lemmas are standard.

Lemma 5.8 *If S is any A-consistent set such that $f(S)$ is finite and $V^* \supseteq f(S)$ is an infinite set of variables, then there exists an A-consistent set S^* such that $S^* \supseteq S$, $f(S^*) \subseteq V^*$, and S^* contains at least one E-formula from each E-form whose antecedent contains as free variables only variables in V^*.*

Lemma 5.9 *If V^* is a set of variables and S is an A-consistent set of wffs such that $f(S) \subseteq V^*$, then there exists a set S^* of wffs such that $S^* \supseteq S$ and S^* is maximally A-consistent with respect to V^*.*

Let V^* be a set of variables and let H^* be a set of wffs which is maximally consistent with respect to V^* and includes at least one E-formula from each E-form whose antecedent's free variables come from V^*. Then (H^*, V^*) satisfies conditions (a)-(f), (i) of the definition of Henkin system. That (H^*, V^*) satisfies (a)-(f) is immediate from Lemmas 5.2-5.5. Suppose $\sim\forall x\alpha \in H^*$. Then $f(\forall x\alpha) \subseteq V^*$. Hence, H^* contains an E-formula $\sim\forall x\alpha \supset \sim\alpha'$ with respect to variable y in V^* where y does not occur in $\forall x\alpha$. Since H^* is closed under modus ponens, $\sim\alpha' \in H^*$. Hence, (H^*, V^*) satisfies (i).

We now show that for every A-consistent wff α a Henkin system Ω can be constructed such that for some (H, V_H) in Ω, $\alpha \in H$. The construction will proceed through a sequence of stages. We note that since the set V of individual variables of L is countably infinite it can repeatedly yield infinitely many variables without becoming depleted.

Suppose α is A-consistent. The 0th stage of the construction consists of a single pair. Let V_0 be an infinite set of variables containing $f(\alpha)$.[8] Since $\{\alpha\}$ is A-consistent and $f(\alpha)$ is finite, it follows from Lemma 5.8 that there is an A-consistent set S containing α and an E-formula from each E-form whose antecedent's free variables come from V_0 such that $f(S) \subseteq V_0$. By Lemma 5.9 here is a set H_0 of wffs containing S which is maximally A-consistent with respect to V_0. The 0th stage of the construction consists of the pair (H_0, V_0).

Let $\beta_1, \beta_2, \ldots$ be the list of wffs such that $\sim\Box\beta_i \in H_0$. For each β_i, form a set of variables $V\beta_i$ containing $f(\beta_i)$ and an infinite number of new variables, different variables for distinct β_i. For each β_i we form a set $H\beta_i$. Begin with the wff $\sim\beta_i$ and for each wff $\Box\gamma \in H_o$ such that $f(\gamma) \subseteq f(\beta_i)$ and each wff $\sim\tau \in H_o$ such that $f(\tau) \subseteq f(\beta_i)$ add the wffs $\Box\gamma$ and $\sim\Box\tau$. By Lemma 5.6 the resulting set of wffs is A-consistent since H_o is. Since only finitely many free variables are involved, by Lemma 5.8 we can consistently add E-formulas with respect to variables in $V\beta_i$ for all E-forms whose antecedent's free variables come from $V\beta_i$. Finally, by Lemma 5.9 this set can be expanded to a set $H\beta_i$ which is maximally consistent with respect to $V\beta_i$. The first stage of the construction consists of the pairs

$$(H\beta_1, V\beta_1), (H\beta_2, V\beta_2), \ldots.$$

For each β_i, let $\beta_{i1}, \beta_{i2}, \ldots$ be the wffs such that $\sim\Box\beta_{ij} \in H\beta_i$. For each β_{ij} a pair $(H\beta_{ij}, V\beta_{ij})$ is constructed as above, where $V\beta_{ij}$ consists of $f(\beta_{ij})$ together with an infinite number of new variables. The second stage of the construction consists of the pairs

$$(H\beta_{11}, V\beta_{11}), (H\beta_{12}, V\beta_{12}), \ldots (H\beta_{21}, V\beta_{21}), (H\beta_{22}, V\beta_{22}), \ldots.$$

The third, fourth, etc., stages are constructed similarly.

It follows from the remarks at the end of the previous section that each pair constructed satisfies conditions (a)-(f), (i) of the definition of *Henkin system*. Also, by construction the system satisfies condition (g). We will argue that it also satisfies (h). Since whenever $f(\gamma) \subseteq V_{H^*}$, $\Box\gamma \supset \gamma \in H^*$, it will be sufficient to show that if $\Box\gamma$ belongs to any set $H\beta_{i_1} \ldots {}_{i_n}$ constructed at the nth stage and $f(\gamma) \subseteq V\beta_{j_1} \ldots {}_{j_k}$, then $\Box\gamma \in H\beta_{j_1} \ldots {}_{j_k}$. We argue by induction on n.

Suppose $n = 0$. Then $\Box\gamma \in H_o$. Suppose $f(\gamma) \subseteq V\beta_{j_1} \ldots {}_{j_k}$. Since all of the variables in this set are new except for those in $f(\beta_{j_1} \ldots {}_{j_k})$ and $f(\gamma) \subseteq V_0$, $f(\gamma) \subseteq f(\beta_{j_1} \ldots {}_{j_k})$. Thus, $f(\gamma) \subseteq V\beta_{j_1} \ldots {}_{j_{k-1}}$. By similar argumentation, $f(\gamma) \subseteq f(\beta_{j_1} \ldots {}_{j_{k-2}}), \ldots, f(\beta_{j_1})$. Since $f(\gamma) \subseteq f(\beta_{j_1})$ and $\Box\gamma \in H_o$, $\Box\gamma \in H\beta_{j_1}$ by construction. Since $f(\gamma) \subseteq f(\beta_{j_1 j_2})$ and $\Box\gamma \in H\beta_{j_1}$, $\Box\gamma \in H\beta_{j_1 j_2}$ by construction. Continuing, it follows that $\Box\gamma \in H\beta_{j_1} \ldots {}_{j_k}$.

Suppose the result holds for $n = m$ and $\Box\gamma \in H\beta_{i_1} \ldots {}_{i_{m+1}}$. If $f(\gamma) \not\subseteq f(\beta_{i_1} \ldots {}_{i_{m+1}})$, then $f(\gamma)$ contains a variable which is new at the stage at which $H\beta_{i_1} \ldots {}_{i_{m+1}}$ was constructed. In this case only variable sets of the form

$$V\beta_{i_1} \ldots {}_{i_{m+1}} \ldots {}_{i_{m+k}}$$

can contain all the variables in $f(\gamma)$ and the argument proceeds like the case for $n = 0$. Suppose $f(\gamma) \subseteq f(\beta_{i_1} \ldots {}_{i_{m+1}})$. Then, $f(\gamma) \subseteq V\beta_{i_1} \ldots {}_{i_m}$. Hence, either $\Box\gamma$ or $\sim\Box\gamma$ is in $H\beta_{i_1} \ldots {}_{i_m}$. If $\sim\Box\gamma \in H\beta_{i_1} \ldots {}_{i_m}$, then $\sim\Box\Box\gamma \in H\beta_{i_1} \ldots {}_{i_{m+1}}$ by construction. But $\vdash \sim\Box\Box\gamma \supset \sim\Box\gamma$, so $\sim\Box\gamma \in H\beta_{i_1} \ldots {}_{i_{m+1}}$. Since $\Box\gamma \in H\beta_{i_1} \ldots {}_{i_{m+1}}$, this is impossible. Thus, $\Box\gamma \in H\beta_{i_1} \ldots {}_{i_m}$. By induction, whenever $f(\gamma) \subseteq V\beta_{j_1} \ldots {}_{j_k}$, $\Box\gamma \in H\beta_{j_1} \ldots {}_{j_k}$.

In summary, the system of all pairs constructed is a Henkin system and $\alpha \in H_0$.

Theorem 5.10 *For every closed wff α, α is A-valid if and only if $\vdash\alpha$.*

Proof: We showed in Section 4 that whenever $\vdash\alpha$, $C\alpha$ is A-valid. Assume α is closed and A-valid. Suppose it is not the case that $\vdash\alpha$. Since $\vdash\alpha$ if and only if $\vdash\sim(\sim\alpha)$, it follows that $\sim\alpha$ is A-consistent. Then there exists a Henkin system Ω and a pair $(H, V_H) \in \Omega$ such that $\sim\alpha \in H$.

Let M_Ω be the A-system associated with Ω. By Theorem 5.1, $M_H \vDash_\theta \sim\alpha$ iff $\sim\alpha \in H$. Thus, $M_H \vDash_\theta \sim\alpha$. Hence it is not the case that $M_H \vDash_\theta \alpha$, so that α is not A-valid.

NOTES

1. We are here considering only *S5* systems, systems in which $\Diamond\alpha \supset \Box\Diamond\alpha$ is a theorem. Equivalently, we are assuming that every possible world is accessible from every other possible world. Our notation differs somewhat from that given by Kripke and Hughes and Cresswell, but this difference is not significant.

2. A partial exception is Kripke's 1963 semantics. In his 1959 semantics quantifiers ranged over the set of all possible objects. The later semantics modified this so that, in a given world, quantifiers range over only those objects existing in the given world. Thus, from a philosophical point of view, the 1963 semantics is a definite improvement over the 1959 semantics.

3. This truth condition for (12) is the same as that resulting from the weak sense of possibility treated by Plantinga in "De Essentia" [6]. If we take 'possibly' to be 'possibly nonfalse' as he suggests, then (12) is true if there is a world in which *Socrates is wise* is not false. This will be the case for the Priorian if in some world either *Socrates is wise* is true or *Socrates is wise* fails to exist. Since *Socrates is wise* fails to exist in a world if and only if Socrates fails to exist in that world, (12) is true if in some world either Socrates is wise or fails to exist. Plantinga rejects this sense of possibility for the same reason that we reject (12c) as a way of understanding (12).

4. More precisely, we assume that the individual variables of L are arranged in a list and the quantifiers are prefixed in the order in which their variables occur in that list.

5. '$\Diamond$' is an abbreviation for '$\sim\Box\sim$'.

6. Thus, not only is the necessitation of every closed theorem a theorem, but whenever the closure of α is a theorem, the closure of $\Box\alpha$ is a theorem.

7. Strictly speaking, $M_{\hat{\Omega}}$ might not be a model structure for A, for it may be the case that some variable in D is not in any V_H. However, in light of Theorem 2.2 we can dispense with this requirement.

8. We assume here that V_0 is chosen so that $V - V_0$ is infinite. We make similar assumptions at each stage of the construction to guarantee that enough variables will be left for the succeeding stage.

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