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There are Denumerably Many Ternary Intuitionistic Sheffer Functions

DJORDJE ČUBRIĆ*

In [1] Došen asks what is the number of mutually nonequivalent ternary indigenous Sheffer functions for $\{\rightarrow,\wedge,\vee,\neg\}$ in the intuitionistic propositional calculus (IPC). The answer is: denumerably many.

Following [2] we shall say that a set of functions F is an indigenous Sheffer set for a set of functions G iff every member of G can be defined by a finite number of compositions from the members of F and vice versa. A function f is an indigenous Sheffer function for G iff $\{f\}$ is an indigenous Sheffer set for G. The n-ary propositional functions f_1 and f_2 are mutually equivalent iff for some permutation P of the sequence $A_1,\ldots,A_n$ in the propositional calculus we can prove $f_1(A_1,\ldots,A_n) \leftrightarrow f_2(P)$. We work all the time in IPC. Expressions of the form $\vdash A$ (or $\nvdash A$) mean that A is provable (or unprovable) in IPC.

Kuznetsov [3] and Hendry [2] have shown that there is no binary indigenous Sheffer function for $\{\rightarrow,\wedge,\vee,\neg\}$ in IPC. The first example of a ternary indigenous Sheffer function was given in [3]. Here we use one of the three examples given in [1].

The Rieger–Nishimura Lattice of one variable X, RNL(X) is recursively defined as follows: $P_0(X) = X \wedge \neg X$, $P_1(X) = X$, $P_2(X) = \neg X, P_\infty(X) = X \rightarrow X$, $P_{2n+3}(X) = P_{2n+1}(X) \vee P_{2n+2}(X)$, $P_{2n+4}(X) = P_{2n+3}(X) \rightarrow P_{2n+1}(X)$, for $n \geq 0$. For every $i > j$, $\nvdash P_i(X) \rightarrow P_j(X)$ (see [5] or [4]).

First, we have one simple lemma:

Lemma *For every* $i \geq 5$:
(1) $\vdash \neg X \wedge P_i(X) \leftrightarrow \neg X$
(2) $\vdash P_i(\perp)$.

Proof: (1) *For every* $i \geq 5$, we have $\neg X \vdash P_i(X)$ directly from RNL(X). We obtain (2) by using $\vdash \neg\perp$ and (1).

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Next, we give the following definition:

Definition $f_i(A,B,C) = t(A,B,C) \wedge P_i(B \wedge C)$, where $t(A,B,C) = ((A \vee B) \leftrightarrow (C \leftrightarrow \neg B)) \vee (A \leftrightarrow (C \leftrightarrow \neg B))$ (see [1]), and $P_i(B \wedge C) \in \mathrm{RNL}(B \wedge C)$.

Theorem *For every $i \geq 5$, f_i is an indigenous Sheffer function for $\{\rightarrow,\wedge,\vee,\neg\}$ and for every $i \geq 5$ and every $j > i$, $\nvdash f_i \leftrightarrow f_j$.*

Proof: For every $i \geq 5$, f_i is an indigenous Sheffer function for $\{\rightarrow,\wedge,\vee,\neg\}$ because:

$$\begin{aligned} f_i(A,A,A) &\leftrightarrow t(A,A,A) \wedge P_i(A) \\ &\leftrightarrow \neg A \wedge P_i(A) \\ &\leftrightarrow \neg A \qquad \text{(by using Lemma (1));} \end{aligned}$$

$$\begin{aligned} f_i(A,B,\neg B) &\leftrightarrow t(A,B,\neg B) \wedge P_i(B \wedge \neg B) \\ &\leftrightarrow (A \vee B) \wedge P_i(\bot) \\ &\leftrightarrow A \vee B \qquad \text{(by using Lemma (2));} \end{aligned}$$

$$\begin{aligned} f_i(A,\neg(A \vee \neg A),B) &\leftrightarrow t(A,\bot,B) \wedge P_i(\bot \wedge B) \\ &\leftrightarrow (A \leftrightarrow B) \wedge P_i(\bot) \\ &\leftrightarrow (A \leftrightarrow B) \qquad \text{(by using Lemma (2));} \end{aligned}$$

and we know that $\{\leftrightarrow,\vee,\neg\}$ is an indigenous Sheffer set for $\{\rightarrow,\wedge,\vee,\neg\}$ (we have: $\vdash(A \rightarrow B) \leftrightarrow ((A \vee B) \leftrightarrow B)$, $\vdash(A \wedge B) \leftrightarrow ((A \vee B) \leftrightarrow (A \leftrightarrow B))$).

If for some $i \geq 5$ and some $j > i$, $\vdash f_i \leftrightarrow f_j$, we have $\vdash f_i(\bot,B,B) \leftrightarrow f_j(\bot,B,B)$ which implies $\vdash T \wedge P_i(B) \leftrightarrow T \wedge P_j(B)$, and that implies $\vdash P_i(B) \leftrightarrow P_j(B)$, which is a contradiction.

Note that this theorem is also valid for $i = 3$.

For every $i \geq 5$ and $j > i$, f_i and f_j are mutually nonequivalent because $f_i(A,B,C)$ is classically equivalent only with $f_j(C,B,A)$, but not in IPC (if it is, we have: $\vdash f_i(\bot,T,C) \leftrightarrow f_j(C,T,\bot)$; then $\vdash t(\bot,T,C) \wedge P_i(C) \leftrightarrow T$, and then $\vdash P_i(C)$, which is a contradiction). Since we have at most denumerably many nonequivalent ternary indigenous Sheffer functions (consider them as words in the alphabet $\{A,B,C,\rightarrow,\wedge,\vee,\neg\}$), we may conclude that there are exactly denumerably many of them.

For every $n > 3$, there exist denumerably many n-ary Sheffer functions for $\{\rightarrow,\wedge,\vee,\neg\}$ (we substitute $A_1 \wedge \ldots \wedge A_{n-2}$ for A in f_i).

We conclude this note with two questions:

(1) Is it true that for every ternary Sheffer function in the classical propositional calculus there exists a classically equivalent function which is a Sheffer function in IPC?

(2) What structure is produced by all ternary Sheffer functions in IPC?

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Matematički Institut
Knez Mihailova 35
Beograd, Yugoslavia