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Many-Sorted Elementary Equivalence

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Introduction Let us consider a many-sorted language $\mathcal{L}$. Any $\mathcal{L}$-theory $\mathfrak{I}$ can be effectively replaced by an equally powerful $\mathcal{L}^*$-theory $\mathfrak{I}^*$, where $\mathcal{L}^*$ is a one-sorted language canonically associated with $\mathcal{L}$ (most often, $\mathcal{L}^*$ contains a unary predicate S^s for each sort s of $\mathcal{L}$). Such a remark appears in the first paragraphs of many texts dealing with many-sorted theories, e.g. [3], p. 13; [5], ch. 5; or [7], ch. XII.

On the other hand, some many-sorted notions cannot be directly transposed to the corresponding one-sorted notions (see, for example, [4]).

In this paper, we will study how the many-sorted elementary equivalence can be transposed into one-sorted elementary equivalence. More precisely, if $\mathfrak{M}$ and $\mathfrak{N}$ are $\mathcal{L}$-structures we will see when $\mathfrak{M} \equiv \mathfrak{N}$ implies, or is implied by, $\mathfrak{M}^* \equiv^* \mathfrak{N}^*$, where $\mathfrak{M}^*$ and $\mathfrak{N}^*$ are $\mathcal{L}^*$-structures canonically associated with $\mathfrak{M}$ and $\mathfrak{N}$ (in a way which will be made more precise later), and $\equiv$ and $\equiv^*$ denote respectively the $\mathcal{L}$- and $\mathcal{L}^*$-elementary equivalence relations.

First, we will study, as an example, the case where $\mathcal{L}$ is $\mathcal{L}_{TT}$, the language of Simple Type Theory, with four different ways to build $\mathcal{L}^*$. Then we will characterize the many-sorted languages for which the results for $\mathcal{L}_{TT}$ can be generalized.

1 An example: $\mathcal{L}_{TT}$

If $\mathcal{L}$ is $\mathcal{L}_{TT}$, then the set of sorts of $\mathcal{L}$ thus is ω, and its only nonlogical symbol is the binary relational symbol $\in$. Hence, if $\mathfrak{A}$ is an $\mathcal{L}$-structure, then $\mathfrak{A}$ will be of the form

$$\mathfrak{A} = (A^0, A^1, \ldots; \in_{\mathfrak{A}}),$$

where

$$\in_{\mathfrak{A}} \subset \bigcup_{i \in \omega} A^i \times A^{i+1}. \tag{1}$$

As usual, we will impose that

$$\forall i \in \omega,\ A^i \neq \emptyset,$$

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and also that

$$\forall i,j \in \omega,\ i \neq j \Rightarrow A^i \cap A^j = \emptyset. \tag{2}$$

Throughout this section, $\mathfrak{M}$ and $\mathfrak{N}$ will denote two $\mathcal{L}$-structures: $\mathfrak{M} = (M^0,M^1,\ldots;\in_{\mathfrak{M}})$, and $\mathfrak{N} = (N^0,N^1,\ldots;\in_{\mathfrak{N}})$.

There will be four cases, each case considering a different way to build $\mathcal{L}^*$. The first and second cases are quite simple; they are mentioned for the sake of completeness.

1.1 Case 1 Here, the nonlogical symbols of $\mathcal{L}^*$ are $\in^0,\in^1,\ldots,\mathrm{S}^0,\mathrm{S}^1,\ldots$ and for every $\mathcal{L}$-structure $\mathfrak{A}$, $\mathfrak{A}^*$ is defined by

$$\mathfrak{A}^* = \left(\bigcup_{i\in\omega} A^i;\in^0_{\mathfrak{A}},\in^1_{\mathfrak{A}},\ldots,S^0_{\mathfrak{A}},S^1_{\mathfrak{A}},\ldots\right),$$

where, for every $i \in \omega$, $\in^i_{\mathfrak{M}} = \in_{\mathfrak{M}} \cap M^i \times M^{i+1}$ and $S^i_{\mathfrak{M}} = M^i$.

It is easy to see that if $\mathfrak{M}^* \equiv^* \mathfrak{N}^*$, then $\mathfrak{M} \equiv \mathfrak{N}$. Indeed, for any $\mathcal{L}$-sentence σ, there exists an $\mathcal{L}^*$-sentence σ^* such that for any $\mathcal{L}$-structure $\mathfrak{A}$, $\mathfrak{A} \models \sigma \Leftrightarrow \mathfrak{A}^* \models \sigma^*$. σ^* is obtained by substituting simultaneously, for each expression of the form $\exists x^i\phi(x^i)$ in σ, an expression of the form $\exists x(S^i(x) \wedge \phi(x))$ (with the understanding that different variables in σ are replaced by different variables in σ^*).

The other implication, $\mathfrak{M} \equiv \mathfrak{N} \Rightarrow \mathfrak{M}^* \equiv^* \mathfrak{N}^*$ is done in a similar way: for any $\mathcal{L}^*$-sentence σ^* we have to find an $\mathcal{L}$-sentence σ such that for any $\mathcal{L}$-structure $\mathfrak{A}$, $\mathfrak{A}^* \models \sigma^* \Leftrightarrow \mathfrak{A} \models \sigma$. The construction of σ will not be given here; it is based on the following lemma:

Lemma 1 *Let ϕ be an $\mathcal{L}^*$-formula, and n_ϕ a natural number such that, if $\in^i$ or S^i occurs in ϕ, then $i < n_\phi$. Also let $\mathfrak{A}$ be an $\mathcal{L}$-structure, and f a permutation of $\bigcup_{i\in\omega} A^i$ such that $f(x) = x$ for every x in $A^0 \cup \ldots \cup A^{n_\phi}$. Then, for every $x,y,\ldots \in \bigcup_{i\in\omega} A^i$, $\mathfrak{A}^* \models \phi[x,y,\ldots] \Leftrightarrow \mathfrak{A}^* \models \phi[f(x),f(y),\ldots]$.*

1.2 Case 2 The nonlogical symbols of $\mathcal{L}^*$ are $\in^0,\in^1,\ldots,$ and for every $\mathcal{L}$-structure $\mathfrak{A}$, $\mathfrak{A}^*$ is defined by

$$\mathfrak{A}^* = \left(\bigcup_{i\in\omega} A^i;\in^0_{\mathfrak{A}},\in^1_{\mathfrak{A}},\ldots\right).$$

As $\mathcal{L}^*$ is a sublanguage of the language $\mathcal{L}^*$ introduced in Case 1, $\mathfrak{M} \equiv \mathfrak{N} \Rightarrow \mathfrak{M}^* \equiv^* \mathfrak{N}^*$ is a consequence of Case 1. But now, $\mathfrak{M}^* \equiv^* \mathfrak{N}^* \Rightarrow \mathfrak{M} \equiv \mathfrak{N}$ is no longer true. This can be shown in the following trivial example:

- $\mathfrak{M} = (\{m_0\},\{m_1\},\{m_2\},\ldots,\{m_i\},\ldots;\in_{\mathfrak{M}})$, where $\in_{\mathfrak{M}} = \emptyset$
- $\mathfrak{N} = (\{n_0,n_1\},\{n_2\},\{n_3\},\ldots,\{n_i\},\ldots;\in_{\mathfrak{N}})$, where $\in_{\mathfrak{N}} = \emptyset$;

$\mathfrak{M}^* \equiv^* \mathfrak{N}^*$, because $\mathfrak{M}^* \cong \mathfrak{N}^*$. But if σ is $\exists x^0\exists y^0\ x^0 \neq y^0$, then $\mathfrak{M} \not\models \sigma$ and $\mathfrak{N} \models \sigma$. Thus $\mathfrak{M} \not\equiv \mathfrak{N}$.

1.3 Case 3 The only nonlogical symbol of $\mathcal{L}^*$ is $\in$, and for every $\mathcal{L}$-structure $\mathfrak{A}$, $\mathfrak{A}^*$ is $\left(\bigcup_{i\in\omega} A^i;\in_{\mathfrak{A}}\right)$. In this case it is not trivial to prove that $\mathfrak{M} \equiv \mathfrak{N} \Rightarrow$

$\mathfrak{M}^* \equiv^* \mathfrak{N}^*$. The easiest proof the author has found, on a suggestion of M. Boffa and M. Crabbé, makes use of Fraïssé's technique of partial isomorphisms (see [6], ch. 26). But such a proof cannot be nicely generalized, since it can only be applied to languages having a finite number of symbols. However, another more general but also more tedious proof has been found. It is given in detail in [2], and its generalization will be given later in this paper.

On the other hand, the argument given in Case 2 can be reproduced here to prove that $\mathfrak{M}^* \equiv^* \mathfrak{N}^* \not\Rightarrow \mathfrak{M} \equiv \mathfrak{N}$.

1.4 Case 4 The nonlogical symbols of $\mathcal{L}^*$ are $\in, \mathrm{S}^0, \mathrm{S}^1, \ldots$, and for every $\mathcal{L}$-structure $\mathfrak{A}$, $\mathfrak{A}^*$ is given by

$$\mathfrak{A}^* = \left(\bigcup_{i \in \omega} A^i; \in_{\mathfrak{A}}, S^0_{\mathfrak{A}}, S^1_{\mathfrak{A}}, \ldots\right),$$

where $S^i_{\mathfrak{A}} = A^i$, for every $i \in \omega$.

An easy adaptation of the proof given in [2] shows that $\mathfrak{M} \equiv \mathfrak{N} \Rightarrow \mathfrak{M}^* \equiv^* \mathfrak{N}^*$. And $\mathfrak{M}^* \equiv^* \mathfrak{N}^* \Rightarrow \mathfrak{M} \equiv \mathfrak{N}$ can be proved by reproducing the argument suggested in Case 1.

2 Notations In the following sections, $\mathcal{L}$ will denote a many-sorted language, whose set of sorts will be Σ. In some sections, there will be constraints on $\mathcal{L}$, which will be indicated. An $\mathcal{L}$-structure $\mathfrak{A}$ will be of the form

$$\mathfrak{A} = ((A^s)_{s \in \Sigma}; (\mathfrak{R}_r)_{r \in R}, (\mathfrak{F}_f)_{f \in F}),$$

where the A_s's are the domains, the $\mathfrak{R}_r$'s the relations, and the $\mathfrak{F}_f$'s the functions. As for $\mathcal{L}_{TT}$-structures, we will impose that

$$\forall s \in \Sigma,\ A^s \neq \emptyset,$$

and that

$$\forall s, t \in \Sigma,\ s \neq t \Rightarrow A^s \cap A^t = \emptyset. \tag{3}$$

To generalize the results produced for $\mathcal{L}_{TT}$, four different one-sorted languages $\mathcal{L}^*$ will be introduced. Each time, $\mathfrak{A}^*$ will be the $\mathcal{L}^*$-structure canonically associated with an $\mathcal{L}$-structure $\mathfrak{A}$, in the same way that this has been done for $\mathcal{L}_{TT}$; in particular, the domain of $\mathfrak{A}^*$ will be $\bigcup_{s \in \Sigma} A^s$.

Two languages, $\mathcal{L}_G$ and $\mathcal{L}^+$ will be also introduced later. $\equiv$, $\equiv^*$, $\equiv_G$, and $\equiv^+$ will denote the elementary equivalence relations corresponding respectively to $\mathcal{L}$, $\mathcal{L}^*$, $\mathcal{L}_G$, and $\mathcal{L}^+$. For example, $\mathfrak{A} \equiv^* \mathfrak{B}$ iff $\mathfrak{A} \models \sigma \Leftrightarrow \mathfrak{B} \models \sigma$ for every $\mathcal{L}^*$-sentence σ.

3 Generalization of Case 1 and Case 2 The generalization of Case 1 is quite straightforward, and thus details will be left as an exercise. Roughly speaking, $\mathcal{L}^*$ is obtained by splitting functional and relational symbols of $\mathcal{L}$, according to the sorts of their arguments, and by adding a unary predicate symbol S^s for each sort s. Then, using a generalization of Lemma 1, one can prove that:

For any $\mathcal{L}^*$-formula ϕ, there exists $\Sigma_\phi \subset \Sigma$, Σ_ϕ finite, such that, for any $\mathcal{L}$-structure $\mathfrak{A}$, and for any $y_1,\dots,y_n \in \bigcup_{s\in\Sigma} A^s$,

$$\exists x \in \bigcup_{s\in\Sigma} A^s\ \mathfrak{A}^* \models \phi[x,y_1,\dots,y_n] \Leftrightarrow \exists x \in \bigcup_{s\in\Sigma_\phi} A^s\ \mathfrak{A}^* \models \phi[x,y_1,\dots,y_n].$$

This is the key to show that $\mathfrak{M} \equiv \mathfrak{N} \Rightarrow \mathfrak{M}^* \equiv^* \mathfrak{N}^*$, for any two $\mathcal{L}$-structures $\mathfrak{M}$ and $\mathfrak{N}$. And $\mathfrak{M}^* \equiv^* \mathfrak{N}^* \Rightarrow \mathfrak{M} \equiv \mathfrak{N}$ is true by exactly the same argument as in Case 1.

Now, in order to generalize Case 2, $\mathcal{L}^*$ will be defined as above, but without the S^s's. As this $\mathcal{L}^*$ is a sublanguage of the preceding $\mathcal{L}^*$, then $\mathfrak{M} \equiv \mathfrak{N} \Rightarrow \mathfrak{M}^* \equiv^* \mathfrak{N}^*$. As we saw in Case 2, $\mathfrak{M}^* \equiv^* \mathfrak{N}^* \Rightarrow \mathfrak{M} \equiv \mathfrak{N}$ is in general false. Nevertheless, it will be true if the S^s's can be "defined" in $\mathfrak{M}$ and $\mathfrak{N}$, i.e. if, for every $s \in \Sigma$, there exists an $\mathcal{L}^*$-formula $\phi^s(x)$, having x as its only free variable, such that

$$\forall x \in \bigcup_{t\in\Sigma} M^t\ \mathfrak{M}^* \models \phi^s[x] \Leftrightarrow x \in M^s$$

and

$$\forall x \in \bigcup_{t\in\Sigma} N^t\ \mathfrak{N}^* \models \phi^s[x] \Leftrightarrow x \in N^s.$$

For example, if $\mathcal{L}$ is $\mathcal{L}_{TT}$, then for every $i \in \Sigma = \omega$, there exists a formula $\phi^i(x)$, such that for every $\mathcal{L}_{TT}$-structure $\mathfrak{A}$ *satisfying the axioms of Simple Type Theory*, $\mathfrak{A}^* \models \phi^i[x] \Leftrightarrow x \in A^i$, for every $x \in \bigcup_{i\in\omega} A^i$. It suffices to define $\phi^i(x)$ as

$$(\exists y(x \in y \wedge \exists x_0 \exists x_1 \dots \exists x_i\ x_0 \in x_1 \wedge x_1 \in x_2 \wedge \dots \wedge x_i \in y))$$
$$\wedge\ (\forall y(x \in y \Rightarrow \neg(\exists x_0 \exists x_1 \dots \exists x_{i+1}\ x_0 \in x_1 \wedge x_1 \in x_2 \wedge \dots \wedge x_{i+1} \in y))).$$

4 Generalization of Case 3

4.1 A counterexample For the generalization of Case 3, $\mathcal{L}^*$ will be the one-sorted language having the same logical, relational, and functional symbols as $\mathcal{L}$.

But first, as a motivation, let us look at a counterexample. Let $\mathcal{L}$ be a many-sorted language whose set of sorts is ω, and whose atomic formulas are of the form $x^i = y^i$ or $x^0 \in y^i$ $(i \in \omega)$. Now let $\mathfrak{M}$ and $\mathfrak{N}$ be the following slightly different $\mathcal{L}$-structures: $\mathfrak{M} = (M^0, M^1, \dots; \in_{\mathfrak{M}})$ and $\mathfrak{N} = (N^0, N^1, \dots; \in_{\mathfrak{N}})$, where

- $M^0 = \{m_1, m_2, \dots\}$
  $M^i = \{\bar{m}_i\}$, for every $i \geq 1$
  $\in_{\mathfrak{M}} = \{\langle m_i, \bar{m}_i\rangle : i \in \omega\}$
- $N^0 = \{n_0, n_1, n_2, \dots\}$
  $N^i = \{\bar{n}_i\}$, for every $i \geq 1$
  $\in_{\mathfrak{N}} = \{\langle n_i, \bar{n}_i\rangle : i \in \omega\}$.

It can be proved that $\mathfrak{M} \equiv \mathfrak{N}$.

But if σ^* is $(\exists x)(\neg\exists y)(x \in y)$, then $\mathfrak{M}^* \not\models \sigma^*$ while $\mathfrak{N}^* \models \sigma^*$. Hence $\mathfrak{M}^* \not\equiv^* \mathfrak{N}^*$.

What is the "fatal difference" between $\mathcal{L}$ and $\mathcal{L}_{TT}$? Well, $\mathcal{L}_{TT}$ is, in a certain sense, "local", while $\mathcal{L}$ is not. This "locality" of $\mathcal{L}_{TT}$ means that there are only a finite number of sorts directly interacting with a given sort. In more technical words, the only atomic formulas into which x^i can occur are of the form $y^{i-1} \in x^i$, $x^i = y^i$ or $x^i \in y^{i+1}$; thus sorts $i-1$, i, and $i+1$ are the only sorts which can directly interact with sort i. This will be the hypothesis we will put on $\mathcal{L}$ to generalize Case 3.

4.2 The hypothesis on $\mathcal{L}$ Let a be an atomic formula of $\mathcal{L}^*$, and suppose that $x_1, \ldots, x_n$ are exactly the variables occurring in $a (n \geq 2)$. Then $\mathcal{S}_a$ will be defined by

$$\mathcal{S}_a = \{\langle s_1, \ldots, s_n \rangle \in \Sigma^n \colon a(x_1^{s_1}/x_1, \ldots, x_n^{s_n}/x_n) \text{ is a well-formed } \mathcal{L}\text{-formula}\}.$$

Then we can define

$$\mathcal{S}_a(s) = \{s' \in \Sigma \colon \exists l \exists l' \neq l\ \exists \langle s_1, \ldots, s_n \rangle \in \mathcal{S}_a\ s_l = s \wedge s_{l'} = s'\}.$$

Now, the constraint we will impose on $\mathcal{L}$ is the following:

for every atomic $\mathcal{L}$-formula a, having at least two variables, and for every $s \in \Sigma$,

$$\mathcal{S}_a(s) \textit{ is finite.}$$

For example, this constraint is satisfied by $\mathcal{L}_{TT}$:

- if a is $x_1 \in x_2$ then $\mathcal{S}_a(0) = \{1\}$, and $\mathcal{S}_a(s) = \{s-1, s+1\}$, if $s > 0$
- if a is $x_1 = x_2$ then $\mathcal{S}_a(s) = \{s\}$.

4.3 Generalized sorts We will now introduce $\mathcal{L}_G$, a many-sorted language in which $\mathcal{L}$ and $\mathcal{L}^*$ can both be "embedded". $\mathcal{L}_G$ has the same logical, relational, and functional symbols as $\mathcal{L}$. But its set of sorts, Σ_G, is bigger than Σ: $\mathrm{s} \in \Sigma_G$ iff

- either $\mathrm{s} \in \Sigma$ (s is called a *"proper sort"*)
- or s is a finite subset of Σ (s is called a *"nonproper sort"*).

Elements of Σ_G are called *"generalized sorts"*. Intuitively, a varaible of sort $\{s_1, \ldots, s_n\}$ will take its value among objects whose sort is neither s_1, nor $s_2, \ldots,$ nor s_n.

Atomic formulas of $\mathcal{L}_G$ are built in the usual way, from the symbols of $\mathcal{L}_G$, without any restriction on the sort of occurring variables. By the way, we can extend the definition of $\mathcal{S}_a$ to cases where a is an atomic $\mathcal{L}_G$-formula, whose variables are exactly $x_1^{\mathrm{s}_1}, \ldots, x_n^{\mathrm{s}_n}$:

$$\mathcal{S}_a = \{\langle s_1, \ldots, s_n \rangle \in \Sigma^n \colon a(x_1^{s_1}/x_1^{\mathrm{s}_1}, \ldots, x_n^{s_n}/x_n^{\mathrm{s}_n}) \text{ is a well-formed } \mathcal{L}\text{-formula}\}.$$

So $\mathcal{S}_a$, and also $\mathcal{S}_a(s)$, remain defined if a is an $\mathcal{L}_G$-formula.

Now, let $\mathfrak{M}$ be an $\mathcal{L}$-structure:

$$\mathfrak{M} = ((M^s)_{s \in \Sigma};\ (\mathcal{R}_r)_{r \in R}, (\mathcal{F}_f)_{f \in F}).$$

The $\mathcal{L}_G$-structure that will be canonically associated with $\mathfrak{M}$ will be

$$\mathfrak{M}_G = ((M^s)_{s\in\Sigma}, (M^S)_{S\in\mathcal{P}^\infty(\Sigma)}; (\mathcal{R}_r)_{r\in R}, (\mathcal{F}_f)_{f\in F}),$$

where, if $S \in \mathcal{P}^\infty(\Sigma)$, $M^S = \bigcup_{s\in\Sigma\setminus S} M^s$. We will suppose also that every M^S is nonempty. This will be the case if Σ is infinite. We will see later what happens if Σ is finite.

We say that $\mathcal{L}$ and $\mathcal{L}^*$ can be "embedded" in $\mathcal{L}_G$ because, if $\mathfrak{M}$ is an $\mathcal{L}$-structure, and if $m_1, \ldots, m_n \in \bigcup_{s\in\Sigma} M^s$, then

- if ϕ is an $\mathcal{L}$-formula, then ϕ is also an $\mathcal{L}_G$-formula and
$$\mathfrak{M} \models \phi[m_1, \ldots, m_n] \Leftrightarrow \mathfrak{M}_G \models \phi[m_1, \ldots, m_n]; \tag{4}$$
- if ϕ is an $\mathcal{L}^*$-formula, then
$$\mathfrak{M}^* \models \phi[m_1, \ldots, m_n] \Leftrightarrow \mathfrak{M}_G \models \phi^\varnothing[m_1, \ldots, m_n], \tag{5}$$
where $\phi^\varnothing$ is ϕ where every variable x has been replaced by $x^\varnothing$ (that is because $M^\varnothing = \bigcup_{s\in\Sigma} M^s$).

Now, let us pause for some useful definitions:

- a variable will be called *proper* (respectively *nonproper*) if its sort is *proper* (respectively *nonproper*);
- an $\mathcal{L}_G$-formula ϕ will be called *proper* (respectively *nonproper*) if all variables occurring (free or bounded) in ϕ are proper (respectively non-proper);
- an $\mathcal{L}_G$-formula will be called *homogeneous* if it is proper or nonproper.

Also, we will use the following conventions: lower-case letters $s, t, \ldots$ will denote proper sorts, upper-case letters $S, T, \ldots$ will denote nonproper sorts, and a bold lower-case s will denote any generalized sort.

Finally, we can introduce the $\mathcal{L}_G$-theory Γ, which will allow us to reason purely syntactically (i.e., independently of the particular structures $\mathfrak{M}$, $\mathfrak{M}^*$, and $\mathfrak{M}_G$). Γ will be satisfied by all $\mathcal{L}_G$-structures of the form $\mathfrak{M}_G$. Its logical axioms are given by the following schemas:

Γ1 $\neg a$, if a is a proper atomic $\mathcal{L}_G$-formula which is *not* an atomic $\mathcal{L}$-formula (this is justified by (3)).

Γ2 $\neg a$, if a is an atomic $\mathcal{L}_G$-formula and $S \supset \mathbb{S}_a(s)$ for some variables x^s and y^S occurring in a.

Γ3 $\exists x^S \phi(x^S) \Leftrightarrow (\exists x^{S\cup\{s\}}\phi(x^{S\cup\{s\}}) \vee \exists x^s \phi(x^s))$, if ϕ is an $\mathcal{L}_G$-formula, and $s \notin S$.

4.4 Working definitions This section contains all the definitions used in the next sections, together with some simple properties.

If ϕ is an $\mathcal{L}_G$-formula, then

- if $s \in \Sigma$, then $\mathcal{S}_\phi(s) =_{df} \bigcup\{\mathcal{S}_a(s)\colon a$ is an atomic subformula of $\phi\}$
- if $S \subset \Sigma$, then $c_\phi(S) =_{df} \bigcup\{\mathcal{S}_\phi(s)\colon s \in S\}$, and we define, by induction,
  $c_\phi^0(S) =_{df} S$
  $c_\phi^{n+1}(S) =_{df} c_\phi^n(S) \cup c_\phi(c_\phi^n(S))$
- if $S \in \Sigma$, then $\mathcal{C}_\phi^n =_{df} c_\phi^{2^n}(S)$
- $\mathrm{pr}(\phi) =_{df} \{s \in \Sigma\colon s$ is the sort of a (proper) variable occurring in $\phi\}$.

From these definitions it is easy to prove that

If $n \leq m$, $S \subset T$, and ψ is a subformula of ϕ,
then $\mathcal{C}_\psi^n(S) \subset \mathcal{C}_\phi^m(T)$, (6)

and also that

If $n \geq 0$, $S \subset \Sigma$, and ϕ is an $\mathcal{L}_G$-formula, then

$$S \subset \mathcal{C}_\phi^n(S) \tag{7}$$
$$\mathcal{C}_\phi^n(\mathcal{C}_\phi^n(S)) = \mathcal{C}_\phi^{n+1}(S). \tag{8}$$

If ϕ is an $\mathcal{L}_G$-formula, then $\mathrm{nqr}(\phi)$, the *"nonproper quantifier rank"* of ϕ is defined by induction as follows:

- $\mathrm{nqr}(\phi) = 0$, if ϕ is atomic;
- $\mathrm{nqr}(\neg\phi) = \mathrm{nqr}(\phi)$;
- $\mathrm{nqr}(\psi \vee \psi') = \max\{\mathrm{nqr}(\psi),\mathrm{nqr}(\psi')\}$;
- $\mathrm{nqr}(\exists x^s\psi) = \mathrm{nqr}(\psi)$;
- $\mathrm{nqr}(\exists x^S\psi) = \mathrm{nqr}(\psi) + 1$.

A key notion, the notion of a *connected $\mathcal{L}_G$-formula*, is also defined by induction:

- if ϕ is atomic, then ϕ is connected
- if ϕ is connected, then $\neg\phi$ is connected
- if ϕ and ψ are connected and have at least one common free variable, then $\phi \vee \psi$ is connected
- if ϕ is connected and if x^s occurs free in ϕ, then $\exists x^s\phi$ is connected.

An easy, but important, property of connected formulas is the following:

If ϕ is a connected $\mathcal{L}_G$-formula such that every atomic
subformula of ϕ is homogeneous, then ϕ is homogeneous. (9)

It is also worth noting that

If σ is a connected $\mathcal{L}_G$-sentence, then σ
is of the form $\exists x^s\phi$ or $\neg \ldots \neg\exists x^s\phi$. (10)

We also have to make more precise our definition of $\mathrm{lg}(\phi)$, the *length* of an $\mathcal{L}_G$-formula ϕ:

- if ϕ is atomic, then $\mathrm{lg}(\phi) = 0$
- $\mathrm{lg}(\psi \vee \psi') = \max\{\mathrm{lg}(\psi),\mathrm{lg}(\psi')\} + 1$
- $\mathrm{lg}(\neg\psi) = \mathrm{lg}(\psi) + 1$
- $\mathrm{lg}(\exists x^s\psi) = \mathrm{lg}(\psi) + 1$.

Finally, here is our definition of a *Boolean combination of a set of $\mathcal{L}_G$-formulas*:

- if ϕ is an $\mathcal{L}_G$-formula, then ϕ is a Boolean combination of $\{\phi\}$
- if Φ is a Boolean combination of the set of $\mathcal{L}_G$-formulas E, then so is $\neg\Phi$
- if Φ and Ψ are respectively Boolean combinations of the sets of $\mathcal{L}_G$-formulas E and F, then $\Phi \vee \Psi$ is a Boolean combination of $E \cup F$.

4.5 Lemmas This section contains four lemmas, used in the proof of Theorem 1 in the next section. It is not necessary to understand the proofs of these lemmas in order to understand the proof of the theorem. Thus, those readers who want to get some motivation before going into the proofs of the lemmas can skip them in a first reading, and see how they are used in the proof of Theorem 1 (Lemma 3 is used only in the proof of Lemma 4).

Lemma 2 (The fundamental lemma) *If ϕ is a nonhomogeneous $\mathcal{L}_G$-formula such that if S is the sort of a free nonproper variable of ϕ then $S \supset \mathcal{C}_\phi^{\mathrm{nqr}(\phi)}(\mathrm{pr}(\phi))$, or if ϕ is a homogeneous $\mathcal{L}_G$-formula, then there exists an $\mathcal{L}_G$-formula ϕ_h such that*

- $\Gamma \vdash \phi \Leftrightarrow \phi_h$
- $\mathrm{nqr}(\phi_h) \leq \mathrm{nqr}(\phi)$
- *every atomic subformula of ϕ_h is homogeneous*
- *every free variable of ϕ_h occurs free in ϕ.*

Proof: The proof will be an induction on $\mathrm{lg}(\phi)$. Only a construction of ϕ_h will be given, with however some indications concerning the reasons that $\Gamma \vdash \phi \Leftrightarrow \phi_h$. The verification of the remainder of the thesis is left as an exercise.

- If ϕ is atomic and homogeneous, then ϕ_h can be set identical to ϕ.
- If ϕ is atomic and nonhomogeneous, then let x^s and y^S be two variables occurring in ϕ. As $\mathrm{nqr}(\phi) = 0$, then, by the hypothesis of the lemma,

  $$S \supset \mathcal{C}_\phi^0(\mathrm{pr}(\phi)) = c_\phi(\mathrm{pr}(\phi)) = \bigcup\{\mathcal{S}_\phi(t) \colon t \in \mathrm{pr}(\phi)\} \supset \mathcal{S}_\phi(s).$$

  By Axiom $\Gamma 2$, ϕ is thus false. Hence, we can set ϕ_h identical to $x^s \neq x^s$.
- If ϕ is $\neg\psi$, then $\mathcal{C}_\phi^{\mathrm{nqr}(\phi)}(\mathrm{pr}(\phi)) = \mathcal{C}_\psi^{\mathrm{nqr}(\psi)}(\mathrm{pr}(\psi))$. So ψ also satisfies the hypothesis of the lemma; by the induction hypothesis, ψ_h exists, and we can set ϕ_h identical to $\neg(\psi_h)$.
- If ϕ is $\psi \vee \psi'$, then if, for example, ψ is not homogeneous, and if S is the sort of a nonproper variable occurring free in ψ, then
  $\mathcal{C}_\psi^{\mathrm{nqr}(\psi)}(\mathrm{pr}(\psi))$
  $\subset \mathcal{C}_\phi^{\mathrm{nqr}(\phi)}(\mathrm{pr}(\phi))$
  (because ψ is a subformula of ϕ, $\mathrm{nqr}(\psi) \leq \mathrm{nqr}(\phi)$, and $\mathrm{pr}(\psi) \subset \mathrm{pr}(\phi)$)
  $\subset S$.

  This is sufficient to show that ψ satisfies the hypothesis of the lemma and, hence, that ψ_h exists. In the same way, ψ'_h exists, and we can set ϕ_h identical to $\psi_h \vee \psi'_h$.

- If ϕ is $\exists x^s\psi$ and that x^s does not occur free in ψ, then, as in the preceding case, it suffices to set ϕ_h identical to ψ_h.

- If ϕ is $\exists x^s\psi$ and that x^s occurs free in ψ, then $\mathcal{C}_\psi^{\mathrm{nqr}(\psi)}(\mathrm{pr}(\psi)) = \mathcal{C}_\phi^{\mathrm{nqr}(\phi)}(\mathrm{pr}(\phi))$ and, as above, ψ satisfies the hypothesis of the lemma, we can set ϕ_h identical to $\exists x^s\psi_h$.
- If ϕ is $\exists x^S\psi(x^S)$, and x^S occurs free in ψ, and ψ has no proper free variable, then ϕ_h can be ϕ.
- If ϕ is $\exists x^S\psi(x^S)$, and x^S occurs free in ψ, and at least one proper variable occurs free in ψ, then we would like to set ϕ_h identical to $\exists x^S\psi_h$. Unfortunately, we cannot prove ψ_h to exist. A solution consists in first defining

$$S' = S \cup \mathcal{C}_\phi^{\mathrm{nqr}(\phi)-1}(\mathrm{pr}(\phi)). \tag{11}$$

By the constraint on $\mathcal{L}$, and by the definition of nonproper sorts, $S'\backslash S$ is finite. We can thus build the following $\mathcal{L}^*$-formula:

$$\exists x^{S'}(\psi(x^{S'}))_h \vee \bigvee_{s\in S'\backslash S} \exists x^s(\psi(x^s))_h. \tag{12}$$

By $\Gamma 3, \phi_h$ can be set identical to this formula. Indeed, we are going to prove that $\psi(x^{S'})$ and the $\psi(x^s)$'s satisfy the hypothesis of the lemma; $(\psi(x^{S'}))_h$ and the $(\psi(x^s))_h$'s thus exist. Let us consider $\psi(x^{S'})$. If T is the sort of a nonproper free variable of $\psi(x^{S'})$, then

a. either T is the sort of a free variable of ϕ,
b. or T is S'.

In the first case,

$$\begin{aligned}\mathcal{C}_{\psi(x^{S'})}^{\mathrm{nqr}(\psi(x^{S'}))}(\mathrm{pr}(\psi(x^{S'}))) &= \mathcal{C}_\phi^{\mathrm{nqr}(\phi)-1}(\mathrm{pr}(\phi)) \\ &\subset \mathcal{C}_\phi^{\mathrm{nqr}(\phi)}(\mathrm{pr}(\phi)) && \text{(by (6))} \\ &\subset T. && \text{(hypothesis of the lemma).}\end{aligned}$$

And in the second case,

$$\begin{aligned}S' &\supset \mathcal{C}_\phi^{\mathrm{nqr}(\phi)-1}(\mathrm{pr}(\phi)) && \text{(by (11))} \\ &= \mathcal{C}_{\psi(x^{S'})}^{\mathrm{nqr}(\psi(x^{S'}))}(\mathrm{pr}(\psi(x^{S'}))).\end{aligned}$$

Thus $\psi(x^{S'})$ satisfies the hypothesis of the lemma. Let us now consider a $\psi(x^s)$. If $\psi(x^s)$ is homogeneous, we are done. And if it is not, then let T be the sort of one of its nonproper free variables. Then

$$\begin{aligned}&\mathcal{C}_{\psi(x^s)}^{\mathrm{nqr}(\psi(x^s))}(\mathrm{pr}(\psi(x^s))) \\ &\quad= \mathcal{C}_{\psi(x^s)}^{\mathrm{nqr}(\psi(x^s))}(\mathrm{pr}(\phi) \cup \{s\}) \\ &\quad\subset \mathcal{C}_{\psi(x^s)}^{\mathrm{nqr}(\psi(x^s))}(\mathrm{pr}(\phi) \cup \mathcal{C}_\phi^{\mathrm{nqr}(\phi)-1}(\mathrm{pr}(\phi))) \\ &\qquad\qquad \text{(by (7), (11), (12), and (6))} \\ &\quad= \mathcal{C}_{\psi(x^s)}^{\mathrm{nqr}(\psi(x^s))}(\mathcal{C}_\phi^{\mathrm{nqr}(\phi)-1}(\mathrm{pr}(\phi))) \qquad \text{(by (7))} \\ &\quad= \mathcal{C}_{\psi(x^s)}^{\mathrm{nqr}(\phi)-1}(\mathcal{C}_\phi^{\mathrm{nqr}(\phi)-1}(\mathrm{pr}(\phi))) \\ &\qquad\qquad \text{(because } \mathrm{nqr}(\psi(x^s)) = \mathrm{nqr}(\phi) - 1) \\ &\quad= \mathcal{C}_\phi^{\mathrm{nqr}(\phi)-1}(\mathcal{C}_\phi^{\mathrm{nqr}(\phi)-1}(\mathrm{pr}(\phi)) \\ &\quad= \mathcal{C}_\phi^{\mathrm{nqr}(\phi)}(\mathrm{pr}(\phi)) \qquad \text{(by (8))} \\ &\subset \mathrm{T} \qquad \text{(hypothesis of the lemma).}\end{aligned}$$

$\phi(x^s)$ thus also satisfies the hypothesis of the lemma.

Connected formulas have been introduced by Crabbé in [1], p. 14, under the name "elementary formulae", together with a weaker version of the following lemma:

Lemma 3 *Any $\mathcal{L}_G$-formula ϕ is equivalent in Γ to a Boolean combination ϕ_c of a set $\{\phi_0, \dots, \phi_n\}$ of $\mathcal{L}_G$-formulas such that, for every $i \leq n$,*

- $\mathrm{nqr}(\phi_i) \leq \mathrm{nqr}(\phi)$
- *ϕ_i is connected*
- *any variable occurring free in ϕ_i occurs free in ϕ*
- *any atomic subformula of ϕ_i is also a subformula of ϕ.*

Proof: The proof is easy; it can be found in [2].

Lemma 4 *If σ is an $\mathcal{L}_G$-sentence all of whose atomic subformulas are homogeneous, then σ is equivalent in Γ to an $\mathcal{L}_G$-sentence σ_c which is a Boolean combination of a set $\{\sigma_0, \dots, \sigma_n\}$ of $\mathcal{L}_G$-sentences such that, for every $i \leq n$,*

- $\mathrm{nqr}(\sigma_i) \leq \mathrm{nqr}(\sigma)$
- *σ_i is connected and homogeneous.*

Proof: This is just a corollary of Lemma 3 and of Property (9).

Lemma 5 *In Γ, every proper $\mathcal{L}_G$-formula ϕ is equivalent to an $\mathcal{L}$-formula ϕ_l such that every free variable of ϕ_l occurs free in ϕ.*

Proof: By Axiom $\Gamma 1$, ϕ_l is obtained by replacing in ϕ every atomic subformula which is not an $\mathcal{L}$-formula by a false $\mathcal{L}$-formula (without adding new variables).

4.6 Conservation of the elementary equivalence In this section, the proof of $\mathfrak{M} \equiv \mathfrak{N} \Rightarrow \mathfrak{M}^* \equiv^* \mathfrak{N}^*$ will be given. This will be Theorem 2, which is an easy consequence of the following Theorem 1:

Theorem 1 *If $\mathfrak{M}$ and $\mathfrak{N}$ are $\mathcal{L}$-structures, then*

$$\mathfrak{M} \equiv \mathfrak{N}$$
$$\Updownarrow$$
$$\mathfrak{M}_G \equiv_G \mathfrak{N}_G.$$

Proof: $\Uparrow$ Suppose that $\mathfrak{M}_G \equiv_G \mathfrak{N}_G$, i.e. for every $\mathcal{L}_G$-sentence σ, $\mathfrak{M}_G \models \sigma \Leftrightarrow \mathfrak{N}_G \models \sigma$. In particular, if σ is an $\mathcal{L}$-sentence, then, by (4), $\mathfrak{M} \models \sigma \Leftrightarrow \mathfrak{M}_G \models \sigma \Leftrightarrow \mathfrak{N}_G \models \sigma \Leftrightarrow \mathfrak{N} \models \sigma$. Hence, $\mathfrak{M} \equiv \mathfrak{N}$.

$\Downarrow$ Suppose that $\mathfrak{M} \equiv \mathfrak{N}$, and let σ be an $\mathcal{L}_G$-sentence. We are going to prove, by induction on $\mathrm{nqr}(\sigma)$, that $\mathfrak{M}_G \models \sigma \Leftrightarrow \mathfrak{N}_G \models \sigma$. As σ has no free variables, Lemma 2 tells us that σ is equivalent, in Γ, to σ_h, which is a sentence all of whose atomic subformulas are homogeneous, and which is such that $\mathrm{nqr}(\sigma_h) \leq \mathrm{nqr}(\sigma)$. Now, by Lemma 4, $\Gamma \vdash \sigma_h \Leftrightarrow \sigma_{hc}$, where σ_{hc} is a Boolean combination of the set $\{\sigma_0, \dots, \sigma_n\}$ of $\mathcal{L}_G$-sentences such that, for every $i \leq n$,

- $\mathrm{nqr}(\sigma_i) \leq \mathrm{nqr}(\sigma_h) \leq \mathrm{nqr}(\sigma)$
- σ_i is connected and homogeneous.

As $\mathfrak{M}_G \models \Gamma$ and $\mathfrak{N}_G \models \Gamma$, the problem is to prove that $\mathfrak{M}_G \models \sigma_{hc} \Leftrightarrow \mathfrak{N}_G \models \sigma_{hc}$. But, roughly speaking, $\models$ commutes with $\neg$ and $\vee$: for example, $(\mathfrak{M}_G \models \neg\psi) \Leftrightarrow \neg(\mathfrak{M}_G \models \psi)$. Thus, it suffices to show that $\mathfrak{M}_G \models \sigma_i \Leftrightarrow \mathfrak{N}_G \models \sigma_i$, for every σ_i. And as each σ_i is homogeneous, there are two cases:

a. either σ_i is proper, and then

$$\begin{aligned}
&\mathfrak{M}_G \models \sigma_i \\
&\Leftrightarrow \mathfrak{M}_G \models (\sigma_i)_l && \text{(by Lemma 5)} \\
&\Leftrightarrow \mathfrak{M} \models (\sigma_i)_l && \text{(by (4))} \\
&\Leftrightarrow \mathfrak{N} \models (\sigma_i)_l && \text{(hypothesis of the theorem)} \\
&\Leftrightarrow \mathfrak{N}_G \models (\sigma_i)_l \\
&\Leftrightarrow \mathfrak{N}_G \models \sigma_i.
\end{aligned}$$

b. or σ_i is nonproper; as σ_i is connected, then, by (10), σ_i is of the form $\exists x^S \phi(x^S)$ or $\neg \ldots \neg \exists x^S\ \phi(x^S)$. If σ_i is of the form $\exists x^S\ \phi(x^S)$, then

$$\mathfrak{M}_G \models \exists x^S\ \phi(x^S)$$

$\Leftrightarrow$ there exists $s \in \Sigma \backslash S$ such that $\mathfrak{M}_G \models \exists x^s\ \phi(x^s)$

$\Leftrightarrow$ there exists $s \in \Sigma \backslash S$ such that $\mathfrak{N}_G \models \exists x^s\ \phi(x^s)$

(by the induction hypothesis, because $\mathrm{nqr}(\exists x^s\ \phi(x^s)) < \mathrm{nqr}(\sigma_i) \leq \mathrm{nqr}(\sigma)$)

$\Leftrightarrow$ $\mathfrak{N}_G \models \exists x^S\ \phi(x^S)$.

And if σ_i is of the form $\neg \ldots \neg \exists x^S\ \phi(x^S)$, the proof is done in the same way.

Thus, $\mathfrak{M}_G \models \sigma \Leftrightarrow \mathfrak{M}_G \models \sigma_{hc} \Leftrightarrow \mathfrak{N}_G \models \sigma_{hc} \Leftrightarrow \mathfrak{N}_G \models \sigma$, and $\mathfrak{M}_G \equiv_G \mathfrak{N}_G$.

Theorem 2 will give the solution to our initial problem. It is an easy consequence of Theorem 1.

Theorem 2 *If $\mathfrak{M}$ and $\mathfrak{N}$ are $\mathfrak{L}$-structures, then $\mathfrak{M} \equiv \mathfrak{N} \Rightarrow \mathfrak{M}^* \equiv^* \mathfrak{N}^*$.*

Proof: Let σ be an $\mathfrak{L}^*$-sentence. Then,

$$\begin{aligned}
&\mathfrak{M}^* \models \sigma \\
&\Leftrightarrow \mathfrak{M}_G \models \sigma^{\varnothing} && \text{(by (5))} \\
&\Leftrightarrow \mathfrak{N}_G \models \sigma^{\varnothing} && \text{(by Theorem 1)} \\
&\Leftrightarrow \mathfrak{N}^* \models \sigma.
\end{aligned}$$

Hence, $\mathfrak{M}^* \equiv^* \mathfrak{N}^*$.

4.7 If Σ was finite As we saw above, if Σ was finite, then some of the $\mathfrak{M}^S$'s could be empty, and then there would be a problem to define $\mathfrak{M}_G$. Nevertheless, it remains true that $\mathfrak{M} \equiv \mathfrak{N} \Rightarrow \mathfrak{M}^* \equiv^* \mathfrak{N}^*$ and, furthermore, there is no constraint on $\mathfrak{L}$ or, if you prefer, the constraint imposed in Section 4.2 is always satisfied. Indeed, for any $\mathfrak{L}^*$-sentence σ^* there exists an $\mathfrak{L}$-sentence σ such that for any $\mathfrak{L}$-structure $\mathfrak{A}$, $\mathfrak{A} \models \sigma \Leftrightarrow \mathfrak{A}^* \models \sigma^*$.

σ is built in two phases. First, σ^* is transformed into $\bar{\sigma}$ by the following procedure:

- if ϕ is atomic, then $\bar{\phi}$ is ϕ
- $\overline{\phi \vee \psi}$ is $\bar{\phi} \vee \bar{\psi}$
- $\overline{\neg\phi}$ is $\neg\bar{\phi}$
- $\overline{\exists x \phi}$ is $\bigvee_{s \in \Sigma} \exists x^s \bar{\phi}(x^s/x)$.

Then σ is obtained by replacing in $\bar{\sigma}$ each atomic subformula which is not an $\mathfrak{L}$-formula by a false $\mathfrak{L}$-formula.

5 Generalization of Case 4 For the generalization of Case 4, $\mathcal{L}$ must also satisfy the constraint introduced in Section 4.2 (if $\mathcal{L}$ is infinite). $\mathcal{L}^*$ will be the one-sorted language having the same logical, relational, and functional symbols as $\mathcal{L}$, plus one unary predicate symbol S^s for each sort $s \in \Sigma$.

Let $\mathfrak{M}$ and $\mathfrak{N}$ be two $\mathcal{L}$-structures. A direct proof of $\mathfrak{M} \equiv \mathfrak{N} \Rightarrow \mathfrak{M}^* \equiv^* \mathfrak{N}^*$ can be obtained by slight modifications in proofs given for the generalization of Case 3. But on the other hand, it can also be seen as a pure consequence of Theorem 2. Indeed, let $\mathcal{L}^+$ be the following extension of $\mathcal{L}$: $\mathcal{L}^+$ is obtained by adding to $\mathcal{L}$ one unary predicate symbol S^s for every sort $s \in \Sigma$. Any $\mathcal{L}$-structure $\mathcal{A}$ can be easily extended to an $\mathcal{L}^+$-structure $\mathcal{A}^+$:

$$\mathcal{A}^+ = ((A^s)_{s\in\Sigma};\ (\mathcal{R}_r)_{r\in R}, (S^s_{\mathcal{A}})_{s\in\Sigma}, (\mathcal{F}_f)_{f\in F}),$$

where, for every $s \in \Sigma$, $S^s_{\mathcal{A}} = A^s$.

Now, atomic formulas of $\mathcal{L}^+$ can be defined to be the atomic $\mathcal{L}$-formulas, plus the formulas of the form $S^s(x^s)$. Thus, $\mathcal{L}^+$ also satisfies the constraint imposed in Section 4.2, and Theorem 2 can be used to prove that

$$\mathfrak{M}^+ \equiv^+ \mathfrak{N}^+ \Rightarrow \mathfrak{M}^* \equiv^* \mathfrak{N}^*.$$

And then it is easy to show that

$$\mathfrak{M} \equiv \mathfrak{N} \Leftrightarrow \mathfrak{M}^+ \equiv^+ \mathfrak{N}^+ \Rightarrow \mathfrak{M}^* \equiv^* \mathfrak{N}^*.$$

On the other hand, it is also easy to show that

$$\mathfrak{M}^* \equiv \mathfrak{N}^* \Rightarrow \mathfrak{M} \equiv \mathfrak{N}.$$

6 Elementary substructures In every result we have shown, the $\equiv$ relation could have been replaced by the $<$ relation (even in the counterexamples). In particular, in the generalization of Case 3, Lemmas 2, 3, and 5 were about formulas, i.e. not specially about sentences. It is thus easy to rewrite Theorems 1 and 2 for the $<$ relation. This will be left as an exercise. (Hint to rewrite Theorem 1: When showing that for any formula ϕ, and for any $m_1, \ldots, m_n \in M$, $\mathfrak{M}^* \models \phi[m_1, \ldots, m_n\} \Leftrightarrow \mathfrak{N}^* \models \phi[m_1, \ldots, m_n]$, you can suppose that free variables of ϕ are proper; then the new version of Lemma 2 can be applied.)

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