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Nonstandard Connectives of Intuitionistic Propositional Logic

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1 Introduction In [7] McCullough examined the question of adding new connectives to the intuitionistic propositional logic. He showed that the connectives called *regular* and defined in terms of Kripke models were expressible in the usual intuitionistic propositional calculus.

Three *new* propositional connectives $\not\subset$ (*converse nonimplication*), $|$ (*not both*), and $\downarrow$ (*neither-nor*) were introduced by Bowen in [1]. He described them by their introduction rules in a Gentzen system. Because of the importance of his approach for what follows (cf. Section 4) we consider one of the above connectives, say $|$:

$|$ has the following introduction rules:

$$|{-}IA \ \frac{\Gamma \to A,\ \Gamma \to B}{A|B, \Gamma \to}$$

and

$$|{-}IS \ \frac{A, \Gamma \to}{\Gamma \to A|B} \qquad \frac{B, \Gamma \to}{\Gamma \to A|B}.$$

Let $LJ_|$ denote the propositional portion of Gentzen's LJ (cf. [4], Section III) added with $|$ and the above introduction rules. Using an external symmetry of $|{-}IA$ and $|{-}IS$ Bowen proved that the Cut-rule is redundant in $LJ_|$. However $LJ_|$ suffers a serious lack: a sequent

$$A_1 \equiv A_2,\ B_1 \equiv B_2,\ A_1|B_1 \to A_2|B_2$$

that must be valid in *any* reasonable semantics is not provable in $LJ_|$! ($A \equiv B$ denotes $(A \supset B) \wedge (B \supset A)$). Moreover, it is not hard to show that the following natural rule of inference does not hold in $LJ_|$:

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$$\frac{\rightarrow A_1 \equiv A_2, \quad \rightarrow B_1 \equiv B_2}{\rightarrow A_1 | B_1 \equiv A_2 | B_2}.$$

$\not\subset$ and $\downarrow$ have the same lack.

In Section 4 we show how to avoid this problem by a marginal modification of *LJ*. Of course, this modification decreases the number of new connectives which can be defined by introduction rules in a Gentzen system.

The above consideration naturally suggests the following definition.

Definition 1 Let **I** be the first-order intuitionistic propositional calculus in the language with $\neg$, $\wedge$, $\vee$, and $\supset$. Consider a symbol # for a new n-ary connectives. Let $\mathbf{I}^{\#}$ be the logical system in the language of **I** added with # obtained by taking all the axioms and rules of inference of **I** together with the axioms and rules of inference for #. # is said to be *extensional* if

$$\mathbf{I}^{\#} \vdash A_1 \equiv B_1 \wedge \ldots \wedge A_n \equiv B_n \;.\!\supset\!.\; \#(A_1,\ldots,A_n) \equiv \#(B_1,\ldots,B_n).$$

is said to be *weakly extensional* if $\mathbf{I}^{\#} \vdash A_1 \equiv B_1 \wedge \ldots \wedge A_n \equiv B_n$ implies $\mathbf{I}^{\#} \vdash \#(A_1,\ldots,A_n) \equiv \#(B_1,\ldots,B_n)$.

Extensionality seems to be a natural necessary condition for accepting a new connective as an intuitionistic connective. In Sections 3 and 4 we give sufficient semantical and syntactical conditions for a connective to be (weakly) extensional.

Some other conditions for accepting a new connective # as an intuitionistic connective were suggested by Gabbay in [2]:

(1) $\mathbf{I}^{\#}$ is a conservative extension of **I**.
(2) # is not expressible by the ordinary connectives, i.e., for all formulas $\psi(p_1,\ldots,p_n)$ of **I** such that the propositional variables of ψ are $p_1,\ldots,p_n$, $\psi(A_1,\ldots,A_n) \equiv \#(A_1,\ldots,A_n)$ is not provable in $\mathbf{I}^{\#}$. ($\psi(A_1,\ldots,A_n)$ denotes the result of the substitution of A_i for p_i, $i = 1,\ldots,n$, in $\psi(p_1,\ldots,p_n)$).
(3) $\mathbf{I}^{\#}$ has the disjunction property.
(4) # is *nonclassical* in $\mathbf{I}^{\#}$, i.e., if we add to $\mathbf{I}^{\#}$ the axiom schema $\neg\neg\phi \supset \phi$, then # becomes expressible by $\neg$, $\wedge$, $\vee$, and $\supset$.
(5) The axioms and rules of inference for # determine "the meaning" of # uniquely, i.e., if we look at a language with two additional connectives $\#_1$ and $\#_2$ which have the "same" axioms and rules of inference as #, and define the system $\mathbf{I}^*$ with the axioms and rules of inference for $\#_1$ and $\#_2$ in addition to the axioms and rules of inference of **I**, then $\mathbf{I}^* \vdash \#_1(A_1,\ldots,A_n) \equiv \#_2(A_1,\ldots,A_n)$.
(6) The connective # is expressible in the second-order intuitionistic propositional calculus.

In the same paper he introduced a new connective that satisfies the Conditions (1)-(6) above.

Conditions (2) and (4) are exactly what we want from a new *intuitionistic* connective and require no motivation.

Condition (3) is a natural property of an intuitionistic system, e.g., **I** has the disjunction property.

In his book ([3]) Gabbay motivates Conditions (1) and (5) by the following example: The intuitionistic propositional calculus **I** is a conservative extension of the calculus $\mathbf{I}^-$ obtained by taking out connective $\vee$ from **I**. Also $\vee$ is uniquely determined in **I** by its axiom schemas:

$$\phi \supset \psi \vee \phi,\ \psi \supset \phi \vee \psi$$
$$(\phi \supset \rho) \wedge (\psi \supset \rho) .\supset. \phi \vee \psi \supset \rho.$$

Whereas Condition (1) is quite natural and can be accepted without the above motivation, Condition (5) seems to be too strong: modal connectives (cf. [3], [8], [9], and [10]) are not uniquely determined by their axioms and rules of inference. Of course, one can argue that modal connectives are not new *intuitionistic* connectives, since they violate Condition (4), but modal connectives are not uniquely determined by their axioms and rules of inference in the classic propositional calculus either. However modal connectives are considered as nonstandard connectives of both intuitionistic and classic logics.

Condition (6) seems to be at least arguable: why a *new* intuitionistic connective has to be expressible in the second-order intuitionistic propositional calculus? And indeed, this condition is omitted in [3].

Also, it seems to be desirable for a new intuitionistic connective to be extensional (cf. Definition 1 above) and interpretable in Kripke models, and for the extended logic to be decidable and to have a Gentzen system with Cut-free deductions which have the subformula property.

Whereas the question: "What is an intuitionistic propositional connective?" is very general and not precise, one can still attempt to answer the question of accepting a *given* connective as a new intuitionistic connective. And this is what we are going to do in this paper.

The paper is organized as follows: In Section 2 we introduce a new propositional connective that satisfies Gabbay's Conditions (1)–(4) above and is *not expressible* in the second-order intuitionistic propositional calculus. This connective is extensional and interpretable in Kripke models. The extended logic has a Gentzen system with the subformula property and Cut-free deductions. (It is easy to see that the logic extended with the connective introduced by Gabbay (cf. [2]) cannot have a Gentzen system with the subformula property.)

In Section 3 we study a possibility of introducing new connectives by their interpretation in Kripke models. We prove that for any n-ary connective # there exists a formula $\psi(A_1,\ldots,A_n)$ with the ordinary connectives only, such that $\mathbf{I} \vdash \psi(A_1,\ldots,A_n)$ implies $\mathbf{I}^{\#} \vdash \#(A_1,\ldots,A_n)$, and $\mathbf{I}^{\#} \vdash \#(A_1,\ldots,A_n)$ implies $\mathbf{I} \vdash \neg\neg\psi(A_1,\ldots,A_n)$. Moreover, we show that if certain conditions on the interpretation of # are satisfied, then # satisfies Gabbay's Condition (4) above.

In Section 4 we consider some extensions $\mathbf{I}^{\#}$ of **I** such that # can be described by rules of inference in a Gentzen system. Under natural and not strong restrictions on rules of inference we prove that $\mathbf{I}^{\#}$ can be interpreted either in $\mathbf{I}^{\circ}$ (where $\circ$ is a new connective introduced in Section 2) or in propositional intuitionistic modal logic **S4** (cf. [3], [8], and [10]).

2 A new connective

Let $\mathbf{I}^{\circ}$ be the logical system in the language of **I** extended with a new unary connective $\circ$ obtained by taking all the axioms and rules of inference of **I** together with the following axiom schemas:

A1 $\phi \supset \circ\phi$
A2 $\circ\phi \supset \neg\neg\phi$
A3 $\circ\circ\phi \supset \circ\phi$
A4 $\circ(\phi \supset \psi) \supset (\circ\phi \supset \circ\psi)$.

Remark 1: It is easy to verify that the deduction theorem holds for $\mathbf{I}^\circ$.

Theorem 1 *Let p be an atomic formula. There is no formula A in the language of* $\mathbf{I}$ *such that* $\mathbf{I}^\circ \vdash \circ p \equiv A$.

Proof: Let A be a formula in the language of $\mathbf{I}^\circ$. By induction on the number of connectives in A we define two translations, A^+ and A^-, of A into the formulas of $\mathbf{I}$ as follows:

1. If A is an atomic formula, then $A^\pm = A$.
2. $(A_1 \otimes A_2)^\pm = A_1^\pm \otimes A_2^\pm$, where $\otimes$ is $\wedge$, $\vee$, or $\supset$; $(\neg A)^\pm = \neg A^\pm$.
3. $(\circ A)^+ = A^+$, and $(\circ A)^- = \neg\neg A^-$.

Obviously, if $\mathbf{I}^\circ \vdash A$, then $\mathbf{I} \vdash A^\pm$, since both of the translations $-$ and $+$ of schemas A1–A4 are provable in $\mathbf{I}$. Consequently, if for some formula A of $\mathbf{I}$, $\mathbf{I}^\circ \vdash \circ p \equiv A$, then $\mathbf{I} \vdash \neg\neg p \equiv A$ and $\mathbf{I} \vdash p \equiv A$, since $(A)^\pm$ is A. Hence $\mathbf{I} \vdash p \equiv \neg\neg p$, but this is impossible.

Remark 2: Exactly the same argument holds for the second-order intuitionistic propositional calculus $\mathbf{I}_2$. However we don't know whether $\mathbf{I}^\circ$ is *interpretable* in $\mathbf{I}_2$. By an interpretation of $\mathbf{I}^\circ$ in $\mathbf{I}_2$ we mean the following:

For a formula $\psi(p)$ in the language of $\mathbf{I}_2$ with only one free variable p define a *translation* t_ψ of formulas of $\mathbf{I}^\circ$ into formulas of $\mathbf{I}_2$ by:

1. If A is an atomic formula (i.e., A is a propositional variable), then $t_\psi(A) = A$.
2. $t_\psi(A_1 \otimes A_2) = t_\psi(A_1) \otimes t_\psi(A_2)$, where $\otimes$ is $\vee$, $\wedge$, or $\supset$; $t_\psi(\neg A) = \neg t_\psi(A)$.
3. $t_\psi(\circ A) = \psi(t_\psi(A))$.

We say that $\mathbf{I}^\circ$ is *interpretable* in $\mathbf{I}_2$ if there exists a translation t_ψ such that for all formulas A of $\mathbf{I}^\circ$,$\mathbf{I}^\circ \vdash A$ iff $\mathbf{I}_2 \vdash t_\psi(A)$.

Let $\mathbf{G}^\circ$ denote the Gentzen system in the language of $\mathbf{I}^\circ$ obtained from the propositional portion of Gentzen's *LJ* (cf. [4], Section III) by extending the axiom schema to all the formulas of $\mathbf{I}^\circ$ and adding the following introduction rules for $\circ$:

$$\circ\!-\!IA \ \frac{A,\Gamma\rightarrow}{\circ A,\Gamma\rightarrow} \qquad \frac{A,\Gamma\rightarrow \circ B}{\circ A,\Gamma\rightarrow \circ B}$$

$$\circ\!-\!IS \ \frac{\Gamma\rightarrow A}{\Gamma\rightarrow \circ A}.$$

Theorem 2 $\mathbf{G}^\circ \vdash \Gamma \rightarrow A$ *iff* $\Gamma \vdash_{\mathbf{I}^\circ} A$.

The proof is similar to the proof of the corresponding theorem in [4], Section V, and is omitted.

In Section 4 (Theorem 11) we prove that the Cut-rule is superfluous in $\mathbf{G}^\circ$. This immediately gives an alternative proof of Theorem 1 and easily implies the following corollaries:

Corollary 1 $\mathbf{I}^\circ$ *is consistent.*
Corollary 2 $\mathbf{I}^\circ$ *is decidable.*
Corollary 3 $\mathbf{I}^\circ$ *is a conservative extension of* $\mathbf{I}$.
Corollary 4 $\mathbf{I}^\circ$ *has the disjunction property.*
Corollary 5 *If* $\mathbf{I}^\circ \vdash \circ\phi$, *then* $\mathbf{I}^\circ \vdash \phi$.

Now we move on to the interpretation of $\circ$ in Kripke models.

Definition 2 Let S be a partially ordered set, $x \subseteq S$. x is said to be a *path* in S if x is a maximal linearly ordered subset of S. The set of all paths of S is denoted by πS:

$$\pi S = \{x | x \text{ is a path in } S\}.$$

Definition 3 A *Kripke model* for $\mathbf{I}^\circ$ is a quadruple $\mathbf{M} = \langle S, \leq, s, V \rangle$, where S is a nonempty set, $\leq$ is a partial order on S, s is the least element of S: $s \leq t$, for any $t \in S$, and V is a function from S into sets of propositional variables such that $V(t) \subseteq V(t')$ if $t \leq t'$.

Let $t \in S$. We can define a relation $t \models A$ for an arbitrary formula A in the language of $\mathbf{I}^\circ$ by induction on the number of connectives of A:

(a) If A has no connectives, then it is a propositional variable, and $t \models A$ iff $A \in V(t)$.
(b) $t \models A \wedge B$ iff $t \models A$ and $t \models B$.
(c) $t \models A \vee B$ iff $t \models A$ or $t \models B$.
(d) $t \models A \supset B$ iff for all $t' \in S$ such that $t' \geq t$, $t' \not\models A$ or $t' \models B$.
(e) $t \models \neg A$ iff for all $t' \in S$ such that $t' \geq t$, $t' \not\models A$.
(f) $t \models \circ A$ iff for all $x \in \pi S$ such that $t \in x$, there exists a $t' \in x$ such that $t' \models A$.

A formula A in the language $\mathbf{I}^\circ$ is said to be *valid* in $\mathbf{M}$ (denoted by $\mathbf{M} \models A$) if for all $t \in S$, $t \models A$.

Proposition 1 (a) *If* $t \models A$, *and* $t' \geq t$, *then* $t' \models A$. (b) $< S, \leq, s, V\rangle \models A$ *iff* $s \models A$.

Proof: (a) Induction on the number of connectives of A using the fact that $V(t) \subseteq V(t')$ for $t \leq t'$. (b) follows from (a).

An intuitive interpretation for Kripke models can be found in [6]. We use this interpretation in Section 3 to motivate some conditions imposed on new connectives.

Next we prove consistency and completeness of $\mathbf{I}^\circ$ with respect to the above semantics.

Theorem 3 (Consistency) *If* $\Gamma \vdash A$, *then for all Kripke models* $\mathbf{M}$ *such that* $\mathbf{M} \models B$ *for all* $B \in \Gamma$, $\mathbf{M} \models A$.

Proof: It is easy to verify that all axioms of $\mathbf{I}^\circ$ are valid in any Kripke model. A straightforward induction on the length of a deduction of A from Γ completes the proof.

It is more convenient to state the contraposition of completeness theorem:

Theorem 4 (Completeness) *If* $\Gamma \nvdash A$, *then there exists a Kripke model* $\mathbf{M}$ *such that* $\mathbf{M} \vDash B$ *for all* $B \in \Gamma$ *and* $\mathbf{M} \nvDash A$.

For the proof we need some preliminaries.

Definition 4 Let Γ be a set of formulas in the language of $\mathbf{I}^\circ$. Γ is said to be *saturated* if it satisfies the conditions below:

(a) Γ is consistent
(b) Γ is deductively closed, i.e., if $\Gamma \vdash A$, then $A \in \Gamma$
(c) if $A \vee B \in \Gamma$, then either $A \in \Gamma$, or $B \in \Gamma$.

Lemma 1 *If* $\Gamma \nvdash A$, *then there exists a saturated set* Δ *such that* $\Gamma \subseteq \Delta$, *and* $\Delta \nvdash A$.

Proof: We claim that any maximal deductively closed set Δ satisfying $\Gamma \subseteq \Delta$, and $A \notin \Delta$ is saturated. Suppose, by contradiction, that there exist B_1 and B_2 such that $B_1 \vee B_2 \in \Delta$, $B_1 \notin \Delta$, $B_2 \notin \Delta$. Since Δ is maximal, $\Delta \cup \{B_1\} \vdash A$, and $\Delta \cup \{B_2\} \vdash A$. But then $\Delta \cup \{B_1 \vee B_2\} \vdash A$. Since $B_1 \vee B_2 \in \Delta$, $\Delta \vdash A$, but this is impossible. The existence of such maximal deductively closed sets is obvious.

Lemma 2 *If* Γ *is saturated, then*:
(1) $A \wedge B \in \Gamma$ *iff* $A \in \Gamma$, *and* $B \in \Gamma$
(2) *If* $\circ A \notin \Gamma$, *then* $A \notin \Gamma$
(3) *If* $A \supset B \notin \Gamma$, *then there exists a saturated set* Δ *such that* $\Gamma \cup \{A\} \subseteq \Delta$, *and* $B \notin \Delta$.

Proof: (1) and (2) are obvious. To prove (3) notice that $\Gamma \cup \{A\} \nvdash B$ and use Lemma 1.

Definition 5 Let Γ be a set of formulas in the language of $\mathbf{I}^\circ$. The *closure* of Γ (is denoted by $[\Gamma]$ and) is defined by

$$[\Gamma] = \Gamma \cup \{A \mid \circ A \in \Gamma\}.$$

Lemma 3 $\Gamma \vdash \circ A$ *iff* $[\Gamma] \vdash \circ A$.

Proof: Since $\Gamma \subseteq [\Gamma]$, part "only if" is obvious. Let $[\Gamma] \vdash \circ A$. By Theorem 2 $\mathbf{G}^\circ \vdash A_1, \ldots, A_n, B_1, \ldots, B_k \rightarrow \circ A$, where $\circ A_i \in \Gamma$, $i = 1, \ldots, n$, and $B_i \in \Gamma$, $i = 1, \ldots, k$. Applying rule $\circ$—AI n times results $\mathbf{G}^\circ \vdash \circ A_1, \ldots, \circ A_n, B_1, \ldots, B_k \rightarrow \circ A$. By Theorem 2, $\Gamma \vdash \circ A$.

Proof of Theorem 4: If Γ is inconsistent, there is nothing to prove. Otherwise, in view of Lemma 1, we may assume that Γ is saturated. Define a Kripke model $\mathbf{M} = \langle S, \leq, s, V \rangle$ by

$S = \{(\Delta, 0, m) \mid \Gamma \subseteq \Delta$ is saturated, $m = 0, 1, \ldots\} \cup \{(\Delta, n, m) \mid [\Gamma] \subseteq \Delta$ is saturated, $n = 1, 2, \ldots,$ $m = 0, 1, \ldots,\}$.

$(\Delta, n, m) \leq (\Delta', n', m')$ iff $(\Delta, n, m) = (\Delta', n', m')$; or $n < n'$, and $[\Delta] \subseteq \Delta'$; or $n = n'$, $m \leq m'$, and $\Delta \subseteq \Delta'$.

$s = (\Gamma,0,0)$
$V((\Delta,m,n)) = \{p \in \Delta \mid p \text{ is a propositional variable}\}$.

Obviously, $\leq$ is a partial order on S; s is the least element of S; and if $(\Delta,n,m) \leq (\Delta',n',m')$, then $V((\Delta,n,m)) \subseteq V((\Delta',n',m'))$. This shows that **M** is well defined.

Let $(\Delta,n,m) \in S$, and let C be an arbitrary formula. We claim that $(\Delta,n,m) \models C$ iff $C \in \Delta$. Since $A \notin \Gamma$, and $\Gamma \subseteq \Delta$ for all $(\Delta,n,m) \in S$, the claim together with Proposition 1 implies Theorem 4.

We prove the claim by induction on the number of connectives of C: For an atomic formula C (i.e., C is a propositional variable) the claim holds by definition. The cases of $\wedge$ and $\vee$ present no difficulties. The case of $\neg$ follows from the case of $\supset$.

Case of $\supset$. Assume that $C_1 \supset C_2 \in \Delta$. If for some $(\Delta',n',m') \geq (\Delta,n,m)$, $(\Delta',n',m') \models C_1$, then, by the inductive hypothesis, $C_1 \in \Delta'$. Since Δ' is deductively closed and $C_1 \supset C_2 \in \Delta \subseteq \Delta'$, $C_2 \in \Delta'$. Applying the inductive hypothesis once more we obtain that $(\Delta',n',m') \models C_2$. Hence $(\Delta,n,m) \models C_1 \supset C_2$.

Assume that $C_1 \supset C_2 \notin \Delta$. By (3) of Lemma 2 there exists a saturated set Δ' such that $\Delta \cup \{C_1\} \subseteq \Delta'$, and $C_2 \notin \Delta'$. Consider $(\Delta',n,m+1)$. By the inductive hypothesis $(\Delta',n,m+1) \models C_1$ and $(\Delta',n,m+1) \not\models C_2$. Since $(\Delta,n,m) \leq (\Delta',n,m+1)$, $(\Delta,n,m) \not\models C_1 \supset C_2$.

Case of $\circ$. Assume that $\circ C_1 \in \Delta$. Let x be a path in S containing (Δ,n,m). First, we contend that there exists $(\Delta',n',m') \in x$, with $n' > n$. Suppose, by contradiction, that for all $(\Delta',n',m') \in x$, $n' \leq n$. Since the first components of the elements of x are linearly ordered by $\subseteq$, their union (call it Θ) is consistent. Let Θ' be a saturated set such that $[\Theta] \subseteq \Theta'$. $(\Theta',n+1,0)$ exceeds all the elements of x, but does not belong to x. This contradicts the maximality of x and proves our contention.

Now, let (Δ',n',m') be an element of x such that $n' > m$. By definition of $\geq$, $[\Delta] \subseteq \Delta'$. Hence $C_1 \in \Delta'$. By the inductive hypothesis $(\Delta',n',m') \models C_1$. Consequently, $(\Delta,n,m) \models \circ C_1$.

Assume that $\circ C_1 \notin \Delta$. We contend that it is sufficient to prove that there exists a path x in S such that $(\Delta,n,m) \in x$, and for all $(\Delta',n',m') \in x$, $\circ C_1 \notin \Delta'$. Indeed, since $\circ C_1 \notin \Delta'$ and Δ' is deductively closed, $C \notin \Delta'$. By the inductive hypothesis $(\Delta',n',m') \not\models C_1$. Hence $(\Delta,n,m) \not\models \circ C_1$, which proves our contention.

To prove that such a path x exists we define a sequence $\Delta_0,\Delta_1,\ldots$ of saturated sets not containing $\circ C_1$ such that $[\Delta_{i-1}] \subseteq \Delta_i$, $i = 1,2,\ldots$.

Let $\Delta_0 = \Delta$.

Assume that Δ_i has been defined. By Lemma 3, $[\Delta_i] \nvdash \circ C_1$. Using Lemma 1 we can extend $[\Delta_i]$ to a saturated set Δ_{i+1} not containing $\circ C_1$. Obviously, the set

$$y = \{(\Delta,n,m') \mid m' \geq m\} \cup \{(\Delta_i,n+i,m') \mid i = 1,2,\ldots,m' = 0,1,2,\ldots\}$$

is a maximal linearly ordered subset of S for which (Δ,n,m) is the least element. Let x be any path in S such that $y \subseteq x$. By the above construction, for all $(\Delta',n',m') \in x$, $\circ C_1 \notin \Delta'$. This completes the proof of Theorem 4.

We conclude this section with the following problem: Is it possible to "embed" $\mathbf{I}^\circ$ into intuitionistic arithmetic (cf. [5], [11], and [12])?

3 Semantical analysis of new connectives In this section we consider two broad classes of potential intuitionistic propositional connectives and establish some of their properties. To proceed we need the following definitions and notations.

Definition 6 Let $\mathbf{M} = \langle S,\leq,s, V\rangle$ be a Kripke model, and let $t \in S$. Define two subsets S_t and S^t of S by

$$S_t = \{u \in S | u \geq t\} \text{ and } S^t = \{u \in S | u = t, \text{ or } u \ngeq t\}.$$

Define two Kripke models $\mathbf{M}_t$ and $\mathbf{M}^t$ by $\mathbf{M}_t = \langle S_t,\leq/S_t,t, V/S_t\rangle$ and $\mathbf{M}^t = \langle S^t,\leq/S^t,s, V/S^t\rangle$, where $/S_t(S^t)$ denotes the restriction of a function (relation) to $S_t(S^t)$.

t is said to be a *maximal* element of $\mathbf{M}$ iff for all $u \geq t$, $V(u) = V(t)$.

Definition 7 Let $\mathbf{I}^\#$ denote an extension of $\mathbf{I}$ with a new n-ary propositional connective #, its axioms and rules of inference. An interpretation of # in Kripke models for which $\mathbf{I}^\#$ is complete and consistent is said to be *weakly invariant* if it satisfies the following conditions:

(1) For all Kripke models $\langle S,\leq,s, V\rangle$, for all formulas $A_1,\ldots,A_n$ of $\mathbf{I}^\#$ for all $t,t' \in S$, such that $t \leq t'$, if $t \vDash \#(A_1,\ldots,A_n)$, then $t' \vDash \#(A_1,\ldots,A_n)$.

(2) If t is a maximal element of a Kripke model $\mathbf{M}$, then for all formulas $A_1,\ldots,A_n$ of $\mathbf{I}^\#$,$\mathbf{M} \vDash \#(A_1,\ldots,A_n)$ iff $\mathbf{M}^t \vDash \#(A_1,\ldots,A_n)$.

(3) Let $\mathbf{M} = \langle S,\leq,s, V\rangle$ and $\mathbf{N} = \langle S,\leq,s, U\rangle$ be Kripke models. Let $A_1,\ldots,A_n$ and $B_1,\ldots,B_n$ be formulas of $\mathbf{I}^\#$ such that for all $t \in S$, $t \vDash_{\mathbf{M}} A$ iff $t \vDash_{\mathbf{N}} B_i$, $i = 1,\ldots,n$. Then for all $t \in S$, $t \vDash_{\mathbf{M}} \#(A_1,\ldots,A_n)$ iff $t \vDash_{\mathbf{N}} \#(B_1,\ldots,B_n)$. (We use $\vDash_{\mathbf{M}}$ and $\vDash_{\mathbf{N}}$ to refer to $\mathbf{M}$ and $\mathbf{N}$ respectively.)

Weakly invariant interpretations which satisfy Condition (4) below are called *invariant*.

(4) If $\mathbf{M}_t$ and $\mathbf{N}_u$ are isomorphic Kripke models, then for all formulas $A_1,\ldots,A_n$ of $\mathbf{I}^\#$

$$t \vDash_{\mathbf{M}} \#(A_1,\ldots,A_n) \text{ iff } u \vDash_{\mathbf{N}} \#(A_1,\ldots,A_n).$$

(Kripke models $\mathbf{M}_1 = \langle S_1\leq_1,s_1, V_1\rangle$ and $\mathbf{M}_2 = \langle S_2,\leq_2,s_2, V_2\rangle$ are said to be *isomorphic* iff there exists an order preserving bijection f from S_1 to S_2 such that for all $t \in S$, $V_1(t) = V_2(f(t))$.)

We call a connective (*weakly*) *invariant* if it has a (weakly) invariant interpretation.

Connectives $\neg$, $\wedge$, $\vee$, $\supset$, $\circ$ and # from [2] are invariant, modal connectives (cf. [3], [8], [9], and [10]) are weakly invariant, but not invariant.

Our intuitive explanation of Definition 7 is based on that of [6] and is as follows:

Let $\mathbf{M} = \langle S,\leq,s, V\rangle$ be a Kripke model. We can consider S as a collection of points in time (which is not necessarily linearly ordered), and $t_1 \leq t_2$ means that time t_2 comes after time t_1. At any point in time we may have various pieces of information which are, roughly speaking, given by function V. If, at a particular point t in time, we have enough information to prove a formula A, we say that $t \models A$; if we lack such information we say that $t \not\models A$. However, $t \not\models A$ does not mean that A has been proved false at t. It simply is not (yet) proved at t, but may be established at a time $t' \geq t$.

Condition (1) of Definition 7 means that if we already have a proof of a formula A at time t, then we can accept A as proved at any later time t', i.e., we don't forget.

Condition (2) of Definition 7 reflects the notion that if the information acquisition stops at time t (i.e., t is a maximal element of $\mathbf{M}$), we can prove nothing new.

Condition (3) of Definition 7 is rather technical: it demands that the verification of $\#(A_1,\ldots,A_n)$ depends only on the verification of $A_1,\ldots,A_n$ (and can be thought as the "subformula property" for $\mathbf{I}^{\#}$).

Condition (4) of Definition 7 means that the validity of a formula at time t depends on its validity at the points of time comparable with t.

Remark 3. In view of the above intuitive interpretation of Kripke models $\circ A$ can be thought as "eventually A will be true".

It seems to be reasonable and desirable for a new connective to be (weakly) invariant. Some properties of such connectives are given by Theorems 5–7 below. For the sake of continuity we first state these theorems and prove them afterwards.

Theorem 5 *If a connective is (weakly) invariant then it is (weakly) extensional (cf. Definition 1).*

Theorem 6
(a) *If connective # is weakly invariant, then there exists a formula $\psi(p_1,\ldots,p_n)$ in the language of* $\mathbf{I}$ *such that the propositional variables of ψ are $p_1,\ldots,p_n$ and*

(1) *if* $\mathbf{I} \vdash \psi(A_1,\ldots,A_n)$, *then* $\mathbf{I}^{\#} \vdash \#(A_1,\ldots,A_n)$, *and*

(2) *if* $\mathbf{I}^{\#} \vdash \#(A_1,\ldots,A_n)$, *then* $\mathbf{I} \vdash \neg\neg\psi(A_1,\ldots,A_n)$.

(b) *If connective # is invariant, then there exists a formula $\psi(p_1,\ldots,p_n)$ in the language of* $\mathbf{I}$ *such that*

(1) $\mathbf{I}^{\#} \vdash \psi(A_1,\ldots,A_n) \supset \#(A_1,\ldots,A_n)$, *and*

(2) $\mathbf{I}^{\#} \vdash \#(A_1,\ldots,A_n) \supset \neg\neg\psi(A_1,\ldots,A_n)$.

(i.e., invariant connectives satisfy Gabbay's "nonclassical" Condition (4); cf. Introduction.)

The following theorem gives a syntactical condition for a connective to be invariant. (It can be considered as a "converse" of Theorem 6.)

Theorem 7 *Let $\mathbf{I}^{\#}$ denote an extension of* $\mathbf{I}$ *with a new n-ary propositional connective #, its axioms and rules of inference. Let # have an interpretation in Kripke models satisfying Conditions (1), (3), and (4) of Definition 7 for which $\mathbf{I}^{\#}$ is complete and consistent. If there exists a formula $\psi(p_1,\ldots,p_n)$ in the language of* $\mathbf{I}$ *such that $\mathbf{I}^{\#} \vdash \psi(A_1,\ldots,A_n) \supset \#(A_1,\ldots,A_n)$ and $\mathbf{I}^{\#} \vdash \#(A_1,\ldots,A_n) \supset \neg\neg\psi(A_1,\ldots,A_n)$, then # satisfies Condition (2) of Definition 7.*

Proof of Theorem 5: Let $\mathbf{I}^{\#} \vdash A_i \equiv B_i$, $i = 1,\ldots,n$. Since $\mathbf{I}^{\#}$ is complete and consistent for Kripke models, it suffices to prove that $\#(A_1,\ldots,A_n) \equiv \#(B_1,\ldots,B_n)$ is valid in all Kripke models. Let $\mathbf{M} = \langle S,\leq,s,V\rangle$ be a Kripke model and for some $t \in S$, $t \models \#(A_1,\ldots,A_n)$. Since $\mathbf{I}^\# \vdash A_i \equiv B_i$, $i = 1,\ldots,n$, for all $\mathbf{u} \in S$, $u \models A_i$ iff $u \models B_i$, $i = 1,\ldots,n$. Therefore by Condition (3) of Definition 7 (letting $U = V$) we conclude that $t \models \#(B_1,\ldots,B_n)$. Since A_i and B_i, $i = 1,\ldots n$, appear symmetrically in the statement of Theorem 5, this completes the proof of the case of weakly invariant connectives.

If # is invariant, it suffices to prove that for all Kripke models $\mathbf{M} = \langle S,\leq,s,V\rangle$ and for all $t \in S$ if $t \models A_1 \equiv B_1 \wedge \ldots \wedge A_n \equiv B_n \wedge \#(A_1,\ldots,A_n)$, then $t \models \#(B_1,\ldots,B_n)$. In view of Condition (4) of Definition 7 and Proposition 1 we may assume that t is the least element of S. Now, by Proposition 1 for all formulas C, $\mathbf{M} \models C$ iff $t \models C$, and we can proceed as in the case of a weakly invariant connective.

To prove Theorem 6 we need some preliminaries.

Definition 8 Let $\{p_i | i = 1,2,\ldots\}$ be a fixed set of propositional variables. A Kripke model $\mathbf{M} = \langle S,\leq,s,V\rangle$ is said to belong to class $\mathbf{K}_m$ if for all $t \in S$, $V(t) \subseteq \{p_1,p_2,\ldots,p_m\}$.

Lemma 4 *Let $\mathbf{M} = \langle S,\leq,s,V\rangle$ be a Kripke model. If for some m, $\mathbf{M} \in \mathbf{K}_m$, then for all $t \in S$ there exists a maximal element u of $\mathbf{M}$ such that $u \geq t$.*

Proof: We prove the lemma by induction on m:

Basis. $m = 1$. If there exists a $u \geq t$ such that $V(u) = p_1$, then this u is maximal, otherwise t is maximal.

Inductive step. Let $\mathbf{M} = \langle S,\leq,s,V\rangle \in \mathbf{K}_{m+1}$. Define a Kripke model $\mathbf{N} = \langle S,\leq,s,U\rangle$ by $u(t) = V(t) - \{p_{n+1}\}$, for $t \in S$. Obviously, $\mathbf{N} \in \mathbf{K}_m$. By the inductive hypothesis, there exists a maximal element (of $\mathbf{N}$) $u \geq t$. If there exists a $v \geq u$ such that $p_{n+1} \in V(v)$, then this v is a maximal element of $\mathbf{M}$ and $v \geq t$, otherwise u is a maximal element of $\mathbf{M}$ and $u \geq t$.

Proof of Theorem 6: Consider a set

$$\mathbf{P} = \{V(t) | \text{ there exists a Kripke model } \mathbf{M} = \langle S,\leq,s,V\rangle \text{ such that } \mathbf{M} \in \mathbf{K}_n, \mathbf{M} \models \#(p_1,\ldots,p_n) \text{ and } t \text{ is a maximal element of } \mathbf{M}\}.$$

Since elements of $\mathbf{P}$ are subsets of $\{p_1,\ldots,p_n\}$, $\mathbf{P}$ is finite. Let $\mathbf{P} = \{E_1,\ldots,E_m\}$. For $j = 1,\ldots,m$ define a formula $\psi_j(p_1,\ldots,p_n)$ by

$$\psi_j = \bigwedge_{i=1}^{n} \bar{p}_i, \text{ where } \bar{p}_i = \begin{cases} p_i \text{ if } p_i \in E_j \\ \neg p_i \text{ if } p_i \notin E_j. \end{cases}$$

Define ψ by

$$\psi = \bigvee_{j=1}^{m} \psi_j \text{ (if } m = 0, \text{ then } \psi \text{ is "false" by definition).}$$

We claim that ψ satisfies the conditions of Theorem 6. To prove this suppose, by contradiction, that $\mathbf{I}^\# \vdash \psi(A_1,\ldots,A_n)$, but $\mathbf{I}^\# \nvdash \#(A_1,\ldots,A_n)$. Since $\mathbf{I}^\#$ is complete with respect to Kripke models, there exists a Kripke model $\mathbf{M} = \langle S,\leq,s,V\rangle$ such that $\mathbf{M} \not\models \#(A_1,\ldots,A_n)$. Let $\mathbf{N} = \langle S,\leq,s,U\rangle$ be a Kripke model defined by $\mathbf{U}(t) = \{p_i | t \models_{\mathbf{M}} A_i, i = 1,\ldots,n\}$, $t \in S$. Since # is weakly invariant, Condition (3) of Definition 7 implies that $\mathbf{N} \not\models \#(p_1,\ldots,p_n)$.

By consistency of $\mathbf{I}^{\#}$ with respect to Kripke models, since $\mathbf{I}^{\#} \vdash \psi(A_1,\ldots,A_n)$, $\mathbf{M} \vDash \psi(A_1,\ldots,A_n)$. Hence for some $j = 1,\ldots,m$, $\mathbf{M} \vDash \psi_i(A_1,\ldots,A_n)$. Thus by Condition (3) of Definition 7, $\mathbf{N} \vDash \psi_i(p_1,\ldots,p_n)$. Therefore s is a maximal element of $\mathbf{N}$ (by construction of $\psi_j(p_1,\ldots,p_n)$). Condition (2) of Definition 7 implies that $\mathbf{N}$ is equivalent to a Kripke model $\mathbf{L} = \langle\{s\},=,s, V/\{s\}\rangle$; i.e., for all formulas A, $\mathbf{L} \vDash A$ iff $\mathbf{N} \vDash A$. Thus $\mathbf{L} \vDash \psi_j(p_1,\ldots,p_n)$ and

(1) $\mathbf{L} \nvDash \#(p_1,\ldots,p_n)$.

In view of definition of $\psi_j, \psi_j(p_1,\ldots,p_h) \nvDash \neg\#(p_1,\ldots,p_n)$. Let $\tilde{\mathbf{L}} = \langle R,\leq, r, W\rangle$ be a Kripke model such that $\tilde{\mathbf{L}} \vDash \psi_j(p_1,\ldots,p_n)$ and $\tilde{\mathbf{L}} \nvDash \neg\#(p_1,\ldots,p_n)$. Obviously, we may assume that $\tilde{\mathbf{L}} \in \mathbf{K}_n$. Thus r is a maximal element of $\tilde{\mathbf{L}}$. Consequently, $\tilde{\mathbf{L}}^r$ is equivalent to $\mathbf{L}$. Since $\tilde{\mathbf{L}}^r$ is a "one point" model, $\tilde{\mathbf{L}}^r \nvDash \neg\#(p_1,\ldots,p_n)$ implies that $\tilde{\mathbf{L}}^r \vDash \#(p_1,\ldots,p_n)$. This contradicts (1) and proves part (a)(1) of Theorem 6.

To prove part (a)(2) suppose that $\mathbf{I}^{\#} \vdash \#(A_1,\ldots,A_n)$, but $\mathbf{I}^{\#} \nvdash \neg\neg\psi(A_1,\ldots,A_n)$. Let $\mathbf{M} = \langle S,\leq,s, V\rangle$ be a Kripke model such that $\mathbf{M} \nvDash \neg\neg\psi(A_1,\ldots,A_n)$. Let $\mathbf{N}$ be as in the proof of (1). Then $\mathbf{N} \vDash \#(p_1,\ldots,p_n)$ and

(2) $\mathbf{N} \nvDash \neg\neg\psi(p_1,\ldots,p_n)$.

By Lemma 4, for all $t \in S$ there exists a $u \geq t$ such that u is a maximal element of $\mathbf{M}$. Then for some $j = 1,\ldots,m$, $u \vDash \psi_j(p_1,\ldots,p_n)$. This contradicts (2) and proves part (a)(2). $\left(\text{We remind the reader that } \psi = \bigvee_{j=1}^{m} \psi_j.\right)$

For part (b) of Theorem 6, let ψ be as above. It suffices to show that for all Kripke models $\mathbf{M} = \langle S,\leq,s, V\rangle$ and for all $t \in S$, $t \vDash \psi(A_1,\ldots,A_n) \supset \#(A_1,\ldots,A_n)$ and $t \vDash \#(A_1,\ldots,A_n) \supset \neg\neg\psi(A_1,\ldots,A_n)$.

In view of Condition (4) of Definition 7, we may assume that t is the least element of S. Then we can proceed as in part (a).

Proof of Theorem 7: By Condition (3) of Definition 7, similarly to the proof of Theorem 6, we can assume that $A_i = p_i$, $i = 1,\ldots,n$ and that any model under the consideration below belongs to class $\mathbf{K}_n$. Moreover, by Condition (4) of Definition 7 it suffices to consider the case in which the least point is maximal.

Now let $\mathbf{M} = \langle S,s,\leq, V\rangle$ be such a model; i.e., for all $t \in S$, $V(t) = V(s) \subseteq \{p_1,\ldots,p_n\}$. We have to prove that $\mathbf{M}^s \vDash \#(p_1,\ldots,p_n)$ iff $\mathbf{M} \vDash \#(p_1,\ldots,p_n)$. Assume that $\mathbf{M}^s \vDash \#(p_1,\ldots,p_n)$. Then $\mathbf{M}^s \vDash \neg\neg\psi(p_1,\ldots,p_n)$. Since $\mathbf{M}^s$ is a "one point" model and ψ does not contain #, $\mathbf{M}^s \vDash \psi(p_1,\ldots,p_n)$. Therefore $\mathbf{M} \vDash \psi(p_1,\ldots,p_n)$, because s is a maximal element of $\mathbf{M}$ (and ψ does not contain #). Consequently, $\mathbf{M} \vDash \#(p_1,\ldots,p_n)$. Part "only if" can be proved in a similar fashion.

The following example shows that we cannot drop out Condition (4) of Definition 7 from the statement of Theorem 7.

Example. Consider a new intuitionistic propositional constant # with the following interpretation in Kripke models: Let $\mathbf{M} = \langle S,s,\leq, V\rangle$ be a Kripke model, $t \in S$. $t \vDash \#$ iff any path in S_t (cf. Definition 2) is infinite. (This is the interpretation of the second-order intuitionistic propositional logic formula $\neg\forall p(p \vee \neg p)$.) Let $\mathbf{I}^{\#}$ consist of all formulas that are valid in every Kripke model with the above interpretation for #. Let ψ be "false". We see that # fulfills all of the conditions of Theorem 7 but (4), and the conclusion of Theorem 7 does not hold.

Remark 4. We don't know whether there exists a weakly extensional connective which is not extensional, nor whether every weakly invariant connective is extensional.

4 New connectives and Gentzen systems We assume that all Gentzen systems below are consistent.

In this section we consider some extensions $\mathbf{I}^{\#}$ of $\mathbf{I}$, where # can be described by introduction rules in a Gentzen system. We prove that under certain conditions imposed on a Gentzen system, $\mathbf{I}^{\#}$ is interpretable either in $\mathbf{I}^{\circ}$ (cf. Section 2) or in intuitionistic propositional modal logic **S4** (cf. [3], [10]), where an interpretation in $\mathbf{I}^{\circ}$ and **S4** is defined similarly to that in $\mathbf{I}_2$ (cf. Remark 2).

To state the results of this section we need some notations and definitions.

Definition 9 Let # be a new n-ary connective.

Let $\mathbf{G}^{\#}$ denote a Gentzen system in the language of **I** added with # that results from Gentzen's *LJ* (cf. [4], Section III) as follows:

1. A "formula" means a formula in the language of **I** added with #.
2. Structural rules and introduction rules for $\vee$, $\wedge$, $\supset$, and $\neg$ are those of *LJ*.
3. $\mathbf{G}^{\#}$ has finitely many introduction rules for #, and each of them has a finite number of premises.
4. The rules for introducing # into an antecedent have the following form:

$$\#\text{–}IA\ \frac{\text{a set of sequents of the form } \Delta,\Gamma \to \Theta}{\#(A_1,\ldots,A_n),\Gamma \to B}$$

where Δ consists of some of A_i's, $i = 1,\ldots,n$, Θ is B, or one of A_i's $i = 1,\ldots,n$, or empty.

5. The rules of introducing # into a succedent have the following form:

$$\#\text{–}IS\ \frac{\text{a set of sequences of the form } \Delta,\Gamma \to \Theta}{\Gamma \to \#(A_1,\ldots,A_n)},$$

where Δ consists of some of A_i's, $i = 1,\ldots,n$, Θ is one of A_i's $i = 1,\ldots,n$, or empty.

(With a given Gentzen system $\mathbf{G}^{\#}$ one can associate a Hilbert system $\mathbf{I}^{\#}$ such that $\mathbf{G}^{\#} \vdash \Gamma \to A$ iff $\Gamma \vdash_{\mathbf{I}^{\#}} A$.)

Let $\mathbf{RG}^{\#}$ denote the system with results from $\mathbf{G}^{\#}$ by imposing the following restrictions on the set of axioms:

6. Axioms of $\mathbf{RG}^{\#}$ are sequents of the form $A \to A$, where A is an atomic formula.

$\mathbf{G}^{\#}$ is said to be *regular* iff for any sequent $\Gamma \to \Theta$ such that $\mathbf{G}^{\#} \vdash \Gamma \to \Theta$, $\mathbf{RG}^{\#} \vdash \Gamma \to \Theta$, i.e., $\mathbf{RG}^{\#} \vdash A \to A$ for all formulas A.

The idea behind the definition of $\mathbf{G}^{\#}$ is quite obvious: we want $\mathbf{G}^{\#}$ to look exactly like *LJ*. Specifically, Conditions (3)–(5) are required to ensure the subformula property for $\mathbf{G}^{\#}$.

Remark 5. Whereas rule $\supset -IA$ of LJ does not satisfy Condition (4) of Definition 9, one can easily show that it can be changed to

$$\supset -IA \frac{\Gamma \rightarrow A \qquad B,\Gamma \rightarrow \Theta}{A \supset B, \Gamma \rightarrow \Theta}$$

without affecting the set of provable sequents. It is also easy to prove that the Cut-rule is superfluous in the resulting system.

One might have the impression that the regularity is a very strong restriction imposed on a Gentzen system. Indeed, it implicitly affects the introduction rules for #. However we shall see later (Theorem 15) that if # is weakly extensional (cf. Section 1, Definition 1), then $\mathbf{G}^{\#}$ is regular, and vice versa: if $\mathbf{G}^{\#}$ is regular, then # is weakly extensional (and even extensional, Corollary to Theorems 8–14). For instance, systems $LJ_{\not\subset}$, $LJ_{\downarrow}$, and $LJ_{|}$ (cf. [1]) are not regular, but LJ (cf. [4], Section III) and $\mathbf{G}^{\circ}$ (cf. Section 2) are.

The following theorem (surprisingly?) shows that no new connectives can be introduced by regular Gentzen systems.

Theorem 8 *Let $\mathbf{G}^{\#}$ be a regular Gentzen system. Then there exists a formula $\psi(p_1,\ldots,p_n)$ in the language of* **I** *such that the propositional variables of ψ are $p_1,\ldots,p_n$ and*

$$\mathbf{G}^{\#} \vdash \rightarrow \psi(A_1,\ldots,A_n) \equiv \#(A_1,\ldots,A_n),$$

i.e., # is expressible by the ordinary connectives.

Theorem 9 *Let $\mathbf{G}^{\#}$ be a Gentzen system. Then the Cut-rule is superfluous in $\mathbf{G}^{\#}$.*

As in the previous section, we postpone all the proofs and proceed with the discussion.

It follows from Theorem 8 that to obtain essentially new connectives Conditions (3)–(5) of Definition 9 need to be changed. If we want a Gentzen system to be "conventional" we cannot give up the finiteness (Condition 3 of Definition 9) nor impose any restrictions on the formulas $A_1,\ldots,A_n$ in (4) and (5). By analogy with Gentzen systems for $\mathbf{I}^{\circ}$ and **S4** we can restrict the introduction rules for # by imposing one of the following conditions:

1. In (4) of Definition 9 either all of the formulas of Γ are of the form $\#(C_1,\ldots,C_n)$ or Γ is empty.
2. In (5) of Definition 9 either Θ is of the form $\#(C_1,\ldots,C_n)$ or Θ is empty.

(Of course, there are other possibilities; cf. [8], say.)

Definition 10 A Gentzen system satisfying Definition 9 and Condition 1(2) above is said to be $A(S)$*-restricted.*

The theorems below show that the connectives which can be defined by means of restricted Gentzen systems are already known to us.

Theorem 10 *Let $\mathbf{G}^{\#}$ be an S-restricted regular Gentzen system. Then there exist a formula $\psi(p_1,\ldots,p_n)$ in the language of* **I** *such that the propositional variables of ψ are $p_1,\ldots,p_n$ and*

$$\mathbf{G}^{\#} \vdash \psi(A_1,\ldots,A_n) \to \#(A_1,\ldots,A_n) \text{ and}$$
$$\mathbf{G}^{\#} \vdash \#(A_1,\ldots,A_n) \to \neg\neg\psi(A_1,\ldots,A_n).$$

Theorem 11 *Let $\mathbf{G}^{\#}$ be an S-restricted Gentzen system. Then the Cut-rule is superfluous in $\mathbf{G}^{\#}$.*
Theorem 12 *Let $\mathbf{G}^{\#}$ be an A-restricted Gentzen system. Then the Cut-rule is superfluous in $\mathbf{G}^{\#}$.*
Theorem 13 *Let $\mathbf{G}^{\#}$ be an A-restricted regular Gentzen system. Then $\mathbf{G}^{\#}$ is interpretable in* **S4.**
Theorem 14 *Let $G^{\#}$ be an S-restricted regular Gentzen system. Then $\mathbf{G}^{\#}$ is interpretable in $\mathbf{G}^{\circ}$.*

A common corollary to Theorems 8–14 is as follows.

Corollary *Let $\mathbf{G}^{\#}$ be a (A or S-) restricted Gentzen system. Then*
(a) *$\mathbf{G}^{\#}$ is decidable*
(b) *$\mathbf{G}^{\#}$ has the subformula property*
(c) *$\mathbf{G}^{\#}$ has the disjunction property*
(d) *$\mathbf{G}^{\#}$ is a conservative extension of LJ*
(e) *if $\mathbf{G}^{\#}$ is regular, then # is extensional.*

Proof of Corollary: (a)–(d) follow from the superfluity of the Cut-rule for $\mathbf{G}^{\#}$. (e) follows from the fact that # is either expressible by the ordinary connectives (cf. Theorem 8), or $\mathbf{G}^{\#}$ is interpretable either in $\mathbf{G}^{\circ}$ or in **S4** (cf. Theorems 13 and 14) and $\circ$ and $\Box$ are extensional.

To prove Theorem 8 we need one more notation:

Definition 11 Let a sequent $\Gamma \to \Theta$ be at a node of some deduction in $\mathbf{G}^{\#}$, and let a formula B occur either in Γ or in Θ. Such an occurrence is said to be *final,* if B is not a *side* formula (cf. [4], 2.511) in any rule of inference applied after $\Gamma \to \Theta$; i.e., B does not explicitly take a part in any rule of inference applied after $\Gamma \to \Theta$. Also we say "B is final in $\Gamma \to \Theta$" or "B is final", if it is understood to which sequent we refer.

Proof of Theorem 8: Let $\#-IS_i$ be a rule for introducing # into a succedent $(i = 1,\ldots,k)$, and let $\{\Delta^i_h,\Gamma \to \Theta_{h_i}\}_{h=1,\ldots,l_i}$ be all its premises. Let Θ_{h_i} be nonempty for $h = 1,\ldots,h_i$ and be empty for $h = h_i + 1,\ldots,l_i$.

Define

$$\psi^i_h(p_1,\ldots,p_n) = \begin{cases} \bigwedge_{A_s\in\Delta^i_h} p_s \supset p_t,\ \Theta_{h_i} = A_t,\ h = 1,\ldots,h_i \\ \neg\left(\bigwedge_{A_s\in\Delta^i_h} p_s\right),\ h = h_i + 1,\ldots,l_i \end{cases}$$

$$\psi^i(p_1,\ldots,p_n) = \bigwedge_{h=1}^{l_i} \psi^i_h(p_1,\ldots,p_n);$$

$$\psi(p_1,\ldots,p_n) = \bigvee_{i=1}^{k} \psi^i(p_1,\ldots,p_n).$$

(Recall that k is the number of rules for introducing # into a succedent, and l_i is the number of premises of the ith rule #-IS_i.)

We claim that $\psi(p_1,\ldots,p_n)$ satisfies the condition of Theorem 8.

To prove that $\mathbf{G}^\# \vdash \psi(A_1,\ldots,A_n) \to \#(A_1,\ldots,A_n)$, it suffices to show that $\mathbf{G}^\# \vdash \psi^i(A_1,\ldots,A_n) \to \#(A_1,\ldots,A_n)$ for each $i = 1,\ldots,k$. First we observe that all the sequents of $\{\Delta_h^i, \psi_h^i(A_1,\ldots,A_n) \to \Theta_{h_i}\}_{h=1,\ldots,l_i}$ are derivable in LJ, and therefore they are derivable in $\mathbf{G}^\#$. (This fact easily follows from the definition of ψ_j^i.) Hence the sequents $\{\Delta_h^i, \psi^i(A_1,\ldots,A_n) \to \Theta_{h_i}\}_{h=1,\ldots,l_i}$ are derivable in $LJ(\mathbf{G}^\#)$. Applying the rule #-IS_i to $\{\Delta_h^i, \psi^i(A_1,\ldots,A_n) \to \Theta_{h_i}\}_{h=1,\ldots,l_i}$ we obtain $\psi^i(A_1,\ldots,A_n) \to \#(A_1,\ldots,A_n)$.

To prove that $\mathbf{G}^\# \vdash \#(A_1,\ldots,A_n) \to \psi(A_1,\ldots,A_n)$ we consider a derivation of $\#(p_1,\ldots,p_n) \to \#(p_1,\ldots,p_n)$ in $\mathbf{RG}^\#$ (cf. (6) of Definition 9), where $p_1,\ldots,p_n$ are propositional variables. We change this derivation in the following manner: in the nodes where $\#(p_1,\ldots,p_n)$ has been introduced into a succedent and its occurrence in the succedent is final, we introduce $\psi(p_1,\ldots,p_n)$ instead of $\#(p_1,\ldots,p_n)$ (using, possibly, a few logical rules and interchanges) and continue with "the previous" derivation. Thus a derivation of $\#(p_1,\ldots,p_n) \to \psi(p_1,\ldots,p_n)$ results. To obtain a derivation of $\#(A_1,\ldots,A_n) \to \psi(A_1,\ldots,A_n)$ in $\mathbf{G}^\#$ we substitute an axiom $A_i \to A_i$ for $p_i \to p_i$ and A_i for $p_i, i = 1,\ldots,n$.

To prove Theorem 9 we need some preliminaries.

Let #-IA_j be a rule for introducing # into an antecedent ($j = 1,\ldots,m$), and let $\{\tilde\Delta_h^j, \Gamma \to \tilde\Theta_h^j\}_{h=1,\ldots,g_j}$ be all of its premises. Let $\tilde\Theta_h^j = B$ for $h = 1,\ldots,h_j'$ and let $\tilde\Theta_h^j$ be one of the A's for $h = h_j' + 1,\ldots,h_j$ and be empty for $h = h_j + 1,\ldots,g_j$. Define

$$\phi_h^j(p_1,\ldots,p_n) = \begin{cases} \bigwedge_{A_s\in\tilde\Delta_h^j} p_s, h = 1,\ldots,h_j' \\ \bigwedge_{A_s\in\tilde\Delta_h^j} p_s \supset p_t,\ \tilde\Theta_h^j = A_t,\ h = h_j' + 1,\ldots,h_j \\ \neg\left(\bigwedge_{A_s\in\tilde\Delta_h^j} p_s\right),\ h = h_j + 1,\ldots,g_j \end{cases}$$

$$\phi^j(p_1,\ldots,p_n) = \bigwedge_{h=h_j'+1}^{g_j} \phi_h^j(p_1\ldots,p_n) \supset \bigvee_{h=1}^{h_j'} \phi_h^j(p_1,\ldots,p_n).$$

$$\phi(p_1,\ldots,p_n) = \bigwedge_{j=1}^{m} \phi^j(p_1,\ldots,p_n).$$

As in the proof of Theorem 8 one can show that $\mathbf{G}^\# \vdash \#(A_1,\ldots,A_n) \to \phi(A_1,\ldots,A_n)$ and $\mathbf{G}^\# \vdash \phi(A_1,\ldots,A_n) \to \#(A_1,\ldots,A_n)$.

To prove Theorem 9 we first show that $LJ \vdash \psi^i(p_1,\ldots,p_n) \to \phi^j(p_1,\ldots,p_n)$, where $i = 1,\ldots,k, j = 1,\ldots,m$ and $p_1,\ldots,p_n$ are propositional variables. We do this in a few steps.

Lemma 5 $\psi^i(p_1,\ldots,p_n) \supset \phi^j(p_1,\ldots,p_n)$ *is provable in the classic propositional calculus.*

Proof: Were $\psi^i(p_1,\ldots,p_n) \supset \phi^j(p_1,\ldots,p_n)$ not provable in the classic propositional calculus we could find a substitution of "true" and "false" for $p_1,\ldots,p_n$ which makes the implication false. This contradicts the consistency of $\mathbf{G}^{\#}$, since

$$\psi(A_1,\ldots,A_n) \to \#(A_1,\ldots,A_n) \text{ and } \#(A_1,\ldots,A_n) \to \phi(A_1,\ldots,A_n)$$

are derivable in $\mathbf{G}^{\#}$ for all formulas $A_1,\ldots,A_n$ $\left(\text{and } \psi = \bigvee_{i=1}^{k} \psi^i,\ \phi = \bigwedge_{j=1}^{m} \phi^j\right)$. Recall that we did not use the regularity of $\mathbf{G}^{\#}$ to derive the above sequents.

Lemma 6 *$LJ \vdash \psi^i(p_1,\ldots,p_n) \to \phi^j(p_1,\ldots,p_n)$, where $p_1,\ldots,p_n$ are propositional variables.*

Proof: It follows from the definition of ψ^i and ϕ^j that Lemma 6 is equivalent to

(3) $LJ \vdash \psi_1^i(p_1,\ldots,p_n),\ldots,\psi_{l_i}^i(p_1,\ldots,p_n),\phi_{h_j'+1}^j(p_1,\ldots,p_n),\ldots,\phi_{g_j}^j(p_1,\ldots,p_n) \to \bigvee_{h=1}^{h_j'} \phi_h^j(p_1,\ldots,p_n)$

Rewriting the succedent into the conjunction normal form one can see that (3) is equivalent to the following assertion: the sequent

(4) $\psi_1^i(p_1,\ldots,p_n),\ldots,\psi_{l_i}^i(p_1,\ldots,p_n),\ \phi_{h_j'+1}^j(p_1,\ldots,p_n),\ldots,\phi_{g_j}^j(p_1,\ldots,p_n) \to \bigvee_{h=1}^{h_j'} p^h,$

where $p^h \in \tilde{\Delta}_h^j$, $h = 1,\ldots,h_j'$, is derivable in *LJ*.

Lemma 5 implies that (4) is derivable in *LK*, where *LK* is a Gentzen system for the classic logic (cf. [4], Section III).

Let Π denote the antecedent of (4). If Π is inconsistent, then, definitely, (4) is derivable in *LJ*. Suppose that Π is consistent. We contend that for each formula of Π the result of the substitution of "true" for those of $p_1,\ldots,p_n$ which are classically derivable from Π and "false" for the others is "true". Indeed, for no formula of Π of the form $\neg\left(\bigwedge_{p\in\Delta} p\right)$ the result of the substitution cannot be "false", since Π is consistent. If for some formula of Π of the form $\bigwedge_{p\in\Delta} p \supset q$ all elements of Δ have been replaced by "true", then $LK \vdash \Pi \to q$, and q has been replaced by "true". This proves our contention.

Since $LK \vdash \Pi \to \bigvee_{h=1}^{h_j'} p^h$, for some $h = 1,\ldots,h_j', p^h$ must be substituted by "true". Hence, for such an h, $LK \vdash \Pi \to p^h$.

We claim that $LJ \vdash \Pi \to p^h$. Obviously, this claim implies that (4) is derivable in *LJ*, and hence proves Lemma 6.

To prove that $LJ \vdash \Pi \to p^h$ it is sufficient to show that for all Kripke models $\mathbf{M} = \langle S,s,\leq, V\rangle$, for all $t \in S$ such that all the formulas of Π are valid at $t, p^h \in V(t)$. It easily follows from the structure of the formulas of Π: $\bigwedge_{p\in\Delta} p$, $\neg\left(\bigwedge_{p\in\Delta} p\right)$ or $\bigwedge_{p\in\Delta} p \supset q$, that if a formula of Π is valid at t, then it is valid in a model for the classic propositional logic that assigns "true" for p iff $p \in V(t)$. Since $LK \vdash \Pi \to p^h$, $p^h \in V(t)$. This proves the claim and completes the proof of Lemma 6.

Since $\mathbf{G}^\#$ is an "extension" of LJ and the Cut-rule is superfluous in LJ (cf. [4]), we immediately obtain the following corollary:

Corollary *For all formulas $A_1, \ldots, A_n$ in the language of* **I** *added with #, there is Cut-free derivation of $\psi^i(A_1, \ldots, A_n) \to \phi^j(A_1, \ldots, A_n)$ in $\mathbf{G}^\#$, $i = 1, \ldots, k$, $j = 1, \ldots, m$.*

Proof of Theorem 9: As in the case of LJ we prove the Cut-elimination theorem not by showing directly that the Cut-rule is redundant, but that an equivalent rule Mix (cf. [4], Section III, §3) can be eliminated. The structure of the proof is the same as that for LJ (cf. [4]) with the following modification: whereas the proof for LJ consists of inductions on the *rank* ρ and *degree* γ of the given derivation, we use the induction on ρ, γ, and ν, the number of occurrences of # in the mix-formula. (Here γ is the number of connectives occurring in the mix-formula, and ρ is the sum of the *left* and *right* ranks. These two terms are defined as follows: The *left* (*right*) *rank* is the largest number of consecutive sequents in a path so that the lowest of these sequents is the *left* (*right*) *hand* upper sequent of the mix and each of the sequents contains the mix-formula in the succedent (antecedent).)

Most of the argument is unaltered; we need only consider the cases in which the last introduction rules before the Mix to be eliminated are $\#{-}IS$ and $\#{-}IA$. The case of $\rho > 2$ can be treated similarly to the cases of the ordinary connectives (cf. [4], Section III, §3, 3.121.23, and 3.122).

Suppose that $\rho = 2$ and the inference is of the form:

$$\cfrac{\cfrac{\{\Delta^i_h, \Gamma_1 \to \Theta_{h_l}\}_{h=1,\ldots,l_i}}{\Gamma_1 \to \#(A_1, \ldots, A_n)}\ \#{-}IS_i \qquad \cfrac{\{\tilde{\Delta}^j_h, \Gamma_2 \to \tilde{\Theta}^j_h\}_{h=1,\ldots,g_j}}{\#(A_1, \ldots, A_n), \Gamma_2 \to B}\ \#{-}IA_j}{\Gamma_1, \Gamma_2 \to B}\ \text{Mix.}$$

This is replaced by Derivation 1, shown on p. 326. (Notice that $\Gamma_1 \to \psi^i(A_1, \ldots, A_n)$ and $\phi^j(A_1, \ldots, A_n)$, $\Gamma_2 \to B$ are derivable without Mix-rules from $\{\Delta^i_h, \Gamma_1 \to \Theta_{h_l}\}_{h=1,\ldots,l_l}$ and $\{\tilde{\Delta}^j_h, \Gamma_2 \to \tilde{\theta}^j_h\}_{h=1,\ldots,g_j}$ respectively.)

The above replacement reduces ν by 1. The case of $\nu = 0$ is exactly that of [4]. This completes the proof of Theorem 9.

Remark 6. A more detailed analysis of the proof shows that if $\mathbf{G}^\#$ is regular, then $\mathbf{G}^\#$ without the Cut-rule is also regular. Indeed, the axioms of the form $A_i \to A_i$, $i = 1, \ldots, n$, required to derive $\psi^i(A_1, \ldots, A_n) \to \phi^j(A_1, \ldots, A_n)$, are eliminated in the subcase of $\rho = 2$ in which the left-hand upper sequent of the Mix is an axiom (cf. [4], Section III, §3, 3.111).

Proof of Theorem 10: Let $\psi(p_1, \ldots, p_n)$ be the same as in the proof of Theorem 8. Exactly the same argument shows that $\mathbf{G}^\# \vdash \psi(A_1, \ldots, A_n) \to \#(A_1, \ldots, A_n)$. In order to prove that $\mathbf{G}^\# \vdash (A_1, \ldots, A_n) \to \neg\neg\psi(A_1, \ldots, A_n)$ we proceed as follows.

Derivation 1.

$$\dfrac{\dfrac{\dfrac{\{\Delta^i_h,\Gamma_1\to\Theta_{h_i}\}_{h=1,\dots,l_i}}{\Gamma_1\to\psi^i(A_1,\dots,A_n)} \qquad \psi^i(A_1,\dots,A_n)\to\phi^j(A_1,\dots,A_n)}{\Gamma_1\to\phi^j(A_1,\dots,A_n)}\,\text{Mix} \qquad \dfrac{\{\tilde{\Delta}^j_h,\Gamma_2\to\tilde{\Theta}^j_h\}_{h=1,\dots,g_j}}{\phi^j(A_1,\dots,A_n),\ \Gamma_2\to}}{\dfrac{\Gamma_1,\Gamma_2^*\to B}{\Gamma_1,\Gamma_2\to B}\ \text{possibly several interchanges and thinnings}}\,\text{Mix.}$$

Derivation 2.

$$\dfrac{\dfrac{\dfrac{\{\Delta^i_h,\Gamma\to\Theta_{h_i}\}_{h=1,\dots,l_i}}{\Gamma\to\psi(A_1,\dots,A_n)} \qquad \psi(A_1,\dots,A_n)\to\psi'(A_1,\dots,A_n)}{\Gamma\to\psi'(A_1,\dots,A_n)}\,\text{Cut} \qquad \psi'(A_1,\dots,A_n)\to\#(A_1,\dots,A_n)}{\Gamma\to\#(A_1,\dots,A_n)}\,\text{Cut.}$$

Consider a derivation of $\#(p_1,\dots,p_n) \to \#(p_1,\dots,p_n)$, where $p_1,\dots,p_n$ are propositional variables (such a derivation exists because $\mathbf{G}^\#$ is regular) and change it in the following manner:

The derivation of $\#(p_1,\dots,p_n) \to \#(p_1,\dots,p_n)$:

$$\frac{\{\Delta^i_h,\Gamma \to \Theta_{h_i}\}_{h=1,\dots,l_i}}{\Gamma \to \#(p_1,\dots,p_n)}\ \#\text{–}IS\ \text{(final)}$$

$$\vdots$$

$$\#(p_1,\dots,p_n) \to \#(p_1,\dots,p_n).$$

The derivation of $\#(p_1,\dots,p_n) \to \neg\neg(p_1,\dots,p_n)$:

$$\frac{\dfrac{\{\Delta^i_h,\Gamma \to \Theta_{h_i}\}_{h=1,\dots,l_i}}{\Gamma \to \psi(p_1,\dots,p_n)}}{\neg\psi(p_1,\dots,p_n),\Gamma \to}\ \neg\text{–}IA$$

$$\vdots$$

$$\frac{\neg\psi(p_1,\dots,p_n),\ \#(p_1,\dots,p_n) \to}{\#(p_1,\dots,p_n) \to \neg\neg\psi(p_1,\dots,p_n)}\ \text{–}\neg IS.$$

If we substitute an axiom $A_i \to A_i$ for $p_i \to p_i$, $i = 1,\dots,n$ in the above derivation, the derivation of $\#(A_1,\dots,A_n) \to \neg\neg\psi(A_1,\dots,A_n)$ results.

Proof of Theorem 11: Similarly to the proof of Theorem 10 one can show that $\mathbf{G}^\# \vdash \phi(A_1,\dots,A_n) \to \#(A_1,\dots,A_n)$ and $\mathbf{G}^\# \vdash \#(A_1,\dots,A_n) \to \neg\neg\phi(A_1,\dots,A_n)$ where ϕ is as in the proof of Theorem 9. Hence $\mathbf{G}^\# \vdash \psi(A_1,\dots,A_n) \to \neg\neg\phi(A_1,\dots,A_n)$ and by Lemmas 5 and 6 $LJ \vdash \psi(A_1,\dots,A_n) \to \phi(A_1,\dots,A_n)$. Thus to prove that the Mix-rule can be eliminated in $\mathbf{G}^\#$ we can proceed exactly as in the proof of Theorem 9 for the case of $\rho = 2$. However we must also consider the case $\rho > 2$, since the following situation may occur:

$$\frac{\dfrac{\{\tilde{\Delta}^j_h,\Gamma_1 \to \tilde{\Theta}^j_h\}_{h=1,\dots,g_j}}{\#(A_1,\dots,A_n),\ \Gamma_1 \to \#(B_1,\dots,B_n)}\ \#\text{–}IA \qquad \Gamma_2 \to C}{\#(A_1,\dots,A_n),\Gamma_1,\Gamma_2^* \to C}\ \text{Mix.}$$

Here we cannot first apply the Mix-rule and then after that introduce $\#(A_1,\dots,A_n)$ into the antecedent, because C is not necessarily of the form #. But we don't have to do this because we actually apply the Mix-rule when $\rho = 2$; i.e., at the node where $\#(B_1,\dots,B_n)$ is introduced into an antecedent (cf. [4], Section III, §3, 3.12). At this node the succedent is either of the form # or empty, and it is possible to introduce $\#(A_1,\dots,A_n)$ into the antecedent.

Proof of Theorem 12: Let ϕ and ψ be as above. If we can prove that $LJ \vdash \psi(A_1,\dots,A_n) \to \phi(A_1,\dots,A_n)$, then we proceed exactly as in the proof of Theorem 11. Since $\mathbf{G}^\#$ is consistent, in view of Lemmas 5 and 6 it is enough to show that for each substitution of "true" and "false" for $A_1,\dots,A_n$, $\mathbf{G}^\# \vdash \bar{\psi} \to \bar{\phi}$, where $\bar{\rho}$ denotes the result of such a substitution into a formula ρ.

If $\bar{\psi}$ is "false", then, definitely, $\mathbf{G}^\# \vdash \bar{\psi} \to \bar{\phi}$.

Otherwise, for some $i = 1,\dots,k$, $\bar{\psi}^i$ must be "true". Since ψ^i has no disjunctions, all the premises of $\#-IS_i$ with the same substitution for $A_1,\dots,A_n$ and empty Γ are derivable in LJ (and hence in $\mathbf{G}^\#$). Therefore $\mathbf{G}^\# \vdash \rightarrow \bar{\#}$. Similarly to the proof of Theorem 8, one can show that $\mathbf{G}^\# \vdash \#(A_1,\dots,A_n) \rightarrow \phi(A_1,\dots,A_n)$. Hence $\mathbf{G}^\# \vdash \rightarrow \bar{\phi}$. This completes the proof.

To prove that $A(S)$-restricted Gentzen systems are interpretable in $\mathbf{S4(G^\circ)}$ we need one more definition.

Definition 12 Let $\mathbf{G}^\#$ be a (A or S-restricted) regular Gentzen system, and let $\mathbf{D}$ be some *fixed* derivation of $\#(p_1,\dots,p_n) \rightarrow \#(p_1,\dots,p_n)$ in $\mathbf{RG}^\#$, where $p_1,\dots,p_n$ are propositional variables. All rules of inference for # used in $\mathbf{D}$ (and only them) are called *essential.*

Lemma 7 *Let formulas ψ' and ϕ' be defined as formulas ψ and ϕ, but only from essential rules of inference. If $\mathbf{G}^\#$ is a (A or S-restricted) regular Gentzen system, then $\mathbf{G}^\# \vdash \rightarrow \psi \equiv \psi' \wedge \phi \equiv \phi'$.*

Proof: (i) If $\mathbf{G}^\#$ is a regular Gentzen system, then, as in the proof of Theorem 8, one can show that $\mathbf{G}^\# \vdash \rightarrow \psi'(A_1,\dots,A_n) \equiv \#(A_1,\dots,A_n)$ and $\mathbf{G}^\# \vdash \rightarrow \phi'(A_1,\dots,A_n) \equiv \#(A_1,\dots,A_n)$. An application of the Cut-rule completes the proof of this case.

(ii) If $\mathbf{G}^\#$ is an A or S-restricted regular Gentzen system, consider a Gentzen system $\tilde{\mathbf{G}}^\#$ obtained from $\mathbf{G}^\#$ by removing A or S-restrictions, respectively. Since the Cut-rule is superfluous in $\mathbf{G}^\#$,$\tilde{\mathbf{G}}^\#$ is consistent, and the proof follows from (i).

Corollary 1 *Let $\mathbf{G}^\#$ be an S-restricted regular Gentzen system. If a sequent is derivable in $\mathbf{G}^\#$, then there exists a derivation of this sequent in which every rule for introducing # into a succedent is essential.*

Proof: Given a derivation in $\mathbf{G}^\#$ we change it as follows: in all nodes where # has been introduced into the succedent by a nonessential rule of inference:

$$\frac{\{\Delta^i_h,\Gamma \rightarrow \Theta_{h_i}\}_{h=1,\dots,h_i}}{\Gamma \rightarrow \#(A_1,\dots,A_n)}\ \#-IS_i$$

we replace a derivation of the sequents at these nodes by Derivation 2 (shown on p. 326).

Corollary 2 *Let $\mathbf{G}^\#$ be an A-restricted regular Gentzen system. Then either $\mathbf{G}^\# \vdash \rightarrow \neg\#(A_1,\dots,A_n)$ or there is only one essential rule for introducing # into a succedent, and if a sequent is derivable in $\mathbf{G}^\#$, then there exists a derivation of this sequent in which every rule for introducing # into a succedent is essential.*

Proof: Let $\mathbf{D}$ be as in Definition 12. Then either $\mathbf{G}^\# \vdash \#(p_1,\dots,p_n) \rightarrow$, and $\#(p_1,\dots,p_n) \rightarrow \#(p_1,\dots,p_n)$ has been obtained by Thinning, or the last operational rule of $\mathbf{D}$ is $\#-IS$.

In the first case $\mathbf{G}^\# \vdash \rightarrow \neg\#(A_1,\dots,A_n)$, and in the second case there is only one essential rule for introducing # into a succedent, and we have to show that if all the sequents of $\{\Delta^i_h,\Gamma \rightarrow \Theta_{h_i}\}_{h=1,\dots,l_i}$ are derivable in $\mathbf{G}^\#$, then $\Gamma \rightarrow$

$\#(A_1,\ldots,A_n)$ is derivable when the only rule for introducing # into a succedent is the essential rule.

As in the proof of Corollary 1 one can show that $\mathbf{G}^\# \vdash \Gamma \to \psi'(A_1,\ldots,A_n)$. Since there is only one essential rule, ψ' does not contain $\vee$. Hence all the premises of this rule are derivable in $\mathbf{G}^\#$ and we can apply this rule to derive $\Gamma \to \#(A_1,\ldots,A_n)$.

Proof of Theorem 13: We claim that $\mathbf{G}^\#$ is interpretable in **S4** with the following translation:

1. $t(A) = A$, if A is an atomic formula
2. $t(A_1 \otimes A_2) = t(A_1) \otimes t(A_2)$, where $\otimes$ is $\wedge$, $\vee$ or $\supset$; $t(\neg A) = \neg t(A)$
3. $t(\#(A_1,\ldots,A_n)) = \Box\psi'(t(A_1),\ldots,t(A_n))$.

If $\mathbf{G}^\# \vdash \to \neg\#(A_1,\ldots,A_n)$, there is nothing to prove, otherwise let $\Gamma \to \Theta$ be a sequent in the language of $\mathbf{G}^\#$. We shall prove by induction on the number of steps in derivations in $\mathbf{G}^\#$ and **S4** that $\mathbf{G}^\# \vdash \Gamma \to \Theta$ iff $\mathbf{S4} \vdash t(\Gamma) \to t(\Theta)$.

Part "only if" is obvious, since given any introduction rule for #, we can first introduce ψ and then, by Lemma 7 (using the Cut-rule), obtain ψ'. After that we can introduce $\Box$.

To prove part "if" it suffices to show that the following two rules of inference are admissible in $\mathbf{G}^\#$ (i.e., do not affect the set of derivable sequents):

$$\frac{\Gamma \to \psi'(A_1,\ldots,A_n)}{\Gamma \to \#(A_1,\ldots,A_n)}, \text{ where all the formulas of } \Gamma \text{ are of the form } \#,$$

and

$$\frac{\psi'(A_1,\ldots,A_n),\Gamma \to \Theta}{\#(A_1,\ldots,A_n),\Gamma \to \Theta}.$$

The admissibility of the first rule easily follows from the proof of Corollary 2 to Lemma 7.

To prove that the second rule is admissible, assume that $\mathbf{G}^\# \vdash \psi'(A_1,\ldots,A_n),\Gamma \to \Theta$. Then, by Lemma 7, $\mathbf{G}^\# \vdash \psi^i(A_1,\ldots,A_n),\Gamma \to \Theta$, $i = 1,\ldots,k$. Now in a derivation of $\psi^i(A_1,\ldots,A_n)$, $\Gamma \to \Theta$, in the nodes where $\psi^i(A_1,\ldots,A_n)$ has been finally introduced into the antecedent, we can use the same premises to derive all the premises of $\#{-}IA_i$, and introduce $\#(A_1,\ldots,A_n)$ instead of $\psi^i(A_1,\ldots,A_n)$. Then we can proceed with the "previous" derivation.

Proof of Theorem 14: If none of the essential rules of inference for introducing # into an antecedent results in a sequent with a nonempty succedent, then, obviously,

$$\mathbf{G}^\# \vdash \to \phi'(A_1,\ldots,A_n) \equiv \neg\neg\phi'(A_1,\ldots,A_n)$$

and, by Theorem 10 and Lemma 7, # is expressible by the ordinary connectives.

Assume now that there exist essential rules of inference for introducing # into an antecedent which result in a sequent with a nonempty succedent (that must be of the form #).

We contend that the following rule for introducing # into an antecedent is admissible in $\mathbf{G}^\#$:

$$\#\text{-}\mathbf{IA}\ \frac{\{A_{i_1},\ldots,A_{i_h},\Gamma\to\Theta \mid \text{in } \mathbf{D} \text{ some } \#\text{-}IS \text{ results } p_{i_1},\ldots,p_{i_h}\to\#(p_1,\ldots,p_n)\}}{\#(A_1,\ldots,A_n),\Gamma\to\Theta},$$

where Θ is either of the form # or empty.

Indeed, #−**IA** can be obtained by the following transformation of **D**:

1. For each axiom $p_i \to p_i$, $i = 1,\ldots,n$ take $A_i \to A_i$ and introduce Γ into the antecedent of $A_i \to A_i$ by thinnings and interchanges. After that substitute $A_i,\Gamma \to A_i$ for the axiom $p_i \to p_i$, $i = 1,\ldots,n$.
2. "Cut off" all the branches of **D** which end with introducing # into a succedent and substitute there an appropriate premise of #−**IA.** (Since **D** is cut-free and it is impossible to introduce # into an antecedent before introducing it into the succedent, there exists such a premise.)
3. Proceed further as in **D** substituting A_i for p_i, $i = 1,\ldots,n$.

Obviously, a derivation of $\#(A_1,\ldots,A_n)$, $\Gamma \to \Theta$ results.

Let $\tilde{\phi}(p_1,\ldots,p_n)$ be defined as $\phi(p_1,\ldots,p_n)$, but only from the premises of #−**IA.** Similarly to the proof of Lemma 7 it can be shown that

(5) $\mathbf{G}^{\#} \vdash \to \phi(A_1,\ldots,A_n) \equiv \tilde{\phi}(A_1,\ldots,A_n)$.

We claim that $\mathbf{G}^{\#}$ is interpretable in G° under the following transformation t:

1. $t(A) = A$, if A is an atomic formula
2. $t(A_1 \otimes A_2) = t(A_1) \otimes t(A_2)$, where $\otimes$ is $\wedge$, $\vee$, or $\supset$; $t(\neg A) = \neg t(A)$
3. $t(\#(A_1,\ldots,A_n)) = \circ\tilde{\phi}(t(A_1),\ldots,t(A_n))$.

Using (5) we can proceed as in the proof of Theorem 13. Obviously, if $\mathbf{G}^{\#} \vdash \Gamma \to \Theta$, then $\mathbf{G}^{\circ} \vdash t(\Gamma) \to (\Theta)$. To prove the converse we use induction on the number of steps in a derivation in $\mathbf{G}^{\circ}$. The only nontrivial step of this induction is to show that the following rule of inference is admissible in $\mathbf{G}^{\#}$:

$$\frac{\tilde{\phi}(A_1,\ldots,A_n),\Gamma \to \Theta}{\#(A_1,\ldots,A_n),\Gamma \to \Theta},$$

where Θ is of the form # or empty.

Since $\tilde{\phi}$ contains neither $\supset$ nor $\neg$, if $\mathbf{G}^{\#} \vdash \tilde{\phi}(A_1,\ldots,A_n)$, $\Gamma \to \Theta$, then all the premises of #−**IA** are derivable in $\mathbf{G}^{\#}$. Hence $\mathbf{G}^{\#} \vdash \#(A_1,\ldots,A_n)$, $\Gamma \to \Theta$.

Finally, we establish one more property of weakly extensional connectives.

Theorem 15 *Let* $\mathbf{G}^{\#}$ *be an (A- or S-restricted) Gentzen system. If # is weakly extensional, then* $\mathbf{G}^{\#}$ *is regular.*

Proof: We shall prove by induction on the number of connectives of a formula A that $A \to A$ is derivable in $\mathbf{G}^{\#}$ from the axioms $\{p \to p \mid p$ is a propositional variable$\}$. The only interesting case is that in which A is of the form $\#(A_1,\ldots,A_n)$. Obviously, we may assume that $A_1,\ldots,A_n$ are atomic formulas; i.e., $A_1 = p_1,\ldots,A_n = p_n$ (cf. the end of the proof of Theorem 8).

Since # is weakly extensional and $\mathbf{G}^{\#} \vdash p_1 \equiv p_1 \wedge p_1$,

$$\mathbf{G}^{\#} \vdash \#(p_1 \wedge p_1, p_2,\ldots,p_n) \to \#(p_1,\ldots,p_n).$$

By Theorems 9, 11, and 12 there is a Cut-free derivation of $\#(p_1 \wedge p_1, p_2, \ldots, p_n) \rightarrow \#(p_1, p_2, \ldots, p_n)$ in $\mathbf{G}^{\#}$. This Cut-free derivation can be transformed to a derivation of $\#(p_1, \ldots, p_n) \rightarrow \#(p_1, \ldots, p_n)$ by ignoring introducing $p_1 \wedge p_1$ in the appropriate nodes. Obviously, each axiom in the last derivation is of the form $p_i \rightarrow p_i$, $i = 1, \ldots, n$.

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