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Partially Generic Formulas in Arithmetic

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Introduction The following problem arose in connection with a question concerning interpretability (cf. [4]). Let T be a consistent recursively enumerable (r.e.) theory containing a sufficient amount of arithmetic. We do not assume that T is Σ_1^0-*sound*, i.e., that only true Σ_1^0 sentences are provable in T. Next, let S be any r.e. theory, set $\mathrm{Th}_T(S) = \{\phi\colon \exists q T \vdash \mathrm{Pr}_{[S\restriction q]}(\bar{\phi})\}$, and let $\sigma(x)$ be any formula numerating S in T. Then if $\phi \in \mathrm{Th}_T(S)$, then $T \vdash \mathrm{Pr}_\sigma(\bar{\phi})$. The problem is now if there is a (Σ_1^0) formula $\sigma(x)$ numerating S in T such that $\mathrm{Pr}_\sigma(x)$ numerates $\mathrm{Th}_T(S)$ in T. (This is, of course, true if T is Σ_1^0-sound.) As was shown in [4] (Lemma 2), the answer is affirmative. The proof uses a result of Guaspari ([1]) on partially conservative sentences. The purpose of this note is to describe a fairly general method (fixed-point construction) by means of which a more direct proof can be obtained and to give some examples of applications of this method including the result of Guaspari just mentioned. This paper may be compared with Smoryński's paper ([8]).

1 ***Preliminaries*** Let T be a consistent r.e. theory. For simplicity we shall assume that T is an extension of Peano arithmetic P. (Notation and terminology not explained here are standard.) Let G be a new predicate. Formulas containing G will be written $\zeta(G;\tilde{x})$, $\chi(G;\tilde{x})$. (Here $\tilde{x}$ is short for $x_0,\ldots,x_{r-1}$. Similarly we write $\tilde{k}$ for $k_0,\ldots,k_{r-1}$ and $\bar{\tilde{k}}$ for $\bar{k}_0,\ldots,\bar{k}_{r-1}$.) For simplicity we assume that G is monadic. The extension of the results of Sections 2 and 4 to formulas containing polyadic predicates is perfectly straightforward. If $\xi(x)$ is any formula, then $\zeta(\xi;\tilde{x})$ is obtained from ζ by replacing G by $\xi(x)$ avoiding clashes of variables in the usual way. To prevent confusion we sometimes use the notation $\lambda x\xi(x)$. In the following we always assume that χ is *positive* in G in the sense that for any arithmetical formulas $\xi_0(x)$ and $\xi_1(x)$,

$$P \vdash \chi(\xi_0;\tilde{x}) \wedge \forall x(\xi_0(x) \rightarrow \xi_1(x)) \rightarrow \chi(\xi_1;\tilde{x}).$$

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Γ is either Σ^0_{n+1} or Π^0_{n+1}. ζ is $\Gamma[G]$ if $\zeta(\xi;\tilde{x})$ is Γ whenever $\xi(x)$ is PR. If X is any set (of natural numbers), then $X \upharpoonright q = \{n \in X: n \leq q\}$. If X is finite, then $[X](x) =: \bigvee\{x = \bar{k}: k \in X\}$. (We use $=:$ to denote equality between formulas.) The obvious fact that $P \vdash \neg[X](\bar{k})$ for $k \notin X$ will be used repeatedly without further comment.

2 *χ-generic formulas*

Let us say that $\xi(x)$ is *χ-generic* in T if for all $\tilde{k}$, if $T \vdash \chi(\xi;\bar{\tilde{k}})$, then there is a q such that $T \vdash \chi([X \upharpoonright q]; \bar{\tilde{k}})$ where X is the set numerated by $\xi(x)$ in T.

Proposition 1 *Suppose $\chi(G;\tilde{x})$ is $\Gamma[G]$ and let X be any r.e. set. There is then a Γ χ-generic numeration $\xi(x)$ of X in T.*

Proof: For simplicity suppose $r = 1$. By Craig's theorem, we may assume that T is primitive recursive. Let $\tau(x)$ be a PR binumeration of T and let $\kappa(x,y)$ be a PR formula such that $X = \{k: \exists m P \vdash \kappa(\bar{k},\bar{m})\}$.

Case 1. $\Gamma = \Sigma^0_{n+1}$. Let $\xi_0(x)$ be such that

$$P \vdash \xi_0(x) \leftrightarrow \exists y(\kappa(x,y) \wedge \forall zu \leq x + y(\mathrm{Prf}_\tau(\ulcorner\chi(\xi_0;\dot{z})\urcorner,u) \rightarrow \chi(\lambda w \exists v(v + w \leq z + u \wedge \kappa(w,v)); z))).$$

Then

(*) if (a) p is a proof of $\chi(\xi_0;\bar{k})$ in T, then (b)

$$T \vdash \chi(\exists y(x + y \leq \overline{k+p} \wedge \kappa(x,y));\bar{k}).$$

For suppose (a) holds. Let $\gamma(x) =: \exists y(x + y \leq \overline{k+p} \wedge \kappa(x,y))$. Then

$$P \vdash \neg\chi(\gamma(w);\bar{k}) \rightarrow (\xi_0(x) \rightarrow \gamma(x)).$$

Since χ is positive in G, it follows that

$$P \vdash \neg\chi(\gamma(w);\bar{k}) \rightarrow (\chi(\xi_0;\bar{k}) \rightarrow \chi(\gamma(x);\bar{k})).$$

But, since $T \vdash \chi(\xi_0;\bar{k})$, this implies (b).

Now let $\xi_1(x)$ be a Σ^0_1 numeration of X in T and let

$$\xi(x) =: \xi_0(x) \wedge \xi_1(x).$$

Then, again since χ is positive in G, it follows at once from (*) that for all k, if $T \vdash \chi(\xi;\bar{k})$, then there is a q such that $T \vdash \chi([X \upharpoonright q];\bar{k})$. Thus to complete the proof we need only show that $\xi(x)$ numerates X in T. To prove this it suffices to show that for all k,p,

$$T \vdash \mathrm{Prf}_\tau(\ulcorner\chi(\xi_0;\bar{k})\urcorner,\bar{p}) \rightarrow \chi(\gamma;\bar{k}).$$

But this too follows at once from (*) and so the proof is complete.

Case 2. $\Gamma = \Pi^0_{n+1}$. Let $\xi_0(x)$ be such that

$$P \vdash \xi_0(x) \leftrightarrow \forall zu(\mathrm{Prf}_\tau(\ulcorner\chi(\xi_0;\dot{z})\urcorner,u) \wedge \neg\chi(\lambda w \exists v(v + w \leq z + u \wedge \kappa(w,v));z) \rightarrow \exists y(x + y \leq z + u \wedge \kappa(x,y))).$$

Next let $\xi_1(x)$ be a Π^0_1 numeration of X in T and define $\xi(x)$ as before. The proof that $\xi(x)$ is as desired is then almost the same as in Case 1.

The following variant of Proposition 1 is occasionally useful but will not be applied in this paper except indirectly via Corollaries 1 and 2. Let us say that χ is *decidable* in T if $\chi([X];\tilde{x})$ is (numeralwise) decidable in T for every finite set X. A set Y is *monoconsistent* with T if $T + \phi$ is consistent for every $\phi \in Y$. $\xi(x)$ will be said to be *χ-generic* with respect to Y in T if for all $\tilde{k}$, if $\chi(\xi;\bar{\tilde{k}}) \in Y$, then there is a q such that $T \vdash \chi([X \upharpoonright q];\bar{\tilde{k}})$ where X is the set numerated by $\xi(x)$ in T.

Proposition 2 *Let χ, Γ, and X be as in Proposition 1. Suppose χ is decidable in T and Y is r.e. and monoconsistent with T. There is then a Γ formula $\xi(x)$ numerating X in T and χ-generic with respect to Y in T.*

Proof: We may assume that Y is *closed* in the sense that if $\phi \in Y$ and $T \vdash \phi \to \psi$, then $\psi \in Y$. Let $\mu(x,y)$ be a PR formula such that $Y = \{k\colon \exists m P \vdash \mu(\bar{k},\bar{m})\}$. Now replace $\mathrm{Prf}_\tau(x,y)$ by $\mu(x,y)$ in the proof of Proposition 1 and use the obvious fact that if $\psi \in Y$ and ψ is decidable in T, then $T \vdash \psi$.

Propositions 1 and 2 and the relevant notation and terminology can be generalized in a straightforward way to formulas $\chi(G_0,\ldots,G_n;\tilde{x})$ containing several new (monadic) predicates and positive in these predicates in the obvious sense. Moreover we can now easily prove the following corollary to the proof of Proposition 2.

Corollary 1 *Suppose $\chi(G_0,\ldots,G_n;\tilde{x})$ is $\Delta^0_1[G_0,\ldots,G_n]$ and Y is r.e. and monoconsistent with T. Let $X_0,\ldots,X_n$ be a sequence of r.e. sets and suppose $m \le n$. There are then formulas $\xi_i(x)$ such that $\xi_i(x)$ is Σ^0_1 for $i < m$, $\xi_i(x)$ is Π^0_1 for $m \le i \le n$, $\xi_i(x)$ numerates X_i in T for $i \le n$, and the sequence $\xi_0(x),\ldots,\xi_n(x)$ is χ-generic with respect to Y in T.*

Proof: For simplicity suppose $m = n = 1$. Let $\mu(x,y)$ be as in the proof of Proposition 2 and let $\kappa_i(x,y)$ be a PR formula such that $X_i = \{k\colon \exists m\, P \vdash \kappa_i(\bar{k},\bar{m})\}$, $i = 0,1$. Next let $\xi_{00}(x)$ and $\xi_{10}(x)$ be such that

$$P \vdash \xi_{00}(x) \leftrightarrow \exists y(\kappa_0(x,y) \wedge \forall zu \le x + y\, \delta(z,u)),$$
$$P \vdash \xi_{10}(x) \leftrightarrow \forall zu(\neg\delta(z,u) \to \exists y(x + y \le z + u \wedge \kappa_1(x,y))),$$

where $\delta(z,u) =:$

$$\mu(\ulcorner \chi(\xi_{00},\xi_{10};\dot{z})\urcorner,u) \to$$
$$\chi(\lambda w \exists v(v + w \le z + u \wedge \kappa_0(w,v)),\ \lambda w\ \exists v(v + w \le z + u \wedge \kappa_1(w,v));z).$$

Finally let $\xi_{01}(x)$ and $\xi_{11}(x)$ be a Σ^0_1 and a Π^0_1 numeration of X in T, respectively, and let

$$\xi_i(x) =: \xi_{i0}(x) \wedge \xi_{i1}(x).$$

Since χ is decidable in T, being $\Delta^0_1[G_0,\ldots,G_n]$, the rest of the proof is now very much the same as the proof of Proposition 2.

We conclude this Section with a result designed to yield Application 4 below.

Corollary 2 *Suppose $\chi(G;x)$ is $\Delta_1^0[G]$, X and Y are r.e., and Y is monoconsistent with P. There are then a Π_1^0 formula $\xi_0(x)$ and a Σ_1^0 formula $\xi_1(x)$ such that*
(i) $P \vdash \xi_1(x) \rightarrow \xi_0(x)$,
(ii) *if* $k \in X$, *then* $P \vdash \xi_1(\bar{k})$,
(iii) *if* $\chi(\xi_0;\bar{k}) \in Y$, *then there is a q such that* $P \vdash \chi([X \upharpoonright q];\bar{k})$.

Proof: Let $\kappa(x,y)$, $\mu(x,y)$, and $\xi_0(x)$ be as in the proof of Case 2 of Proposition 2 and let $\xi_1(x) =:$

$$\exists y(\kappa(x,y) \wedge \forall zu \leq x + y(\mu(\ulcorner\chi(\xi_0;\dot{z})\urcorner,u) \rightarrow \chi(\lambda w \exists v(v + w \leq z + u \wedge \kappa(w,v));z))).$$

Then (i) is obvious, (iii) is proved in the same way as before using the fact that χ is decidable in P, and (ii) follows from the proof of (iii).

*3 **Some applications*** We can now easily solve the problem mentioned in the introduction.

Application 1 If S is r.e., then there is a Σ_1^0 numeration $\sigma(x)$ of S in T such that $\mathrm{Pr}_\sigma(x)$ numerates $\mathrm{Th}_T(S)$ in T.

Proof: Let $\chi(G;x) =: \mathrm{Pr}_G(x)$ and apply Proposition 1.

A sentence ϕ is Γ-*conservative* over T if $T + \phi \vdash \psi$ implies $T \vdash \psi$ for every Γ sentence ψ. The following result is due to Guaspari ([1]) (cf. also [3], [7], [8]).

Application 2 To any r.e. set X, there is a Γ formula $\xi(x)$ such that
(i) if $k \in X$, then $T \vdash \xi(\bar{k})$,
(ii) if $k \notin X$, then $\neg\xi(\bar{k})$ is Γ-conservative over T.

Proof: Let $\Gamma\text{-true}(x)$ be a Γ partial truth-definition for Γ sentences and let $\chi(G;x,y) =: Gx \vee \Gamma\text{-true}(y)$. Then any Γ χ-generic numeration of X in T is as desired.

Let $\xi(x)$ be as in Application 2 with $X = \{\gamma(x): T \vdash \neg\gamma(\bar{\gamma})\}$ and let $\theta =: \neg\xi(\bar{\xi})$. Then $T \nvdash \theta$ and θ is Γ-conservative over T. This can be improved as follows (cf. [1]). (The somewhat stronger results proved in [3] and [8] can be obtained in a similar way. See also Application 7 below.)

Application 3 There is a Π_{n+1}^0 sentence θ such that θ and $\neg\theta$ are Σ_{n+1}^0- and Π_{n+1}^0-conservative over T, respectively.

Proof: There is a Γ formula $\Gamma\text{-sat}(x,y)$ such that for every Γ formula $\gamma(x)$,

$$P \vdash \gamma(x) \leftrightarrow \Gamma\text{-sat}(x,\bar{\gamma}).$$

Let

$$\chi_0(G;x,y) =: \exists z(Gz \wedge \neg\, \Pi_{n+1}^0\text{-sat}(z,x)) \vee \Sigma_{n+1}^0\text{-true}(y),$$
$$\chi_1(G;x,y) =: \forall z(\Sigma_{n+1}^0\text{-sat}(z,x) \rightarrow Gz) \vee \Pi_{n+1}^0\text{-true}(y).$$

By Proposition 1, there are a Σ_{n+1}^0 formula $\xi_0(x)$ and a Π_{n+1}^0 formula $\xi_1(x)$ such that $\xi_i(x)$ is a χ_i-generic numeration of ω in T, $i = 0,1$. Let

$$\theta =: \forall x(\xi_0(x) \rightarrow \xi_1(x)).$$

Now let ϕ be a Σ^0_{n+1} sentence such that $T + \theta \vdash \phi$. Then $T \vdash \chi_0(\xi_0;\bar{\xi}_1,\bar{\phi})$. Hence there is a q such that $T \vdash \chi_0(x \leq \bar{q};\ \bar{\xi}_1,\bar{\phi})$, whence $T \vdash \exists x \leq \bar{q} \neg \xi_1(x) \vee \phi$. But clearly $T \vdash \neg \exists x \leq \bar{q} \neg \xi_1(x)$ and so $T \vdash \phi$. Thus θ is Σ^0_{n+1}-conservative over T. The proof that $\neg\theta$ is Π^0_{n+1}-conservative over T is similar.

The following result is proved in [3] (Lemma 5).

Application 4 Let X and Y be r.e. sets and suppose Y is monoconsistent with P. There are then a Π^0_1 formula $\xi_0(x)$ and a Σ^0_1 formula $\xi_1(x)$ such that
(i) $P \vdash \xi_1(x) \rightarrow \xi_0(x)$
(ii) if $k \in X$, then $P \vdash \xi_1(\bar{k})$,
(iii) if $k_s \notin X$, $s \leq n$, then $\bigvee\limits_{s \leq n} \xi_0(\bar{k}_s) \notin Y$.

Proof: Let $\chi(G;x) =: \exists y < lh(x) G(x)_y$. By Corollary 2, there are a Π^0_1 formula $\xi_0(x)$ and a Σ^0_1 formula $\xi_1(x)$ such that (i) and (ii) hold and if $\chi(\xi_0;\bar{k}) \in Y$, then there is a q such that $P \vdash \chi([X \restriction q];\bar{k})$. But then it is straightforward to show that (iii) is satisfied, since we may assume that Y is closed.

Let X_0 and X_1 be disjoint r.e. sets. It is an old result of Putnam and Smullyan (cf. [7]) that there is then a Σ^0_1 formula $\xi(x)$ such that $\xi^i(x)$ numerates X_i in T, $i = 0,1$. (Here and in what follows $\xi^0(x) =: \xi(x)$ and $\xi^1(x) =: \neg\xi(x)$.) This can be improved as follows.

Application 5 Suppose X_0 and X_1 are disjoint r.e. sets. There is then a Σ^0_1 formula $\xi(x)$ such that
(i) if $k \in X_i$, then $P \vdash \xi^i(\bar{k})$, $i = 0,1$,
(ii) if $k_s \notin X_0 \cup X_1$, $k_s \neq k_t$ for $t < s \leq n$, and $f \in {}^{n+1}2$, then $T \nvdash \bigvee \{\xi^{f(s)}(\bar{k}_s)\colon s \leq n\}$.

Proof: By a well-known result of Mostowski (cf. [7]), there is a Σ^0_1 formula $\mu(x)$ which is independent over T. Let

$$Y = \bigcup\{\mathrm{Th}(T \cup \{\mu^{g(k)}(\bar{k})\colon k \leq n\})\colon n \in \omega \ \&\ g \in {}^{n+1}2\}.$$

Then Y is r.e. and monoconsistent with P. Let

$$\chi(G_0,G_1;x_0,x_1) =: \bigvee_{i=0,1} \exists y < lh(x_i) G_i(x_i)_y.$$

By Corollary 1, there are a Σ^0_1 formula $\xi_0(x)$ and a Π^0_1 formula $\xi_1(x)$ such that $\xi_i(x)$ numerates X_i in P and $\xi_0(x)$, $\xi_1(x)$ is χ-generic with respect to Y in P. Let $\rho_i(x,y)$ be a PR formula such that $X_i = \{k\colon \exists m P \vdash \rho_i(\bar{k},\bar{m})\}$ and let $\nu(x) =: \forall y(\rho_0(x,y) \rightarrow \exists z \leq y \rho_1(x,z))$. Finally let

$$\xi(x) =: (\xi_0(x) \vee \mu(x)) \wedge \neg(\xi_1(x) \wedge \nu(x)).$$

If $k \in X_i$, then $P \vdash \nu^{1-i}(\bar{k})$ and so (i) holds. To prove (ii), suppose $T \vdash \bigvee \{\xi^{f(s)}(\bar{k}_s)\colon s \leq n\}$. Then, by propositional logic, $T \vdash \bigvee \{\xi_i(\bar{k}_s)\colon i = 0,1 \ \&\ s \leq n\} \vee \bigvee \{\mu^{f(s)}(\bar{k}_s)\colon s \leq n\}$, whence $\bigvee \{\xi_i(\bar{k}_s)\colon i = 0,1 \ \&\ s \leq n\} \in Y$. But then there is a q such that $P \vdash \bigvee \{[X_i \restriction q](\bar{k}_s)\colon i = 0,1 \ \&\ s \leq n\}$. It follows that $k_s \in X_0 \cup X_1$ for some $s \leq n$. This proves (ii) and so concludes the proof.

There are a number of simple formulas χ, in addition to those already mentioned, that naturally come to mind. One of them is

$$\chi_0(G;x) =: \exists y(Gy \wedge \Sigma^0_{n+1}\text{-sat}(x,y)).$$

Applying Proposition 1 to this formula we get the case $\Gamma = \Sigma^0_{n+1}$ of the following

Application 6 Let X be any r.e. set of Γ formulas and let

$$Y = \left\{k\colon \exists q T \vdash \bigvee \{\xi(\bar{k})\colon \xi(x) \in X \restriction q\}\right\}.$$

There is then a Γ formula $\eta(x)$ such that $T \vdash \xi(x) \to \eta(x)$ for every $\xi(x) \in X$ and $\eta(x)$ numerates Y in T.

Proof: Case 1. $\Gamma = \Sigma^0_{n+1}$. Let $\mu(x)$ be a Σ^0_{n+1} χ_0-generic numeration of X in T and let $\eta(x) =: \chi_0(\mu;x)$.

Case 2. $\Gamma = \Pi^0_{n+1}$. Let $\rho(x,y)$ be a PR formula such that $X = \{k\colon \exists m P \vdash \rho(\bar{k},\bar{m})\}$ and let $\chi_1(G;x) =:$

$$\forall y(\forall zu \le y(\rho(z,u) \to \neg\Pi^0_{u+1}\text{-sat}(x,z)) \to Gy).$$

Next let $\nu(x)$ be a Π^0_{n+1} χ_1-generic numeration of ω in T and let $\eta(x) =: \chi_1(\nu;x)$.

4 *Γ-generic formulas*

Let us say that $\xi(x)$ is *Γ-generic* in T if $\xi(x)$ is Γ and χ-generic in T for every $\Gamma[G]$ formula χ.

Proposition 3 *If X is r.e., then there is a Γ-generic numeration of X in T.*

Proof: There is a $\Gamma[G]$ formula Γ-true$(G;x)$ such that for every $\Gamma[G]$ sentence $\zeta(G)$ (not necessarily positive in G) and every arithmetical formula $\gamma(x)$,

$$P \vdash \zeta(\gamma) \leftrightarrow \Gamma\text{-true}(\gamma;\overline{\zeta(G)}).$$

Let $\chi_n(G)$, $n = 0,1,2,\ldots$, be a primitive recursive enumeration of all $\Gamma[G]$ sentences $\chi(G)$ positive in G and let $\delta(x,y)$ be a PR formula such that $P \vdash \forall y(\delta(\bar{n},y) \leftrightarrow y = \bar{\chi}_n)$ for every n. Next let

$$\chi_\Gamma(G;x) =: \exists y(\delta(x,y) \wedge \Gamma\text{-true}(G;y)).$$

χ_Γ is not necessarily positive in G but for every k, $\chi_\Gamma(G;\bar{k})$ is positive in G and this is sufficient to show that there is a Γ χ_Γ-generic numeration $\xi(x)$ of X in T (see the proof of Proposition 1). Clearly $\xi(x)$ is χ-generic in T for every $\Gamma[G]$ sentence $\chi(G)$. It follows that $\xi(x)$ is Γ-generic in T.

Proposition 3 can, of course, be used to give somewhat simplified proofs of applications of Proposition 1. We illustrate this by proving the following result which is proved in [8] (Application 4) and is also an immediate consequence of Lemma 3 of [3].

Application 7 Let X_0 and X_1 be disjoint r.e. sets. There is then a Π^0_{n+1} formula $\xi(x)$ such that

(i) if $k \in X_i$, then $T \vdash \xi^i(\bar{k})$, $i = 0,1$,
(ii) if $k \notin X_0 \cup X_1$, then $\xi(\bar{k})$ and $\neg\xi(\bar{k})$ are Σ^0_{n+1}- and Π^0_{n+1}-conservative over T, respectively.

Proof: Let $\mu_i(x,y)$ be a PR formula such that $X_i = \{k: \exists m P \vdash \mu_i(\bar{k},\bar{m})\}$. Let $\xi_0(x)$ and $\xi_1(x)$ be a Σ^0_{n+1}- and a Π^0_{n+1}-generic numeration of ω in T, respectively. Finally let $\xi(x) =:$

$$\forall z(\xi_0(z) \wedge \forall y \leq z \neg \mu_0(x,y) \rightarrow \xi_1(z) \wedge \neg \mu_1(x,z)).$$

It is then straightforward to check that $\xi(x)$ is as claimed.

5 ***A model-theoretic application*** Let M be any model of P and let $\text{nst}(M)$ be the set of nonstandard elements of M. We define the sets of Γ-*isolated* and Γ-*small* elements of M as follows.

$$\Gamma\text{-isol}(M) = \{a \in \text{nst}(M)\text{: there is a } \Gamma \text{ formula } \gamma(x) \text{ such that } a \text{ satisfies } \gamma(x) \wedge \forall y < x \neg \gamma(y) \text{ in } M\},$$
$$\Gamma\text{-small}(M) = \{a \in \Gamma\text{-isol}(M)\text{: } a \leq_M b \text{ for every } b \in \check{\Gamma}\text{-isol}(M)\},$$

where $\check{\Gamma}$ is the dual of Γ. A set $X \subseteq \text{nst}(M)$ is *coinitial* in M if for every $a \in \text{nst}(M)$, there is a $b \in X$ such that $b \leq_M a$. (For results on the existence of models M in which Γ-isol(M) is coinitial see [5] and [6].) Clearly if Γ-small$(M) \neq \varnothing$, then $\check{\Gamma}$-isol(M) is not coinitial in M and $\check{\Gamma}$-small$(M) = \varnothing$.

Application 8 There is a set B of Γ sentences such that $T \cup B$ is consistent and Γ-small$(M) \neq \varnothing$ for every model M of $T \cup B$.

Proof: Case 1. $\Gamma = \Sigma^0_{n+1}$. Let $\xi(x)$ be a Π^0_{n+1}-generic numeration of ω in T and let $\{\gamma_s(x): s \in \omega\}$ be a maximal set of Σ^0_{n+1} formulas such that $S = T \cup \{\exists x \geq \bar{m}\gamma_s(x): m,s \in \omega\}$ is consistent. Finally let

$$B = \{\exists x(\gamma_s(x) \wedge \neg \xi(x)): s \in \omega\}.$$

Then $T \cup B$ is consistent. For suppose not. Then there is a p such that $T \vdash \bigvee_{s \leq p} \forall x(\gamma_s(x) \rightarrow \xi(x))$. But then, since $\xi(x)$ is Π^0_{n+1}-generic in T, there is a q such that $T \vdash \bigvee_{s \leq p} \forall x(\gamma_s(x) \rightarrow x \leq \underline{q})$. It follows that S is inconsistent, contrary to hypothesis. Thus $T \cup B$ is consistent.

Now let M be any model of $T \cup B$ and let a be the member of M satisfying $\neg \xi(x) \wedge \forall y < x \xi(y)$ in M. Then $a \in \text{nst}(M)$, since $M \vDash \xi(\bar{k})$ for every k. To show that $a \in \Sigma^0_{n+1}$-small(M) suppose $b \in \Pi^0_{n+1}$-isol(M). Let $\beta(x)$ be a Π^0_{n+1} formula such that b satisfies $\beta(x) \wedge \forall y < x \neg \beta(y)$ in M. Since b is nonstandard, $\forall y < x \neg \beta(y)$ is $\gamma_s(x)$ for some s. But then there is a c satisfying $\forall y < x \neg \beta(y) \wedge \neg \xi(x)$ in M. Clearly $a \leq_M c \leq_M b$ and so $a \leq_M b$. Thus $a \in \Sigma^0_{n+1}$-small(M) as was to be shown.

Case 2. $\Gamma = \Pi^0_{n+1}$. We may assume that T is not Σ^0_1-sound. Let $\delta(x)$ be a PR formula such that $T \vdash \neg \delta(\bar{k})$ for every k and $T \vdash \exists x \delta(x)$. Let $\xi(x)$ be a Σ^0_{n+1}-generic numeration of ω in T and let $\{\gamma_s(x): s \in \omega\}$ be a maximal set of Σ^0_{n+1} formulas such that $T \cup B$ is consistent where

$$B = \{\forall x(\gamma_s(x) \rightarrow \neg \xi(x)): s \in \omega\}.$$

Now let M be any model of $T \cup B$. Then for every Σ^0_{n+1} formula $\sigma(x)$,

(+) if $M \vDash \neg \sigma(\bar{k})$ for every k, then $\forall x(\sigma(x) \rightarrow \neg \xi(x)) \in B$.

For suppose $\forall x(\sigma(x) \to \neg\xi(x)) \notin B$. Then $T \cup B \vdash \exists x(\sigma(x) \wedge \xi(x))$. But then there is a q such that $T \cup B \vdash \exists x \leq \bar{q}\sigma(x)$ and so $M \models \sigma(\bar{k})$ for some k.

By (+), $\forall x(\delta(x) \to \neg\xi(x)) \in B$ and so $M \models \exists x \neg\xi(x)$. Let a be the member of M satisfying $\neg\xi(x) \wedge \forall y < x\xi(y)$ in M. Then $a \in \Pi^0_{n+1}$-isol(M). Suppose $b \in \Sigma^0_{n+1}$-isol(M) and let $\beta(x)$ be a Σ^0_{n+1} formula such that b satisfies $\beta(x) \wedge \forall y < x \neg\beta(y)$ in M. Then $M \models \neg\beta(\bar{k})$ for every k and so, by (+), $M \models \forall x(\beta(x) \to \neg\xi(x))$. Thus b satisfies $\neg\xi(x)$ in M and so $a \leq_M b$. It follows that $a \in \Pi^0_{n+1}$-small(M) as was to be shown.

Application 8 is a partial refinement of a result of [2] (Theorem 4.2.8). It can be generalized without difficulty to theories T that are Γ-*selfbinumerable* in the sense that there is a Γ formula $\tau(x)$ binumerating T in T. (Thus we no longer assume that T is r.e.) This result is, of course, derived from a suitable extension of Proposition 1 to Γ-selfbinumerable theories. Adding an argument due to McAloon ([5] and [6]) it can be shown that if T is Γ-selfbinumerable, then T has a model M in which Γ-isol(M) is coinitial but $\check{\Gamma}$-isol(M) is not. Finally, by repeated applications of the strengthened version of Application 8, we obtain a set C of sentences such that $T \cup C$ is consistent and if M is any model of $T \cup C$, then Σ^0_{n+1}-small$(M) \neq \varnothing$ for every n. It follows that T has a pointwise definable nonstandard model M such that for every Γ, Γ-isol(M) is not coinitial in M. This is essentially Theorem 3.4 of [6].

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