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The Number of Nonnormal Extensions of $S4$

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Thanks to the work of Jankov ([3]) and Fine ([2]), we know that there are uncountably many normal extensions of $S4$—the most thoroughly studied of all modal logics. Likewise, Segerberg ([7]) has shown that there are uncountably many nonnormal extensions of $K4$ (indeed, even $K4Grz$). But his method of proof does not cover $S4$, and it is natural to wonder how many nonnormal extensions that logic has. That such things exist at all was established nearly forty years ago by McKinsey and Tarski ([4]), though not long thereafter Scroggs ([5]) showed that no nonnormal logics extend $S5$, and, more recently, Segerberg ([6]) has proved that none extend even $S4.3$. Are they, then, just isolated curiosities, or are there enough of them to form a potentially worthy topic of investigation? Curiosities or not, there are in fact a slew of them.

Theorem *There exist $2^{\aleph_0}$ nonnormal extensions of $S4$.*

Proof: Fine shows how to construct reflexive transitive frames $\mathfrak{F}_i = (W_i, R_i)$ and formulas α_j such that $\mathfrak{F}_i$ validates α_j iff $i \neq j$. Since each $\mathfrak{F}_i$ is finite, we can suppose that these frames are pairwise disjoint and $W_i \subset \{j \mid j \geqq 6\}$. For any nonempty $\Gamma \subseteq \omega$, let $\mathfrak{F}_\Gamma = (W_\Gamma, R_\Gamma)$ where

$$W_\Gamma = \bigcup_{i \in \Gamma} W_i \cup \{3,4,5\},$$

$$R_\Gamma = \bigcup_{i \in \Gamma} R_i \cup \{(5,j) \mid j \in W_\Gamma\} \cup \{3,4\}^2.$$

The frame $\mathfrak{F}_\Gamma$ is nothing more than a jazzed-up version of the one used by McKinsey and Tarski (see [4], Theorem 3.1), in which their world 2 has been replaced by the family of frames $\{\mathfrak{F}_i \mid i \in \Gamma\}$. The trick now will be to show that each $\mathfrak{F}_\Gamma$ determines a distinct nonnormal extension of $S4$.

Let

$$L(\Gamma) = \{\alpha \mid (\mathfrak{A}, 5) \models \alpha \text{ for all } \mathfrak{A} \text{ based upon } \mathfrak{F}_\Gamma\}.$$

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Then:

(1) $L(\Gamma)$ is an extension of $S4$.

$\mathfrak{F}_\Gamma$ is a reflexive and transitive frame, and $L(\Gamma)$ is closed under both modus ponens and substitution.

(2) For distinct $\Gamma, \Delta \subseteq \omega$, $L(\Gamma) \neq L(\Delta)$.

Suppose $\Gamma \neq \Delta$. Then $i \in \Gamma - \Delta$, say, for some $i \in \omega$. Let ζ be the formula

$$(p \wedge \sim\Box\beta) \rightarrow \Box((\sim p \wedge \beta) \rightarrow \alpha_i)$$

where β is $\Diamond(\Diamond p \rightarrow \Box p)$ and p is any variable not appearing in α_i. Now suppose $(\mathfrak{A},5) \models p \wedge \sim\Box\beta$ for $\mathfrak{A}$ based upon $\mathfrak{F}_\Delta$. Then $(\mathfrak{A},5) \models p$ and $(\mathfrak{A},j) \not\models \beta$ for some $j \in W_\Delta$. But each of Fine's frames validates β and Δ is nonempty, so we have $(\mathfrak{A},k) \models \beta$ for $k \in W_\Delta - \{3,4\}$. So $j = 3$ or 4, from which it follows that $(\mathfrak{A},n) \not\models \beta$ for both $n = 3$ and $n = 4$. Thus, if $k \in W_\Delta$ and $(\mathfrak{A},k) \models \sim p \wedge \beta$, then $k \in W_h$ for some $h \neq i$, so $(\mathfrak{A},k) \models \alpha_i$. Hence $(\mathfrak{A},5) \models \Box((\sim p \wedge \beta) \rightarrow \alpha_i)$. But then $(\mathfrak{A},5) \models \zeta$, so $\zeta \in L(\Delta)$. On the other hand, $(\mathfrak{M},j) \not\models \alpha_i$ for some model $\mathfrak{M} = (W_i, R_i, \phi)$ based upon $\mathfrak{F}_i$ and $j \in W_i$. Now to see that $\zeta \notin L(\Gamma)$, let $\mathfrak{B} = (W_\Gamma, R_\Gamma, \psi)$ where $\psi(p) = \{3,5\}$ and $\psi(q) = \phi(q)$ for each variable q in α_i. Then $(\mathfrak{B},3) \not\models \beta$, so $(\mathfrak{B},5) \models p \wedge \sim\Box\beta$. But $(\mathfrak{B},j) \models \sim p \wedge \beta$ and $(\mathfrak{B},j) \not\models \alpha_i$, so $(\mathfrak{B},5) \not\models \Box((\sim p \wedge \beta) \rightarrow \alpha_i)$. Hence $(\mathfrak{B},5) \not\models \zeta$.

(3) $L(\Gamma)$ is nonnormal.

Letting β be as before, pick $\mathfrak{F}_i \in \Gamma$ and $j \in W_i$. Since $(\mathfrak{A},j) \models \beta$, we have $(\mathfrak{A},5) \models \beta$ for all models $\mathfrak{A}$ based upon $\mathfrak{F}_\Gamma$. But then $\beta \in L(\Gamma)$. On the other hand, let $\mathfrak{B} = (W_\Gamma, R_\Gamma, \phi)$ where $\phi(p) = \{3\}$. Then $(\mathfrak{B},3) \not\models \beta$. It follows that $(\mathfrak{B},5) \not\models \Box\beta$, so $\Box\beta \notin L(\Gamma)$.

(1)–(3) give the result.[1]

NOTE

1. An alternative proof can be derived from the work of Blok and Köhler ([1]) using results from [2] and [4]. In fact, [1] provides a powerful framework in which to conduct the study of nonnormal logics generally and already sheds considerable light upon such extensions of $S4$. I am indebted to the referee for this and other excellent comments.

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