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De Re and *De Dicto*

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It is widely recognized that the modal sentence 'Necessarily, Socrates is wise' can be interpreted in two different ways. Under the first interpretation, the *de re* interpretation, it is understood as saying that Socrates has the property of being wise essentially; i.e., that Socrates has the property of being wise in every possible world in which Socrates exists. Under the second interpretation, the *de dicto* interpretation, it is understood as saying that the proposition *Socrates is wise* is true in every possible world. In [3] and [4] Plantinga has suggested a way of understanding the notion of necessity using the concept of *essence*: a property E which is exemplified in some possible world and is such that, in every possible world, for every x, if x has E then: (a) x has E essentially and (b) in no world does anything distinct from x have E. Using this notion of essence, an applied semantics can be introduced for both *de re* and *de dicto* necessity which satisfies the doctrine of *serious actualism*, that, necessarily, there are no objects that do not exist and objects can have properties only in worlds in which they do exist. For a first-order modal language L, a corresponding formal semantics for *de re* necessity (and denial) and *de dicto* necessity (and denial) can be introduced (the systems A and A^* of [2]). Because L contains a single necessity operator and a single denial operator, it does not allow for the simultaneous treatment of both kinds of interpretations. In this paper the language L is extended by introducing a new class of operators; the result is a language rich enough to support a semantics treating *de re* and *de dicto* notions simultaneously. The formal semantics introduced is characterized axiomatically.

Initially one might think that the problem of formally representing the two senses of necessity simultaneously can be solved by introducing two necessity operators, $\Box_1$ and $\Box_2$, so that

(1) $\Box_1$ (Socrates is wise)

is interpreted as *Socrates is essentially wise*, and

(2) $\Box_2$ (Socrates is wise)

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is interpreted as *'Socrates is wise' is necessarily true.* Such a solution, however, will not cover all of the possible constructions.

Consider the sentence

(3) Socrates is the teacher of Plato

and its necessitation

(4) Necessarily, Socrates is the teacher of Plato.

The latter can be interpreted in four ways:

(4a) The pair (Socrates, Plato) essentially stands in the relation 'is the teacher of'.
(4b) *Socrates is the teacher of Plato* is necessarily true.
(4c) Socrates has essentially the property of being Plato's teacher.
(4d) Plato has essentially the property of being Socrates' student.

Versions (4a) and (4b) are pure *de re* and *de dicto* versions of (4), while (4c) and (4d) are hybrids. In worlds in which both Plato and Socrates exist, (4a) is the weakest claim (requiring (3) to be true only in those worlds in which both Socrates and Plato exist), (4b) is the strongest claim (requiring (3) to be true in all worlds), and (4c) and (4d) are intermediate (requiring (3) to be true in all worlds in which Socrates or Plato exists).

Negation displays the same complex behavior. Consider the denial of (3):

(5) It is not the case that Socrates is the teacher of Plato.

There are four possible readings:

(5a) The pair (Socrates, Plato) stand in the relation 'not the teacher of'.
(5b) *Socrates is the teacher of Plato* is false.
(5c) Socrates has the property of not being the teacher of Plato.
(5d) Plato has the property of not being Socrates' student.

In any world in which both Socrates and Plato exist (5a)–(5d) have the same truth value, but in worlds in which one or the other fails to exist the truth values may differ. For example, assuming the doctrine of serious actualism, if w is a world in which Socrates exists but Plato does not, (5a) and (5d) must be false while (5b) and (5c) must be true. Versions (5a) and (5b) are pure *de re* and *de dicto* versions of (5), while (5c) and (5d) are hybrids.

In general, if F is an n-ary predicate, the necessitation and denial of the statement $Fa_1a_2 \dots a_n$ can each be interpreted in 2^n different ways. The following formal system is rich enough to represent all of these possibilities.

1 The formal semantics D We first define the language L' for the semantics. Its primitive symbols include those typically found in first-order modal languages: individual variables $x_1, x_2, \dots,$ predicate symbols, connective and quantifier symbols, and parentheses. In addition, there is a *dictafier symbol* ∇, and for each individual variable x_i there are infinitely many *position variants* $x_i^1, x_i^2, x_i^3, \dots$. The formation rules are:

(1) if F is an n-ary predicate symbol and $y_1, \ldots, y_n$ are position variants, then $Fy_1 \ldots y_n$ is an atomic wff,
(2) if α and β are wffs, so are $(\sim\alpha)$, $(\alpha \wedge \beta)$, and $(\Box\alpha)$,
(3) if α is a wff and x_i is an individual variable, then $(\forall x_i \alpha)$ is a wff,
(4) if α is a wff and x_i^k a position variant, then $(\nabla x_i^k \alpha)$ is a wff. (An expression of the form '∇x_i^k' is called a *dictafier*.)

If α is a wff and x_i^k a position variant, an occurrence of x_i^k in α is *free* if it is not within the scope of a quantifier $\forall x_i$. Hence, the quantifier $\forall x_i$ covers all free occurrences of all position variants of x_i within its scope. A variable x_i is *free* in α if some position variant of x_i has a free occurrence in α. An occurrence of x_i^k is ∇-*free* in α if it is free in α and is not within the scope of a dictafier ∇x_i^k. A variable x_i is ∇-*free* in α if some position variant of x_i has a ∇-free occurrence in α. A dictafier ∇x_i^k binds only ∇-free occurrences of the position variant x_i^k itself. We let $f(\alpha)$ be the set of variables which are free in α and let $d(\alpha)$ be the set of variables which are ∇-free in α. If $d(\alpha)$ is empty, we say that α is ∇-*closed*. If $\{x_1, x_2, \ldots, x_n\} = d(\alpha)$, then $\forall x_1 \forall x_2 \ldots \forall x_n \alpha$ is the ∇-*closure* of α. Obviously, the ∇-closure of α is ∇-closed but not necessarily closed in the quantifier sense.

A *model structure* for L' is a quadruple $M = (D, W, \psi, \phi)$, where D and W are nonempty sets (of essences and possible worlds, respectively), ψ is a function from W to the nonempty subsets of D (for $w \in W$ we write 'D_w' for '$\psi(w)$'), and ϕ is a function which assigns to each pair (F, w), where F is an n-ary predicate symbol and $w \in W$, a set of n-tuples in D_w. In addition, we require that $\bigcup_{w \in W} D_w = D$. If M is a model structure and $w \in W$, then the pair (M, w) is a *model* for L' and will be denoted by 'M_w'.

If M is a model structure, a function θ from the individual variables of L' to D is called an *essence assignment*. We will assume that each essence assignment θ is extended by the rule $\theta(x_i^k) = \theta(x_i)$, so that its domain includes all position variants. We now define for each model M_w, assignment θ, and L'-wff α the notion that M_w satisfies α relative to θ:

(a) $M_w \models_\theta Fy_1 \ldots y_n$ iff $(\theta(y_1), \ldots, \theta(y_n)) \in \phi(F, w)$
(b) $M_w \models_\theta \alpha \wedge \beta$ iff $M_w \models_\theta \alpha$ and $M_w \models_\theta \beta$
(c) $M_w \models_\theta \forall x_i \alpha$ iff $M_w \models_{\theta'} \alpha$ for every θ' such that $\theta'(x_i) \in D_w$ and θ' has the same values as θ for all variables other than x_i
(d) $M_w \models_\theta \nabla x_i^k \alpha$ iff $M_w \models_\theta \alpha$
(e) $M_w \models_\theta \sim\alpha$ iff $\theta(x_i) \in D_w$ for all $x_i \in d(\alpha)$ and not $M_w \models_\theta \alpha$
(f) $M_w \models_\theta \Box\alpha$ iff $\theta(x_i) \in D_w$ for all $x_i \in d(\alpha)$ and $M_{w'} \models_\theta \alpha$ for every w' such that $\theta(x_i) \in D_{w'}$ for all $x_i \in d(\alpha)$.

A wff α is *valid* in D if $M_w \models_\theta \alpha$ for all M, w, and θ.

To see how the system D allows the treatment of the various *de re* and *de dicto* combinations, consider the following examples. Suppose Fz represents *Socrates is wise.* By (f) and (d)

$$M_w \models_\theta \Box \nabla z Fz \text{ iff } M_{w'} \models_\theta Fz \text{ for all } w',$$

so that $\Box\nabla zFz$ represents the *de dicto* interpretation of 'Necessarily, Socrates is wise'. By (f)

$$M_w \models_\theta \Box Fz \quad \text{iff } \theta(z) \in D_w \text{ and } M_{w'} \models_\theta Fz \text{ for all } w' \text{ such that } \theta(z) \in D_{w'},$$

so that $\Box Fz$ represents the *de re* interpretation. Similarly, $\sim\nabla zFz$ represents the *de dicto* denial

(6) *Socrates is wise* is false,

while $\sim Fz$ represents the *de re* denial

(7) Socrates is nonwise.

If Fzy represents *Socrates is the teacher of Plato*, the four varieties of necessitation (4a)–(4d) are represented by $\Box Fzy$, $\Box\nabla z\nabla yFzy$, $\Box\nabla yFzy$, and $\Box\nabla zFzy$, respectively; and the four varieties of denial (5a)–(5d) by $\sim Fzy$, $\sim\nabla z\nabla yFzy$, $\sim\nabla yFzy$, and '$\sim\nabla zFzy$.

The system D contains as fragments the pure *de re* and *de dicto* systems A and A^* introduced in [2]. The set of dictafier free wffs in L' containing only the position variants $x_1^1, x_2^2, \ldots$, obviously gives the system A, and the set of L'-wffs generated from formulas of the form $\nabla y_1 \ldots \nabla y_n Fy_1 \ldots y_n$, where y_j is a position variant of the form x_i^j, by denial, conjunction, quantification, and necessitation gives the system A^*.

It follows easily by induction that if $M_w \models_\theta \alpha$, then $\theta(x_i) \in D_w$ for all $x_i \in d(\alpha)$. Consequently, every valid wff must be ∇-closed (though not necessarily closed). For example, of the wffs $\forall x_i(Fx_i^1 \vee \sim Fx_i^1)$, $\nabla x_i^1 Fx_i^1 \vee \sim\nabla x_i^1 Fx_i^1$, $Fx_i^1 \vee \sim Fx_i^1$, and $\nabla x_i^1(Fx_i^1 \vee \sim Fx_i^1)$, only the first two are D-valid. The last wff, $\nabla x_i^1(Fx_i^1 \vee \sim Fx_i^1)$, has an interesting and useful property. Since $M_w \models_\theta \nabla x_i^1(Fx_i^1 \vee \sim Fx_i^1)$ iff $\theta(x_i) \in D_w$, $\nabla x_i^1(Fx_i^1 \vee \sim Fx_i^1)$ is a ∇-closed wff which functions as an internally defined exemplification predicate; i.e., it is true of an essence in a world iff that essence is exemplified in that world. In what follows, the wff $\nabla x_i^1(Fx_i^1 \vee \sim Fx_i^1)$ (for a fixed predicate symbol F) will be abbreviated by ϵx_i.

Let α be a wff and x_i and x_j individual variables. We say that x_i is *free for* x_j in α if no free occurrence of a position variant of x_i is within the scope of a quantifier $\forall x_j$. If x_i is free for x_j in α, then a substitution of variants of x_j for all the free occurrences of variants of x_i is a *good substitution* if: (1) no position variant of x_j which is used has a free occurrence in $\forall x_i \alpha$, (2) distinct variants of x_i are replaced by distinct variants of x_j, and (3) multiple occurrences of x_i^k are replaced by the same variant of x_j. Any result of such a good substitution will be denoted by $\alpha[x_i|x_j]$. That $\alpha[x_i|x_j]$ is ambiguous will present no difficulty in what follows.

If α is a wff and x_i is an individual variable, then $\nabla x_i \alpha$ will be an abbreviation of the wff resulting from prefixing α with dictafiers with respect to all variants of x_i which have ∇-free occurrences in α.

2 *Axiomatics for D*

If α if an L'-wff, we write $\vdash\alpha$ if the ∇-closure of α is a theorem. We have the following axiom formation rules:

(D1) If α is an instance of a truth functional tautology, then $\vdash\alpha$.
(D2) For any α, β, x_i, $\vdash\forall x_i(\alpha \supset \beta) \supset (\forall x_i\alpha \supset \forall x_i\beta)$.
(D3) For any α, x_i^k, $\vdash\alpha \equiv \nabla x_i^k\alpha$.
(D4) If $x_i \in d(\alpha)$, then $\vdash\nabla x_i^k\alpha \supset \epsilon x_i$.
(D5) If $x_i \notin f(\alpha)$, then $\vdash\alpha \equiv \forall x_i\alpha$.
(D6) If α is a wff and x_i is free for x_j in α and $\alpha[x_i|x_j]$ is any good substitution, then $\vdash(\forall x_i\alpha) \wedge \epsilon x_j \supset \nabla x_j\alpha[x_i|x_j]$.
(D7) If F is any predicate letter and y_i and y_i^* are variants of the same variable, then $\vdash Fy_1 \ldots y_n \equiv Fy_1^* \ldots y_n^*$.
(D8) For any wff α, $\vdash\Box\alpha \supset \alpha$.
(D9) For any α, β, if $d(\alpha) - d(\beta) = \{z_1, \ldots, z_n\}$, then $\vdash\Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box(\epsilon z_1 \wedge \ldots \wedge \epsilon z_n \supset \beta))$.
(D10) For any α, $\vdash\Diamond\alpha \supset \Box\Diamond\alpha$.
(D11) For any x_i, $\vdash\Diamond\epsilon x_i$.

In addition, there are three inference rules: modus ponens, necessitation, and generalization. Formally:

(MP) If $d(\alpha) \subseteq d(\beta)$, $\vdash\alpha \supset \beta$, and $\vdash\alpha$, then $\vdash\beta$.
(N) If $\vdash\alpha$, then $\vdash\Box\alpha$.
($\forall$) If $\vdash\alpha$, then for any x_i, $\vdash\forall x_i\alpha$.

That the axioms are D-valid and that the inference rules preserve validity are easy to show. Some examples will illuminate the necessity for the unusual forms of (D9) and (MP). First, consider the wff

(8) $\forall x_i(\Box(Gx_i^k \supset \epsilon x_j) \supset (\Box Gx_i^k \supset \Box\epsilon x_j))$.

If $M = (D, W, \psi, \phi)$, where $D = \{a,b,c\}$, $W = \{w_1,w_2\}$, $D_{w_1} = \{a,b\}$, $D_{w_2} = \{c\}$, $\phi(G, w_1) = \{a\}$, and $\phi(G, w_2)$ is empty, then for $\theta(x_i) = a$ and $\theta(x_j) = b$, it is false that $M_{w_1} \models_\theta \Box(Gx_i^k \supset \epsilon x_j) \supset (\Box Gx_i^k \supset \Box\epsilon x_j)$. Hence, (8) is not valid and axiom schema (D9) cannot be replaced by the more familiar

(D9′) For any α, β, $\vdash\Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta)$.

Second, since $\forall x_i((Fx_i^k \vee \sim Fx_i^k) \supset \epsilon x_i)$ and $\forall x_i(Fx_i^k \vee \sim Fx_i^k)$ are valid, but ϵx_i is not we cannot conclude from the validity of the ∇-closures of $(Fx_i^k \vee \sim Fx_i^k) \supset \epsilon x_i$ and $(Fx_i^k \vee \sim Fx_i^k)$ the validity of the ∇-closure of ϵx_i (which is ϵx_i itself). Hence, modus ponens in its usual form does not work for D.

One further example of the quirks of D is useful. There is no analog to the quantifier generalization rule ($\forall$) for dictafiers; i.e., the rule

(∇) If $\vdash\alpha$, then $\vdash\nabla x_i^k\alpha$,

does not preserve validity. For example, the ∇-closure of $(Fx_i^k \vee \sim Fx_i^k)$ is valid, but the ∇-closure of $\nabla x_i(Fx_i^k \vee \sim Fx_i^k)$ is not.

Some consequences of the axioms and inference rules follow.

Theorem 2.1 (Generalized modus ponens) *If* $\vdash\alpha \supset \beta$, $\vdash\alpha$, *and* $d(\alpha) - d(\beta) = \{z_1, \ldots, z_n\}$, *then* $\vdash\epsilon z_1 \wedge \ldots \wedge \epsilon z_n \supset \beta$.

Proof: If $n = 1$, by ($\forall$), (D2), and (MP) we get $\vdash\forall z_1\beta$. By (D6), $\vdash\forall z_1\beta \wedge \epsilon z_1 \supset \nabla z_1\beta$; and by (D3) $\vdash\nabla z_1\beta \supset \beta$. Hence, by (D1) and (MP), $\vdash\forall z_1\beta \wedge \epsilon z_1 \supset$

β and $\vdash \forall z_1 \beta \supset (\epsilon z_1 \supset \beta)$. By (MP), $\vdash \epsilon z_1 \supset \beta$. The general case follows easily by induction on n.

Theorem 2.2 (Substitution) *If the position variants with ∇-free occurrences in λ are exactly those with ∇-free occurrences in μ and α is like α' except for containing an occurrence of μ where α contains λ, then $\vdash \lambda \equiv \mu$ implies $\vdash \alpha \equiv \alpha'$.*

Proof: By induction on the formation rules for α. If α is $\Box\beta$ and α' is $\Box\beta'$, then, since $d(\beta) = d(\beta')$, $\vdash \Box\beta \equiv \Box\beta'$ follows from $\vdash \beta \equiv \beta'$. Suppose α is $\nabla x_i^k \beta$, α' is $\nabla x_i^k \beta'$, and $\vdash \beta \equiv \beta'$. Because $d(\beta) = d(\beta')$ and $\vdash \nabla x_i^k \beta \supset \beta$, $\vdash \nabla x_i^k \beta \supset \beta'$. Also, $\vdash \beta' \supset \nabla x_i^k \beta'$. If $x_i \notin d(\beta')$, then $\vdash \nabla x_i^k \beta \supset \nabla x_i^k \beta'$ follows from (MP). If $x_i \in d(\beta')$, then $\vdash \epsilon x_i \supset (\nabla x_i^k \beta \supset \nabla x_i^k \beta')$ follows from Theorem 2.1. But, by (D4) $\vdash \nabla x_i^k \beta \supset \epsilon x_i$. Hence, $\vdash \nabla x_i^k \beta \supset \nabla x_i^k \beta'$. The converse follows similarly. The proofs for the rest of the cases are standard.

Theorem 2.3 *For any x_i, $\vdash (\forall x_i) \epsilon x_i$.*

Proof: By (D3), $\vdash (Fx_i^1 \vee \sim Fx_i^1) \supset \epsilon x_i$. By ($\forall$), (D2), and (MP), $\vdash \forall x_i (Fx_i^1 \vee \sim Fx_i^1) \supset (\forall x_i) \epsilon x_i$. By (D1) and ($\forall$), $\vdash \forall x_i (Fx_i^1 \vee \sim Fx_i^1)$. Hence, $\vdash (\forall x_i) \epsilon x_i$.

Theorem 2.4 *If α and α' are alphabetic variants, then $\vdash \alpha \equiv \alpha'$.*

Proof: By induction on the formation rules. Since for two alphabetic variants the same position variants have free occurrences in each, all of the cases follow from Theorem 2.2 except for the case that α is $\forall x_i \beta$ and α' is $\forall x_j \beta' [x_i | x_j]$, where β' is an alphabetic variant of β in which x_j is not free and $\beta'[x_i | x_j]$ is the good substitution which replaces each x_i^k in β' with x_j^k. By the induction hypothesis and substitution, $\vdash \forall x_i \beta \equiv \forall x_i \beta'$. By (D6) and (D3), $\vdash \forall x_i \beta' \wedge \epsilon x_j \supset \beta'[x_i | x_j]$. Since x_j is not free in β', Theorem 2.3 along with (D1), (D2), (D5), and Theorem 2.1 gives $\vdash \forall x_i \beta' \supset \forall x_j \beta' [x_i | x_j]$. The converse follows similarly.

All of the usual modal redundancy results of the modal (*S5*) propositional calculus hold. Some of the less familiar results which are necessary for what follows are included in

Theorem 2.5 *If $d(\alpha) = d(\delta)$,*
(a) *If $\vdash \Diamond\delta \supset \Diamond\alpha$, then $\vdash \Diamond(\Diamond\delta \supset \alpha)$*
(b) *If $\vdash \delta$, then $\vdash \Diamond\alpha \supset \Diamond(\alpha \wedge \delta)$*
(c) $\vdash \Diamond(\Diamond\alpha \supset \Diamond\delta) \equiv (\Diamond\alpha \supset \Diamond\delta)$.

The following results are somewhat technical in nature, but will be used in proving completeness.

Theorem 2.6 *If $d(\delta) = d(\alpha)$ and z is not free in δ, then $\vdash \Box\forall z \Box(\delta \supset \Box\alpha) \supset \Box(\delta \supset \Box\forall z \Box\alpha)$.*

Proof:
$$\begin{aligned}
\vdash \Box\forall z \Box(\delta \supset \Box\alpha) &\supset \Box\forall z\, \Box(\sim\Box\alpha \supset \sim\delta) \\
&\supset \Box\forall z(\Box\sim\Box\alpha \supset \Box\sim\delta) \\
&\supset \Box\forall z(\Diamond\delta \supset \Box\alpha) \\
&\supset \Box(\Diamond\delta \supset \forall z\Box\alpha) \\
&\supset \Box(\Box(\Diamond\delta \supset \forall z\Box\alpha)) \\
&\supset \Box(\Box\Diamond\delta \supset \Box\forall z\Box\alpha) \\
&\supset \Box(\delta \supset \Box\forall z\Box\alpha).
\end{aligned}$$

Theorem 2.7 *If* $d(\delta) = d(\lambda)$ *and* z *is not free in* δ, *then if* $\vdash \Diamond \exists z \Diamond \lambda$, *then* $\vdash \Diamond \exists z \Diamond (\Diamond \delta \supset \Diamond (\delta \wedge \Diamond \lambda))$.

Proof: By Theorem 2.5(b), $\vdash \Diamond \delta \supset \Diamond (\delta \wedge \Diamond \exists z \Diamond \lambda)$. Replacing α in Theorem 2.6 with $\sim\lambda$ and taking the contrapositive produces $\vdash \Diamond (\delta \wedge \Diamond \exists z \Diamond \lambda) \supset \Diamond \exists z \Diamond (\delta \wedge \Diamond \lambda)$. Hence, $\vdash \Diamond \delta \supset \Diamond \exists z \Diamond (\delta \wedge \Diamond \lambda)$. By Theorem 2.5(a), $\vdash \Diamond (\Diamond \delta \supset \exists z \Diamond (\delta \wedge \Diamond \lambda))$. Since z is not free in δ, $\vdash \Diamond \exists z (\Diamond \delta \supset \Diamond (\delta \wedge \Diamond \lambda))$. The conclusion now follows from Theorem 2.5(c).

Theorem 2.8 *If* $d(\lambda) = d(\delta_1) = \ldots = d(\delta_n)$ *and* $\vdash \sim (\lambda \wedge \Box \delta_1 \wedge \ldots \wedge \Box \delta_n)$, *then* $\vdash \sim (\Diamond \lambda \wedge \Box \delta_1 \wedge \ldots \wedge \Box \delta_n)$.

Proof: Suppose $\vdash \sim (\lambda \wedge \Box \delta_1 \wedge \ldots \wedge \Box \delta_n)$. Then, $\vdash \Box \delta_1 \supset (\ldots \supset (\Box \delta_n \supset \sim\lambda) \ldots)$. By (N) and (D8), and redundancy of double necessitation, $\vdash \Box \delta_1 \supset (\ldots (\Box \delta_n \supset \Box \sim \lambda) \ldots)$. Hence, $\vdash \sim (\Diamond \lambda \wedge \Box \delta_1 \wedge \ldots \wedge \Box \delta_n)$.

Let β be a ∇-closed wff and x_j a variable not occurring in β. If x_i is any other variable and $\beta[x_i|x_j]$ is the good substitution which replaces each free x_i^k in β with x_j^k, then $(\exists x_i \beta \supset \beta[x_i|x_j]) \wedge \epsilon x_j$ is a 0-*level E-formula with respect to* x_j. If δ is an n-level E-formula with respect to x_j and λ is a ∇-closed wff in which x_j is not free, then $\Diamond \lambda \supset \Diamond (\lambda \wedge \delta)$ is an $n + 1$-*level E-formula with respect to* x_j.[1]

Theorem 2.9 *If* λ *is a* 0-*level E-formula with respect to* z, *then* $\vdash \exists z \lambda$.

Proof: The usual argument works because of Theorem 2.3.

Theorem 2.10 *If* λ *is a k-level E-formula with respect to* z, $k > 0$, *then* $\vdash \Diamond \exists z \Diamond \lambda$.

Proof: Suppose $k = 1$ and λ is $\Diamond \delta \supset \Diamond (\delta \wedge \mu)$, where μ is 0-level. Then

$$\begin{aligned} \vdash \Diamond \delta &\supset \Diamond (\delta \wedge \exists z \mu) \text{ (Theorem 2.9 and Theorem 2.5(b))} \\ &\supset \Diamond \exists z (\delta \wedge \mu) \\ &\supset \Diamond \exists z \Diamond (\delta \wedge \mu). \end{aligned}$$

By Theorem 2.5(a), $\vdash \Diamond (\Diamond \delta \supset \exists z \Diamond (\delta \wedge \mu))$, so that $\vdash \Diamond \exists z (\Diamond \delta \supset \Diamond (\delta \wedge \mu))$. The conclusion follows trivially.

Suppose $k > 1$, so that λ is $\Diamond \delta \supset \Diamond (\delta \wedge \mu)$ where μ has the form $\Diamond \delta^* \supset \Diamond (\delta^* \wedge \mu^*)$. By Theorem 2.5(c), $\vdash \mu \equiv \Diamond \mu$. By the induction hypothesis and Theorem 2.7, $\vdash \Diamond \exists z \Diamond (\Diamond \delta \supset \Diamond (\delta \wedge \Diamond \mu))$. Using substitution of μ for $\Diamond \mu$ produces $\vdash \Diamond \exists z \Diamond \lambda$.

Theorem 2.11 *If* α *is* ∇-*closed*, z *is not free in* α, *and* λ *is an E-formula with respect to* z, *then* $\vdash \sim (\alpha \wedge \lambda)$ *implies* $\vdash \sim \alpha$.

Proof: Suppose λ is 0-level and $\vdash \sim (\alpha \wedge \lambda)$. Then $\vdash (\alpha \supset \sim \lambda)$. Since z is not free in α, $\vdash \alpha \supset \forall z \sim \lambda$. Hence, $\vdash \exists z \lambda \supset \sim \alpha$. By Theorem 2.9, $\vdash \sim \alpha$. Suppose λ is k-level $k > 0$. Then, $\vdash \lambda \equiv \Diamond \lambda$. Suppose $\vdash \sim (\alpha \wedge \lambda)$. By substitution, $\vdash \alpha \supset \sim \Diamond \lambda$. Hence, $\vdash \Diamond \alpha \supset \Diamond \sim \Diamond \lambda$, so that $\vdash \Diamond \alpha \supset \Box \sim \lambda$. Hence, $\vdash \Diamond \alpha \supset \forall z \Box \sim \lambda$ and $\vdash \Box \Diamond \alpha \supset \Box \forall z \Box \sim \lambda$. It now follows that $\vdash \Diamond \exists z \Diamond \lambda \supset \Diamond \Box \sim \alpha$. Using Theorem 2.10 gives $\vdash \Diamond \Box \sim \alpha$, from which $\vdash \sim \alpha$ obviously follows.

Theorem 2.12 *If* $d(\lambda) = \{z_1, \ldots, z_n\}$ *then* $\vdash \Box (\epsilon z_1 \wedge \ldots \wedge \epsilon z_n \supset \nabla z_1 \ldots \nabla z_n \lambda) \equiv \Box \lambda$.

Proof: By (D4), $\vdash \nabla z_i \sim \lambda \supset \epsilon z_i$. Combining this with (D3) gives $\vdash \sim\lambda \supset \epsilon z_i$ for all i. Hence, $\vdash \sim\lambda \supset \epsilon z_1 \wedge \ldots \wedge \epsilon z_n$. Combining this with $\vdash \nabla z_1 \ldots \nabla z_n \lambda \supset \lambda$ produces $\vdash (\epsilon z_1 \wedge \ldots \wedge \epsilon z_n \supset \nabla z_1 \ldots \nabla z_n \lambda) \supset \lambda$. Application of (N) and (D9) gives one of the desired conditionals. The converse conditional follows directly from (D9), (N), and $\vdash \lambda \supset \nabla z_1 \ldots \nabla z_n \lambda$.

3 Completeness A *Henkin system* for D is a nonempty set Ω of pairs (H, V_H), where H is a nonempty set of L'-wffs and V_H is a nonempty set of variables satisfying

(a) If $\alpha \in H$, then $d(\alpha) \subseteq V_H$
(b) If $d(\alpha) \subseteq V_H$, then exactly one of α, $\sim\alpha$ is in H
(c) If $d(\alpha) \subseteq V_H$ and $\vdash\alpha$, then $\alpha \in H$
(d) If $\alpha \supset \beta$, $\alpha \in H$, then $\beta \in H$
(e) If $x_i \in V_H$, then $\epsilon x_i \in H$
(f) If $x_i \notin V_H$, then $\sim\epsilon x_i \in H$
(g) If $\forall x_i \alpha \in H$, $x_j \in V_H$, and x_i is free for x_j in α, then for every good substitution, $\alpha[x_i|x_j] \in H$.
(h) If $\exists x_i \alpha \in H$ and α is ∇-closed, then there is an $x_j \in V_H$ such that x_j does not occur in α and for some good substitution, $\alpha[x_i|x_j] \in H$
(i) $\nabla x_i^k \beta \in H$ iff $\beta \in H$
(j) If α is ∇-closed and $\Box\alpha \in H$, then $\alpha \in H'$ for every H'
(k) If α is ∇-closed and $\Diamond\alpha \in H$, then $\alpha \in H'$ for some H'.

Any pair (H, V_H) satisfying (a)–(i) will be called a *Henkin set*.

If Ω is a Henkin system, the quadruple $M = (D, W, \psi, \phi)$, where D is the set of variables of L, $W = \Omega$, $\psi(H, V_H) = V_H$, and $\phi(H,F) = \{(z_1, \ldots, z_n) | Fz_1^* \ldots z_n^* \in H$ for some position variants z_i^* of $z_i\}$ is a model structure. Observe that (D11) guarantees that $D = \bigcup_H V_H$. If θ is the identity mapping from the set of variables of L to D, the following theorem holds:

Theorem 3.1 *For any wff* α, $M_H \models_\theta \alpha$ *iff* $\alpha \in H$.

Proof: Define the level L of a wff by

(a) $L(Fy_1 \ldots y_n) = n + 1$
(b) $L((\alpha \wedge \beta)) = L(\alpha) + L(\beta) + 1$
(c) $L((\sim\alpha)) = L(\alpha) + 1$
(d) $L((\nabla x_i^k \alpha)) = L(\alpha) + 1$
(e) $L((\forall x_i \alpha)) = 2^{L(\alpha)}$
(f) $L((\Box\alpha)) = 14^{L(\alpha)}$.

By induction on the level of α. Basically the cases have standard proofs. The argument for the case that α *is* $\forall x_i \beta$ follows from (h) applied to the ∇-closed wff $\forall x_i \nabla z_1 \ldots \nabla z_n \beta$ (where $d(\beta) = \{z_1, \ldots, z_n\}$), (i), and (g). The argument for the case that α is $\Box\beta$ follows from (i) and (k) applied to the ∇-closed wff $\Box\beta^*$ given by $\Box(\epsilon z_1 \wedge \ldots \wedge \epsilon z_n \supset \nabla z_1 \ldots \nabla z_n \beta)$ and Theorem 2.12. For example, suppose $M_H \models_\theta \Box\beta$ and $\Box\beta \notin H$. Then $d(\beta) = \{z_1, \ldots, z_n\} \subseteq V_H$ and by Theorem 2.12 $\vdash \Box\beta \equiv \Box\beta^*$. By (c), $\Box\beta^* \supset \Box\beta$ is in H. Since $\Box\beta \notin H$, $\Box\beta^* \notin H$. But, $\vdash \Box\beta \supset \Box\beta^*$, so that $M_H \models_\theta \Box\beta \supset \Box\beta^*$. Hence, $M_H \models_\theta \Box\beta^*$. Thus,

β^* is a ∇-closed wff such that $L(\beta^*) < L(\Box\beta)$, $M_H \models_\theta \Box\beta^*$, and $\Box\beta^* \notin H$. The usual argument now produces a contradiction from (k).

A wff α is *D-consistent* if not $\vdash\sim\alpha$. A set S of wffs is *D-consistent* if every finite conjunction of wffs in S is D-consistent. If V^* is a set of variables, a set S of wffs is *maximally V^*-consistent* if

1. If $\alpha \in S$, then $d(\alpha) \subseteq V^*$
2. If $z \in V^*$, then $\epsilon z \in S$
3. *If* $z \notin V^*$, then $\sim\epsilon z \in S$
4. If $d(\alpha) \subseteq V^*$, then either $\alpha \in S$ or $\sim\alpha \in S$
5. S is consistent.

Theorem 3.2 *If S is maximally V^*-consistent, then (S, V^*) satisfies (a)–(g), (i) of the definition of Henkin set. If, in addition, S satisfies*

(∗) *For every ∇-closed wff β, there is a variable x_j not occurring in β such that for some good substitution $(\exists x_i\beta \supset \beta[x_i|x_j]) \wedge \epsilon x_j$ is in S.*

then (S, V^*) is a Henkin set.

Proof: (a)–(f) are trivial.

(i) If $\beta \in S$, since $\vdash\beta \supset \nabla x_i^k\beta$ we have $\nabla x_i^k\beta \in S$. Conversely, suppose $\nabla x_i^k\beta \in S$, so that $d(\nabla x_i^k\beta) \subseteq V^*$. Since $\vdash\nabla x_i^k\beta \supset \beta$, if $d(\beta) \subseteq V^*$ then $\beta \in S$. If $x_i \notin d(\beta)$, then $d(\beta) \subseteq V^*$. If $x_i \in d(\beta)$, by (D4) $\vdash\nabla x_i^k\beta \supset \epsilon x_i$, so that $\epsilon x_i \in S$. But this forces $x_i \in V^*$, so that $d(\beta) \subseteq V^*$.

(g) If $\forall x_i\alpha \in S$ and $x_j \in V^*$, then by (2) $\epsilon x_j \in S$. Using (D6) and (i), $\alpha[x_i|x_j] \in S$.

(h) Follows trivially from (∗).

Theorem 3.3 *If α is any ∇-closed consistent wff, then there exists a Henkin system Ω containing a Henkin set (H, V_H) such that $\alpha \in H$.*

Proof: As usual, we say that two E-formulas belong to the same E-form if they differ only with respect to the variable z. There are countably many E-forms: $E_1, E_2, \ldots$, each of which contains an infinite number of wffs.

We construct the set S as follows. Beginning with α we add a sequence of E-formulas from the E-forms, at each step choosing an E-formula with respect to a variable not occurring in any previous wff. It follows from Theorem 2.11 that the resulting set is D-consistent. Next we pass through the list of all wffs β, adding β to the set if it can be consistently added. The resulting set is S and is consistent by construction. Let $V_S = \{z|\epsilon z \in S\}$. Since S contains a 0-level E-formula from each 0-level E-form, if S is maximally V_S-consistent, (S, V_S) is a Henkin set. First, suppose $\beta \in S$ and $z \in d(\beta)$. If $z \notin V_S$ then $\epsilon z \notin S$. Hence, $S \cup \{\epsilon z\}$ is inconsistent. Since ϵz is ∇-closed and S is consistent, it follows that $S \cup \{\sim\epsilon z\}$ is consistent, so that $\sim\epsilon z \in S$. But, by (D3) and (D4), $\vdash\beta \supset \epsilon z$, so that $\vdash\sim(\beta \wedge \sim\epsilon z)$. This conflicts with the consistency of S. Hence, (S, V_S) satisfies condition (1) of the definition. (2) follows from the definition of V_S. The proof of (3) is included in the proof of (1). Next, suppose $d(\beta) \subseteq V_S$ and neither β nor $\sim\beta$ is in S. Then, there are conjunctions γ and τ in S such that $\vdash\sim(\gamma \wedge \beta)$ and $\vdash\sim(\tau \wedge \sim\beta)$. (D1) and (MP) give $\vdash\sim(\tau \wedge \sim\beta) \supset \sim(\tau \wedge \gamma)$. If $\{z_1, \ldots, z_n\} = d(\beta) \cup d(\tau) - d(\gamma) \cup d(\tau)$, by Theorem 2.1 we get $\vdash\epsilon z_1 \wedge \ldots \wedge$

$\epsilon z_n \supset \sim(\tau \wedge \gamma)$, or $\vdash\sim(\epsilon z_1 \wedge \ldots \wedge \epsilon z_n \wedge \tau \wedge \gamma)$. Since $d(\beta) \subseteq V_S$, this conflicts with S's consistency. Hence, (S, V_S) is a Henkin set.

Next, if $\Diamond\beta \in S$ and β is ∇-closed, we define the set S_β as follows. We begin with β. If the first E-form is $E_1 = \{\delta_{11}, \delta_{12}, \ldots\}$, then, since S contains a wff from each E-form, S contains a wff of the form $\Diamond\beta \supset \Diamond(\beta \wedge \delta_{1n_1})$, where the variable z for which δ_{1n_1} is an E-formula is not free in β. By Theorem 2.11, δ_{1n_1} can be consistently added to $\{\beta\}$. Similarly, if $E_2 = \{\delta_{21}, \delta_{22}, \ldots\}$, S contains a wff of the form $\Diamond(\beta \wedge \delta_{1n_1}) \supset \Diamond(\beta \wedge \delta_{1n_1} \wedge \delta_{2n_2})$, where the variable z for which δ_{2n_2} is an E-formula is not free in $\beta \wedge \delta_{1n_1}$. By Theorem 2.11, $\{\beta, \delta_{1n_1}, \delta_{2n_2}\}$ is consistent. Continue this process through the list of E-forms. Now, for any wff $\Diamond\lambda$ in S such that λ is ∇-closed, add $\Box\lambda$ to the set. The result is still consistent. If $\vdash\sim(\beta \wedge \delta_{i_1 n_{i_1}} \wedge \ldots \wedge \delta_{i_k n_{i_k}} \wedge \Box\lambda_{j_1} \wedge \ldots \wedge \Box\lambda_{j_n})$, then by Theorem 2.8

$$(**) \qquad \vdash\sim(\Diamond(\beta \wedge \delta_{i_1 n_{i_1}} \wedge \ldots \wedge \delta_{i_k n_{i_k}}) \wedge \Box\lambda_{j_1} \wedge \ldots \wedge \Box\lambda_{j_n}).$$

But, $\Diamond\beta \in S$, $\Diamond(\beta) \supset \Diamond(\beta \wedge \delta_{i_1 n_{i_1}}) \in S$, etc., so that $\Diamond(\beta \wedge \delta_{i_1 n_{i_1}} \wedge \ldots \wedge \delta_{i_k n_{i_k}}) \in S$. But, $\Box\lambda_i \in S$ for each i, so that (**) cannot be true since S is consistent. Finally, we go through the list of all wffs σ, adding σ to the set if σ can be consistently added. The resulting set is S_β. If $V_\beta = \{z \mid \epsilon z \in S_\beta\}$, then (S_β, V_β) is a Henkin set.

Finally, $\{(S, V_S)\} \cup \{(S_\beta, V_\beta) \mid \Diamond\beta \in S,\ \beta$ is ∇-closed$\}$ is a Henkin system. This follows easily from the construction and the fact that $\Box\Diamond\sigma$ is in a Henkin set iff $\Diamond\sigma$ is.

Theorem 3.4 *Every D-valid wff is a theorem.*

Proof: From Theorems 3.1 and 3.3.

NOTE

1. This definition of *E-formula* and its subsequent use follows [1], pp. 165–168. However, since the Barcan formula, $\forall x_i \Box\alpha \supset \Box\forall x_i \alpha$, is not valid in D (even for ∇-closed wffs α), some modifications are necessary. In particular, $\vdash\exists x_i \lambda$ holds only for 0-level E-formulas. For higher levels, we have the weaker result Theorem 2.10.

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