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Operational Semantics for Positive *R*

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1 Two kinds of formal semantics for intensional logics It is convenient to begin with a few remarks about the distinction between model-theoretic (sometimes called 'set-theoretic') semantics and algebraic semantics for sentential logics containing non-truth-functional connectives. Both issue in definitions of validity on a structure or over a class of structures in terms of which completeness theorems are sought, to the effect that provability in this or that logic coincides with a certain such notion of validity. I take the hallmark of the model-theoretic approach to be that it characterizes the validity notion in question via an inductively defined notion of truth of a formula at a point in a model, while the algebraic approach features no such intermediate level of description.[1] The considerable appeal of the Kripke relational semantics for normal (and some non-normal) modal logics and for intuitionistic and intermediate logics over earlier algebraic accounts was due no doubt in part to its supplying this intermediate level of description, with something recognizably analogous to the informal notion of truth restored to center stage. This feature of the relational semantics for modal logic is shared by the operational semantics suggested for certain normal systems by Garson ([9]), as well as by the neighborhood semantics for these (and weaker) systems. Accordingly, in such cases, even when the stuctures are algebraic structures (carrier set + operation(s)) what we have is model-theoretic rather than algebraic semantics.

The above distinction is somewhat stipulatively drawn, articulating just one significant difference often marked by the terminological contrast. It certainly ignores, in particular, a tendency on the part of some writers to speak of any proposed formal semantics as 'merely algebraic' as opposed to 'genuine' semantics when they are not persuaded that it throws any light on the intended meanings of the expressions involved (see [4] for example).

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The reason I open with this distinction is that I shall be concerned in what follows with a variation on the semilattice semantics originally proposed by Urquhart (in [15]) for relevant logics, which is by our present lights clearly *model-theoretic*, though operational, semantics. We consider, in particular, as did Urquhart, the negation-free fragment of the system R, and will be suggesting that troubles—over disjunction—his semantics got into as a semantics *for* that system may be solved if we see an additional binary operation as called for to deal with disjunction. This yields algebras of the same similarity type as bounded lattices and rings. Since such structures have figured prominently in the algebraic semantics for relevant logics, I repeat that it is model-theoretic semantics that is being offered here.

The question raised in the second paragraph above about the intelligibility of the semantics itself has of course always been important in relevant logic and I address it here with the following comment. The operation used in the Urquhart semantics in defining truth for implicational formulas deserves no doubt somewhat more attention than it gets in [15], where there are just a few informal remarks, the gist of which is summarized below in Section 2. The additional operation invoked in this paper to deal with disjunctive formulas makes more immediate—or so I hope my informal remarks will show—intuitive sense. Thus the claim is that a slight variation, itself independently motivated, on the Urquhart semantics can be seen to handle disjunction in positive R without introducing any *further* murkiness.

2 The Urquhart semilattice semantics In [15] there may be found a semantic treatment of the $\{\to,\wedge\}$-fragment of R, which is axiomatized by the following schemata:

$$A \to A \qquad \text{(Id)}$$
$$(A \to (B \to C)) \to (B \to (A \to C)) \qquad \text{(Perm)}$$
$$(A \to B) \to ((B \to C) \to (A \to C)) \qquad \text{(Suff)}$$
$$(A \to (A \to B)) \to (A \to B) \qquad \text{(Contrac)}$$
$$((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C)) \qquad (\wedge\text{Intr})$$
$$(A \wedge B) \to A \qquad (\wedge\text{Elim1})$$
$$(A \wedge B) \to B. \qquad (\wedge\text{Elim2})$$

with the two rules of proof:

$$\frac{\vdash A \qquad \vdash B}{\vdash A \wedge B} \qquad \text{(Adjunction)}$$

$$\frac{\vdash A \to B \qquad \vdash A}{\vdash B.} \qquad \text{(MP)}$$

The "frame" role, to adapt (with [6]) this term from the relational model theory of modal logic, was played in Urquhart's models by groupoids $\langle S,\cdot,1\rangle$ with a left-identity element (i.e., $\cdot$ is a binary operation on S and $1 \in S$ satisfies: $1x = x$ for all $x \in S$; unlike Urquhart, I use the multiplicative notation here). A model on such a frame—here again departing somewhat in manner of presentation from Urquhart—is specified when a valuation function V from atomic formulas[2] paired with elements of S to truth-values $(\mathbf{0,1})$ is supplied. Such an assignment of truth-values is extended to arbitrary formulas by an inductive definition of the relation $\langle S, \cdot, 1, V\rangle$, $x \models A$. We suppress mention of the model:

$x \models A$ iff $V(A,x) = \mathbf{1}$ for A atomic
$x \models A \wedge B$ iff $x \models A$ and $x \models B$
$x \models A \to B$ iff for all $y \in S$, if $y \models A$ then $xy \models B$.

A formula A is valid on a frame $\langle S, \cdot, 1\rangle$ if for any model $\langle S, \cdot, 1, V\rangle$ on that frame, we have: $\langle S, \cdot, 1, V\rangle$, $1 \models A$. By an ingenious argument Urquhart managed to show that the theorems of the $\{\wedge, \to\}$-fragment of R, as axiomatized above, are precisely the formulas valid on every frame $\langle S, \cdot, 1\rangle$ which is a semilattice (commutative idempotent semigroup) with identity. His idea was we could think of the sets S as collections of pieces of information combinable with the aid of the operation $\cdot$, with 1 as the 'empty' piece of information, and he noted that this represented a generalization of the Kripke semantics for intuitionistic logic, which could be recovered by adding the persistence (or 'no forgetting') condition: if $V(A,x) = \mathbf{1}$ then $V(A,xy) = \mathbf{1}$ for atomic A. Note that this is a condition on models rather than on frames, as is the intermediate condition Urquhart provided for the extension of the above system by the Mingle schema $A \to (A \to A)$: if $V(A,x) = V(A,y) = \mathbf{1}$ then $V(A,xy) = \mathbf{1}$.[3] For at least the original $\{\wedge, \to\}$-fragment of R, Urquhart suggested reading '$x \models A$' as something along the lines of 'on the basis of the information in x, all of which is relevant for this purpose, we may conclude that A'.

When Urquhart extended the above semantic account to accommodate disjunction, he suggested extending the definition of truth by the 'classical' clause:

$$x \models A \vee B \text{ iff } x \models A \text{ or } x \models B$$

and found that while every theorem of the $\to$, $\wedge$, $\vee$-fragment of R was valid on all semilattice frames, so also were some additional formulas not belonging to this fragment. To axiomatize the fragment in question, add to the axiom-schemata listed already the following four:

$A \to (A \vee B)$	$\vee$Intr1
$B \to (A \vee B)$	$\vee$Intr2
$((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$	$\vee$Elim
$(A \wedge (B \vee C)) \to ((A \wedge B) \vee C)$.	$\wedge/\vee$-Distrib

Then one example (due to Meyer and Dunn, independently) of a formula-schema not all of whose instances are provable from this extended set of axioms but all of which are valid on all semilattices is:

$$((A \to (B \vee C)) \wedge (B \to C)) \to (A \to C).$$

(Actually this is a simplification, from [5], of the formula mentioned as due to Meyer and Dunn in [15].) Another schema in the same category is the similar-looking:

$$(A \to (B \vee C)) \to ((A \wedge (B \to C)) \to C).$$

The extent to which these formulas or other similarly supernumerary results could be held to violate motivating constraints on relevant logics does not appear to have been addressed in the literature, though Dunn ([5]) has some remarks on its interaction with negation and conjunction. In the absence of a definitively unfavorable verdict here, it would be open to Urquhart to take the same line about disjunction as he does about negation itself (not a topic for treatment in the present paper), namely that there is nothing sacrosanct about the particular choice of axioms made by (Ackermann and) Anderson and Belnap.[4] However, in the meantime, Fine and Charlwood have gone on to find complete (if somewhat inscrutable) axiomatizations of the formulas in $\rightarrow$, $\wedge$, and $\vee$ valid on all semilattices with the above definition of truth in place (in [8], [2]).

It is my intention instead to modify the semantic account of disjunction in order to adapt Urquhart's model theory in such a way as to make available a completeness proof for positive R, or R^+ as we call it from now on. Thus, the system R^+ will here be taken to be that axiomatized by the schemata Id, Perm, Suff, Contrac, $\wedge$Intr, $\wedge$Elim1,2 $\vee$Intr1,2, $\vee$Elim, and $\wedge/\vee$-Distrib, with the rules MP and Adjunction.

3 Dealing with disjunction We begin by contrasting Urquhart's treatment of disjunction with that provided by the Routley–Meyer (ternary-)relational semantics and Fine's combined operational and (binary-)relational semantics ([14], [6], respectively). The Routley–Meyer account manages to match validity with provability in R^+ (indeed R itself) in spite of using a classical clause for disjunction like Urquhart's. This it achieves by exploiting the extra flexibility of a ternary relation over a binary operation (especially the absence of a uniqueness requirement), along with the use of conditions on models analogous to those resorted to by Urquhart only for the Mingle and intuitionist extensions of the system. For more by way of comparison, [5] may be consulted. Fine's models come equipped with a distinguished subset of "saturated" points, an operation like the "$\cdot$" we have been using, as well as a binary relation "is an extension of" (in terms of which a persistence condition on models is stated), with a disjunction true at an arbitrary point just in case each saturated extension of that point verifies one disjunct or the other. The force of the "is an extension of" relation will be capturable with the aid of the new operation to be introduced below; its introduction turns out to render unnecessary the appeal to saturated points. (Though we do not take the relation in question as primitive, it will be convenient to define it below, where it appears in the notation "$\leq$".) One small price to be paid for these simplifications over Fine's account is that we shall need, alongside the distinguished element 1, another such element, 0, which may be thought of as a maximally undiscriminating piece of information: at 0, all formulas will be true (in any model).

Continuing, then, with something like Urquhart's "pieces of information" picture in mind, we note (with Copeland [3]) how very *inappropriate* to that picture is the classical clause for "$\vee$" adopted by Urquhart. A piece of information may notoriously not be sufficiently informative to determine as true one disjunct or the other of a disjunction it does determine as true. There is a close analogy here with the "possibilities" semantics for modal logic here (cf. [10]), understood as regions of logical space rather than (as possible worlds are) points therein. Indeed, a piece of information may be regarded as a possibility in epistemic dress: all information is information that you are in such-and-such a region of logical space. And, in contrast with the case of possible worlds, a disjunctive statement may hold throughout a region, without this being so for either disjunct, for the same reason that a barnyard can be full of things each of which is a sheep or a goat, without being full of sheep or full of goats. In the latter case this can only be so if the yard can be divided without remainder into a (perhaps spatially scattered) part which is full of sheep and a part which is full of goats. Similarly, we are led to require that an $(A \vee B)$-verifying region should be exhaustively composed out of an A-verifying and a B-verifying subregion. We denote this mode of composition by "+". The subregions may in general overlap, of course, should the disjuncts be compatible. Some will want to think of regions of logical space as sets of possible worlds, and think of "+" in this context as representing set union. In fact, though (cf. [10]), there is no need to do so, and we may prefer to think more mereologically, with $x + y$ as the smallest whole of which both x and y are parts. The difference between these two views does not matter for what follows, however. Either way, what is suggested is the following clause for disjunction in the definition of truth, once we have incorporated the + operation into our models:

(*) $x \models A \vee B$ iff there exist y, z such that $x = y + z$ and $y \models A$ and $z \models B$

Below, we shall be using "$\leq$" to represent the *subregion* relation here borne by y and z to x. You may prefer to think of pieces of information more *linguistically*. Then instead of taking our x, y, etc. as regions of logical space, they are thought of as sets of sentences, those true over the whole of the region in question. In this case, the $\leq$-relation is somewhat confusingly named since when $x \leq y$, x contains more *sentences* than y: the smaller the region the more there will be that is true over the whole of it. The + operation, thought of as set-theoretic union or mereological aggregation on the "ontic" approach to information, may here be thought of as acting on x and y to "dilute" each with the other, in the following sense: $x + y$ consists of all those sentences $B \vee C$ for which B belongs to x and C to y. When we come to consider the consequences of such a set, we find them to be precisely the common consequences of x and y, a fact exploited in the completeness proof of the following section, where in general the 'linguistic' approach is dominant. On either way of thinking of information, what we are thinking of is information in a rather abstract sense, rather than information as actually possessed by a knowing subject. Care is needed in transferring concepts across to this area: for example, when $x = y + z$, a person in possession of information x need not be in possession of y or in possession of z: there are in general more specific pieces of information of which he may be ignorant.

Now (*) may look suspiciously *conjunctive* for a treatment of disjunction. For example, mightn't A already be determined as true by x—true, that is, over the whole region x (to revert to this way of conceptualizing information) without there being any subregion of x to play the role of the z in (*) and verify B? This is where the new element 0, mentioned above as an identity element for the operation +, comes to our assistance. We can apply (*) in a case like this by taking x itself to be the promised y and take 0 as the z. So this does not after all raise a difficulty for regarding a disjunction $A \vee B$ as holding over a region just when that region can be broken down into A- and B-verifying parts. (Compare the difference between Urquhart's clause and (*) with that between the treatment of disjunction in the Beth semantics and in the Kripke semantics for intuitionistic logic.)

The reader may feel somewhat uneasy at the role 0 is here called on to play, either because of dissatisfaction with the idea of an empty region or because of the fact that, until we consider (in Section 5) the addition of the sentential constant F, 0 does not play a role in the truth-definition (or, like 1, in the explanation of what it is to be valid). The latter uneasiness has been expressed to me by Kit Fine, who also suggested the following alteration to (*) designed to circumvent appeals to the presence of 0 and the associated apparatus deployed below (the conditions (C0) and Zeroing, and the Zero Lemma): count a disjunction $A \vee B$ true at x just in case A is true at x, or B is true at x, or the right hand side of (*) holds. I leave the exploration of this suggestion to any reader affected by such qualms, preferring to work with the simpler (*) itself.

The clause (*) has an interesting relation to Urquart's own clause for disjunction which was drawn to my attention by Chris Brink. Recall that to any algebra there corresponds a *power algebra* (also called: subset algebra, global algebra, complex algebra) whose underlying set consists of that algebra's subsets and whose fundamental operations are defined by 'lifting' the original algebra's corresponding operations in the following way. Where f is an n-ary operation on the original algebra, we understand $ⓕ(X_1, \ldots X_n)$ to be the set of all those elements y of the original algebra for which there exist $x_1 \in X_1, \ldots,$ $x_n \in X_n$ such that $y = f(x_1, \ldots, x_n)$. Now denote by $\|A\|$ the set of points in an a model, as defined precisely below, at which the formula A is true. Then Urquhart's clause for disjunction puts $\|A \vee B\| = \|A\| \cup \|B\|$ whereas (*) amounts to a power-algebraic lifting of the + operation: $\|A \vee B\| = \|A\| \oplus \|B\|$. (In [1] Brink gives other applications of power algebras and related constructions to the semantics of relevant logic.) In fact, [15] also suggests (what amounts to) a lifting of the product operation for determining the truth-sets of fusion-formulas, but as we shall have occasion to note in Section 5, this results in validating formulas even in just the connectives of conjunction, implication, and fusion, which are not theorems of R. We turn now to a more precise description of the proposed model theory.

An *R-frame* is to be an algebra $\langle S, \cdot, +, 1, 0\rangle$ of similarity type (2,2,0,0) in which $\langle S, +, 0\rangle$ is a semilattice with zero (0), possessing a certain Decomposition Property to be described below, $\langle S, \cdot, 1\rangle$ is a commutative semigroup with unit (1), satisfying the following conditions on the interaction between the additive and multiplicative notions:

Ring Distribution	$x(y + z) = xy + xz$
Pseudo-idempotence	$x(x + 1) = x^2$
Zeroing	$0x = 0.$

This third condition insists that in R-frames the additive identity 0 really lives up to its name in doubling also as the annihilator for the product operation. A corresponding condition requiring that $x + 1 = 1$ for all $x \in S$ would be of no relevant-logical interest as a condition on R-frames in that the class of formulas valid on every such frame (with validity on a frame understood as defined below) coincides with the class of negation-free theorems of intuitionistic propositional logic. However, as a condition on individual points x in the frames, it will be seen in Section 5 to play a role in connection with the addition of sentential constants (in particular, with "t").

The condition of pseudo-idempotence is perhaps not very attractive; it is so named because, as the reader will notice, the idempotence demanded of the operation $\cdot$ on Urquhart's account has been abandoned here and this weaker condition does the same work (basically validating Contrac). In the light of the first condition, the ring-theoretic distribution law, it may be rewritten as requiring that $x^2 = x^2 + x$, or again equivalently as: for some y, $x^2 + y = x$. The $\leq$-relation introduced informally above may now conveniently be officially defined by: $u \leq v$ iff for some y, $u + y = v$; with its aid we may further rewrite the condition of pseudo-idempotence as: $x \leq x^2$. In terms of the 'logical space' conceptualization of information, this may be read as saying that the possibility x is a subregion of the possibility x^2: from the point of view of its formal appearance it is a condition familiar from other model-theoretic and algebraic work on the semantics of R. One reason we do not demand idempotence is simply that the style of completeness proof to be found in the following section, which is a minor adaptation of the argument of [14] and (especially) [6], simply does not provide us with an idempotent $\cdot$ in the canonical frame. But more importantly, it can be shown that the interaction between $+$ and an idempotent $\cdot$ validates, given the other features of our semantics, some formulas which are not theorems of R. (See Appendix A for details.)

Finally, having now officially introduced the semilattice ordering notation $\leq$, we can spell out not too long-windedly what is required for the additive operation to satisfy the decomposition property:

Decomposition If $x \leq u + v$ then there exist $u' \leq u$, $v' \leq v$ with $u' + v' = x$.

In other words, if x is a subregion of the sum of u and v, then x can be decomposed into regions which are subregions of these summands (u' and v' being "x-parts" of u and v respectively). This is reasonable on any mereological reading of "$\leq$". For example, if an area of the earth's surface is a subregion of the aggregate consisting of North and South America, it must itself be the aggregate of a subregion of North America and a subregion of South America. (In this case, because North and South America have no common nonempty subregion, the representation of the region in question as a sum of their subregions is unique; this will not generally be so.) For those who prefer to think more linguistically of pieces of information and the operations on and relations between them, the plausibility of this condition may be ascertained from the way in which it is established to hold for the canonical model in the following section.

The obvious observation to make here is that with Decomposition we lose the chance, hitherto alive, of regarding R-frames as an equational class. While this does cost us the availability of certain techniques from universal algebra, it does make for a simpler definition of truth, in particular as regards disjunctive formulas, than would otherwise be possible. It is true that one could do without it by complicating the notion of a frame, adding a further binary operation, lattice-theoretically dual to $+$. Write this as #. Then the idea would be that the reducts $\langle S,+,\#\rangle$ of the resulting structures would be required to be distributive lattices. A supplementary condition analogous to Ring Distribution would be imposed, to the effect that $x(y \# z) = xy \# xz$. The clause for "$\wedge$"

given in Section 2 in the definition of truth would be replaced by one exactly analogous to (*) but with the role of $+$ played instead by #. (The Zero Lemma and the Plus Lemma, stated exactly as below, though in the latter case it pays to observe that if A is a formula for which $x + y \models A$ implies $x \models A$, all x,y, then A is a formula for which $u \models A$ implies $u \# v \models A$, all u, v. The completeness proof proceeds as in Section 4, taking $x \# y$ to be the deductive closure of the union of x and y.) We do not take this path simply because we can avoid the complication of having a third operation in the frames by sticking with the truth-definition in its present form. There is a sharp contrast here with the case of disjunction, for which we have noted the classical clause to be both motivationally astray and ill-suited to semantically characterizing R^+. The # alternative is mentioned simply so as to note that under it there would be no need separately to impose the condition of Decomposition, since when $x \leq y + z$, the required $y'\,(\leq y)$ and $z'\,(\leq z)$ summing to x can be taken simply as $x \# y$ and $x \# z$ respectively. Having said this much to make the condition appear reasonable, we continue the development of the semantics.

If $\langle S,\cdot,+,1,0\rangle$ is an R-frame, then $\langle S,\cdot,+,1,0,V\rangle$ is a *model* on $\langle S,\cdot,+,1,0\rangle$ if V is a function from formulas paired with elements of S to truth-values $\{\mathbf{1},\mathbf{0}\}$ (which appear in boldface here to avoid confusion with the unit and zero of the R-frames) such that for each atomic formula A these two conditions are satisfied:

(C+) $V(A, x + y) = \mathbf{1}$ iff $V(A,x) = V(A,y) = \mathbf{1}$ for all $x,y \in S$
(C0) $V(A,0) = \mathbf{1}$.

These conditions will not appear unreasonable in the light of our previous remarks. (C+) is an orthodox persistence condition from left to right (thinking of $x + y$ as a less specific piece of information, in general, than x and y) and from right to left corresponds to the condition of resolution in [7], refinability in [10], cumulativity in [12], etc. (C0) is the embodiment of the idea, mentioned above, of the zero as a maximally undiscriminating piece of information, recalling the doctrine popularized by Popper and others, that degree of informativeness may be measured by how much is *excluded*.[5]

As before, we shall say a formula A is *valid* on an R-frame $\langle S,\cdot,+,1,0\rangle$ when for any model $\langle S,\cdot,+,1,0,V\rangle$ on that frame, we have $\langle S,\cdot,+,1,0,V\rangle$, $1 \models A$. The inductive definition of truth (the $\models$ relation) is as in Section 2 for atomic, conjunctive, and implicational formulas, with disjunctive formulas interpreted in accordance with (*) above. The remainder of this section establishes that the theorems of R^+ are valid on every R-frame; the following section shows that every formula (of the present language) valid on all R-frames is a theorem of R^+. We then consider some extensions of the machinery. As a preliminary to the soundness argument, we must first observe that the conditions (C+) and (C0) imposed on models generalize from atomic to arbitrary formulas. We deal with the two cases in that order.

Plus lemma *For any model* $\langle S,\cdot,+,1,0,V\rangle$ *and any formula* A:

$$\langle S,\cdot,+,1,0,V\rangle\ x + y \models A \textit{ iff } x \models A \textit{ and } y \models A, \textit{ all } x,y \in S.$$

(We revert here to suppressing mention of the model, as on the right-hand side, when the context makes it clear which model is intended.)

Proof: By induction on formula complexity. We work through the case where A is an implication and the case in which A is a disjunction, for illustration. (i) A is $B \to C$ and the Lemma holds for B, C. Suppose it fails for $B \to C$, say left to right. It will be sufficient to derive a contradiction from supposing $x + y \models B \to C$ while $x \not\models B \to C$, so let us suppose this. Then for some z, $z \models B$ and $xz \not\models C$. Since $x + y \models B \to C$ and $z \models B$, $(x + y)z = xz + yz \models C$ (by commutativity and ring distribution), which contradicts the inductive hypothesis since we have already been forced to suppose that $xz \not\models C$. Next, we check the right to left direction. Suppose:

$$x \models B \to C \qquad y \models B \to C \qquad x + y \not\models B \to C.$$

The third assumption means there exists z such that $z \models B$ with $(x + y)z \not\models C$. But the first two assumptions then imply $xz \models C$ and $yz \models C$, so (ind. hypoth.) $xz + yz \models C$. Since $(x + y)z = xz + yz$ we have our contradiction. (ii) A is $B \vee C$ and the Lemma holds for B, C. First, left to right. Suppose $x + y \models B \vee C$ (to show $x \models B \vee C$, the case of y being similar since $+$ is commutative). Then for some u, v $x + y = u + v$, $u \models B$ and $v \models C$. Since $x \leq u + v$, by Decomposition there exist $u' \leq u$, $v' \leq v$, with $u' + v' = x$. Then by the inductive hypothesis, $u' \models B$ and $v' \models C$, because $u \models B$ and $v \models C$. So, since $x = u' + v'$, $x \models B \vee C$. Next, the converse direction. Suppose that $x \models B \vee C$ and $y \models B \vee C$. Then x and y can be represented as $x_1 + x_2$ and $y_1 + y_2$ respectively, with B true at each of the first summands and C true at each of the second. So by the inductive hypothesis, $x_1 + y_1 = B$ and $x_2 + y_2 \models C$; then since $x = (x_1 + y_1) + (x_2 + y_2)$, $x \models B \vee C$.

Zero lemma $\quad$ *$\langle S,+,\cdot,0,1,V\rangle$ and any formula A:*

$$\langle S,+,\cdot,0,1,V\rangle\ 0 \models A.$$

Proof: Again by induction on the complexity of A. The details are routine; the case where A is of the form $B \to C$ requires appeal to the condition on frames we called Zeroing.

We are now ready for the

Soundness theorem $\quad$ *Every formula provable in R^+ is valid on every R-frame.*

Proof: By induction on the length of proofs in the axiomatization presented in the preceding section. The rules present no novelty; nor do most of the axioms. We explicitly examine Contrac, $\vee$Intr1, and $\vee$Elim; $\wedge/\vee$-Distrib requires the left-to-right direction of the Plus Lemma.

Contrac: If an instance of this schema is false at the element 1 in model on an R-frame, then for the model in question there is an element x of the frame, and formulas A and B with:

$$x \models A \to (A \to B) \qquad x \not\models A \to B.$$

From this second fact, we know there is a y with $y \models A$ and $xy \not\models B$; so by the first fact, $xy \models A \rightarrow B$, so $(xy)y = xy^2 \models B$. But since $y^2 = y^2 + y$ (pseudo-idempotence in distributed form), $xy^2 = xy^2 + xy$ and since $xy^2 \models B$, by the Lemma, left to right, $xy \models B$: a contradiction.

$\vee$Intr1: Suppose $x \models A$; then since $x = x + 0$ and (by the Zero Lemma) $0 \models B$, x is the sum of an A-verifying and a B-verifying region, so $x \models A \vee B$.

$\vee$Elim: Here we are to suppose that $x \models A \rightarrow C$, $x \models B \rightarrow C$, but $x \not\models (A \vee B) \rightarrow C$; so for some y

$$y \models A \vee B \qquad xy \not\models C.$$

Thus for some u, v, w: $w + y = u + v$, and $u \models A$, $v \models B$, and by the original suppositions, $xu \models C$ and $xv \models C$, so the Lemma (right to left) gives $xu + xv \models C$; since $xu + xv = x(u + v) = x(w + y) = xw + xy$, we have $xw + xy \models C$, which contradicts (by the left to right direction of the Lemma) $xy \not\models C$.

4 The completeness of R^+ This section establishes that every formula which is valid on every R-frame is a theorem of R^+; it consists of a routine adaptation of the argument to be found, e.g., in [6]. We will provide a single frame and a model on that frame, called respectively the canonical frame and the canonical model for the system R^+ with the property that the formulas true at its 1 element are precisely the theorems of R^+: thus any nontheorem fails to be true at the 1 element in this model, and hence is not valid on the frame in question.

For the remainder of this section $\langle S, \cdot, +, 1, 0\rangle$ will denote a certain algebraic structure to be called the canonical frame for R^+, though it will of course have to be verified after the definition is given that the algebra concerned really is an R-frame. A set of formulas which contains B whenever it contains $A_1, \ldots, A_n$ and $R^+ \vdash (A_1 \wedge \ldots \wedge A_n) \rightarrow B (n \geq 1)$ will be called *deductively closed.* Then the elements of the set S are to be all of the nonempty deductively closed sets of formulas of the language of R^+. The operation $\cdot$ is defined by:

$$xy = \{B: \text{for some } A \in y,\ A \rightarrow B \in x\}.$$

It is easy to verify (cf. [6][6]) that for deductively closed x, y, the set xy so defined is also deductively closed; so to show that whenever $x, y \in S$, the product $xy \in S$, it suffices to check that if x and y are not only deductively closed but also nonempty, xy is nonempty. Here we note that permuting twice on an instance $(A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow (B \rightarrow C))$ of the schema Id gives $R^+ \vdash A \rightarrow (B \rightarrow [(A \rightarrow (B \rightarrow C)) \rightarrow C]$. Thus given some $A \in x$, $B \in y$, we conclude that xy is also nonempty, containing for every formula C at least the formula $(A \rightarrow (B \rightarrow C)) \rightarrow C$. It is similarly straightforward to verify (again cf. [6]) that the operation $\cdot$ is commutative (appealing to Perm) and that it satisfies $(xy)z \subseteq x(yz)$ (appealing to Suff); from these last two facts it follows that $\cdot$ is associative. The identity element for $\cdot$, 1, is defined to be the set of theorems of R^+. Since MP and Adjunction guarantee that this set is deductively closed, it belongs to S. The definition of deductive closure and the axiom schema Id further ensure that $1 = x$ for all $x \in S$. The operation $+$ is defined by:

$$x + y = x \cap y.$$

The required semilattice properties are then immediate. 0 is to be the set of all formulas of the language (giving $x + 0 = x$). (Recall that while thinking of x and y as regions of logical space, $x + y$ is in general a larger region including both of them, here + contracts them to their intersection. This is because we here 'identify' such a region with the set of formulas true over it, and since the formulas true over the larger region are precisely those true over each of the regions summed, the set of formulas true shrinks as the region grows. Thus $x \leq y$ holds when $y \subseteq x$.)

We now show $x(x + 1) = x^2$ in the unreduced form (Ring Distribution being justified next) $x^2 + x = x^2$. This means simply that $x^2 \cap x = x^2$, i.e., $x^2 \subseteq x$, which follows by appeal to Contrac. Turning now to Ring Distribution we show:

$$x(y + z) = xy + xz.$$

First, $x(y + z) \subseteq (xy + xz)$: since + is $\cap$, by symmetry it suffices to show $x(y + z) \subseteq xy$. So suppose $A \in x(y + z)$, i.e., for some $B \in y + z$, $B \to A \in x$. Since $B \in y$, it follows immediately that $A \in xy$. Next, we show $(xy + xz) \subseteq x(y + z)$. Suppose $A \in xy + xz$; then there are formulas B, C with $B \in y$, $B \to A \in x$, $C \in z$, $C \to A \in x$. By deductive closure and $\vee$Elim, then, $(B \vee C) \to A \in x$. But by $\vee$Intr1,2 $B \vee C \in y, z$. So $B \vee C \in y + z$ and hence $A \in x(y + z)$. The nontrivial direction of the third of our separately listed conditions on R-frames (Zeroing), namely $0 \subseteq 0x$, follows from the fact that x is nonempty.

This leaves the Decomposition Property to check. Our argument makes use of the concept of the deductive closure ("*dc*") of a set of formulas, where if Γ is a set of formulas $dc(\Gamma)$ denotes the smallest deductively closed set of formulas including Γ. What we have to show is that if $x \leq u + v$ then there are $u' \leq u$, $v' \leq v$ such that $u' + v' = x$ (all quantifiers ranging over S, of course). That is, we must show that if $u \cap v \subseteq x$ then there are supersets u', v', of u, v, respectively, with $u' \cap v' = x$. So suppose that $u \cap v \subseteq x$. Then let $u' = dc(u \cup x)$, $v' = dc(v \cup x)$. We must show that the intersection of these two sets is precisely x. But it is clear immediately that if $A \in x$, then A belongs to each of these sets. Suppose, conversely, that: (i) $A \in dc(u \cup x)$ and (ii) $A \in dc(v \cup x)$; then there is, by (i) a formula $B \in u$ and a formula $D \in x$ such that $R^+ \vdash (B \wedge D) \to A$. [An initial reaction might be that we should allow for formulas $B_1, \ldots, B_m \in u$ and similarly with v; but since these sets are deductively closed, we may just as well consider the conjunction of these B_i as the single formula B.] And by (ii), there are formulas $C \in v$ and $E \in x$ with $R^+ \vdash (C \wedge E) \to A$. From these two theorems of R, we easily get that

$$R^+ \vdash (B \wedge (D \wedge E)) \to A \text{ and } R^+ \vdash (C \wedge (D \wedge E) \to A.$$

Hence, by Adjunction and $\vee$Elim:

$$R^+ \vdash [(B \wedge (D \wedge E)) \vee (C \wedge (D \wedge E))] \to A.$$

So appealing to $\wedge/\vee$ Distrib:

$$R^+ \vdash [(B \vee C) \wedge (D \wedge E)] \to A.$$

Now since $B \in u$ and $C \in v$, by $\vee$Intr1,2 the formula $B \vee C \in u \cap v$, and as we are supposing that $u \cap v \subseteq x$, we have $B \vee C \in x$. But also $D, E \in x$ so that

$D \wedge E$, and hence also $(B \vee C) \wedge (D \wedge E)$, belongs to x. So by the theorem of R^+ last cited, we get the desired conclusion that $A \in x$.

Having shown the canonical frame for R^+ to be an R-frame, we proceed to supply it with a V to yield the canonical model for the system; we stipulate that for $x \in S$, $V(A,x) = \mathbf{1}$ if $A \in x$, for all atomic A, and $V(A,x) = \mathbf{0}$ otherwise. Note that the conditions (C+), (C0) on models are satisfied because of the definitions of 0 and +. We now show that what was said for V and atomic formulas holds for $\models$ and formulas in general.

Fundamental theorem for R^+ *For every formula A, and all $x \in S$:*

$$\langle S, \cdot, +, 1, 0, V\rangle x \models A \textit{ iff } A \in x.$$

Proof: As usual, by induction on the complexity of A. The case of A atomic and the inductive steps for $A = B \wedge C$ or $A = B \rightarrow C$ are familiar from the literature so we deal only with the case of $A = B \vee C$.

"Only if" direction: suppose $x \models B \vee C$; then for some y, z, $x = y + z$, $y \models B$, $z \models C$. So (induct. hyp.) $B \in y$, $C \in z$. Then $B \vee C \in y, z$ by $\vee$Intr1,2; therefore $B \vee C \in y \cap z = y + z = x$.

"If" direction: suppose $B \vee C \in x$. Then by $\vee$Elim $dc(\{B\}) \cap dc(\{C\}) \subseteq x$. We now appeal to the Decomposition Property, established above for the canonical frame, conclude that there are u', v' including respectively $dc(\{B\})$ and $dc(\{C\})$, and therefore containing, and so by induct. hyp. verifying, respectively B and C, and which are further such that $u' + v' = x$. Therefore $x \models B \vee C$.

As a corollary to the Fundamental Theorem we have the

Completeness theorem *If a formula A is valid on all R-frames, it is a theorem of R^+.*

Proof: If $R^+ \nvdash A$, then in the canonical model just defined $1 \not\models A$ because $A \notin 1$; thus A is not valid on the canonical frame for R^+, which is an R-frame.

5 Additional vocabulary Four sentential constants, normally written T, t, F, and f, have played a prominent role in studies in relevant logic ([5] provides a pleasant discussion). Because of its intimate connections with (relevant) negation, a discussion of the last of these, "f", has no place in a paper on positive R. The other three are usually governed by axiom schemata along the following lines:

(T) $A \rightarrow T$
(t1) t
(t2) $A \rightarrow (t \rightarrow A)$
(F) $F \rightarrow A$.

To core with these enrichments of the language of positive R together with the corresponding extensions of the logic R^+, taken either severally or individually, we extend the definition of truth by the stipulation that: In any model $\langle S, \cdot, +, 0, 1, V\rangle$, for all $x \in S$ we have

$$x \models T \text{ iff } x = x$$
$$x \models t \text{ iff } x + 1 = 1$$
$$x \models F \text{ iff } x = 0.$$

Soundness and completeness results for these extensions of R^+ are provided by methods like those of the preceding two sections. For example, the valid formulas of the language with just t added are precisely the theorems of system whose axiomatization is that of R^+ (with schematic letters now ranging over formulas of the enriched language) with (t1) and (t2) as added axioms; the valid formulas of the language also having F in its vocabulary are those provable from the extension of this system with (F), and so on. The soundness parts of these claims are left to the reader. A point to bear in mind is that it is not just a matter of checking that the axioms are all valid: an eye must be kept on retaining the Lemmas from Section 3 intact as we pass to richer languages. In this connection, note that the right-hand sides of the biconditionals above extending the truth-definition above by the new atomic cases for the sentential constants impose in each case a certain condition—call it $\Phi(x)$—on a point x which it requires the point to meet in order to verify the constant in question. Thus it suffices to check for the Plus Lemma that for each of the three choices of Φ we have $\Phi(x + y)$ iff $\Phi(x)$ and $\Phi(y)$, and similarly for the Zero Lemma that we always have $\Phi(0)$. Completeness uses, for the various systems, canonical models whose S component consists of all the nonempty deductively closed sets of formulas of the language concerned, where of course deductive closure is understood in terms of provability in the particular system. If (F) is present, the observation that the zero of the canonical frame is $dc(\{F\})$ is all that is called for to show that the Fundamental Theorem of the previous section extends to this case. Similarly, if (T) is present, we show that T belongs to every element of the canonical frame by the fact that these are all *nonempty* as well as deductively closed, so that any formula in such a set implies, by (T), that T also belongs to the set. The remaining case is slightly more interesting and so is left for the reader's amusement.

We should not leave the subject of these sentential constants without pausing to remark in connection with the various extension of R^+ by a selection of the above axioms which includes (F) that the system in question is not the fragmenting the constants involved of the full system R, being nonconservatively extended by the axioms governing (De Morgan) negation or the constant f. This observation is due to Meyer, who at p. 17 of [11] cites the following example of a "missing theorem-schema" (provable in the full system): $(A \to F) \vee ((A \to F) \to F)$. Meyer also has some interesting comments on when one should, and when one should not, be alarmed at the prospect of nonconservative extension.

We turn now from the sentential constants to the treatment of fusion, to be written "$\circ$". This is the binary connective added to the vocabulary of R in order to have a form of the exportation/importation principles for relevant implication. A common pair of schemata which are used to obtain this extension are:

$$A \to (B \to (A \circ B)) \qquad (\circ\mathrm{Intr})$$
$$(A \to (B \to C)) \to ((A \circ B) \to C) \qquad (\circ\mathrm{Elim})$$

It is well known that the first of these schemata can be replaced by the converse of the second and that either of these three-variable schemata can be taken in "rule" rather than axiom form without deductive loss given the implicational axioms of R.

Some interest attaches to the extension of R^+ itself by ($\circ$Intr,Elim); in view of [15] one may expect things to work out satisfactorily by defining truth for fusion formulas thus:

(**) $x \models A \circ B$ iff for some y, z, such that $x = yz$, $y \models A$ and $z \models B$.

But with such a clause, I have not been able to show that the Plus Lemma continues to hold, in order to secure which I opt instead for:

(***) $x \models A \circ B$ iff for some w, y, z, $w + x = yz$ and $y \models A$ and $z \models B$.

In terms of our earlier abbreviation, we could express the right hand side here by: for some y, z such that $x \leq yz$, $y \models A$ and $z \models B$. This makes the proof of the left to right direction of the Lemma (that if $u + v \models B \circ C$ then $u \models B \circ C$ and $v \models B \circ C$) immediate. For the converse, suppose $u \models B \circ C$ and $v \models B \circ C$, so that there exist u', v', y, z, y', z' with $u + u' = yz$, $v + v' = y'z'$, $y \models B$, $z \models C$, $y' \models B$, $z' \models C$. Then by the inductive hypothesis $y + y' \models B$ and $z + z' \models C$; so by the truth-definition $(y + y')(z + z') \models B \circ C$. But, distributing, $(y + y')(z + z') = yz + yz' + y'z + y'z'$. What we want is a point w such that $w + (u + v)$ is the product of a B-verifying and a C-verifying point; then the truth-definition allows us to conclude that $u + v \models B \circ C$. But by the application just cited of the ring-distributive law, we may take the desired w to be: $(y'z + yz') + (u' + v')$. The remaining details of the soundness and completeness proof for this extension of R^+ again present some interest but no essential difficulty and are accordingly left for the reader to supply.

The difficulties with Urquhart's clause (**) for fusion do not essentially involve disjunction. It was mentioned in Section 3 that the formulas in $\rightarrow, \wedge, \circ$ valid on Urquhart's semantics do not coincide with the theorems of R involving just these connectives. Here is an example of a schema valid when (**) is employed in the truth-definition, but not provable in R^+ extended by the fusion axioms:

$$((A \rightarrow B) \wedge (A \circ C)) \rightarrow B.$$

Interestingly, the techniques of [14] allow one easily to show that extending R^+ by this schema gives a system whose theorems are precisely the formula verified in all Routley–Meyer ternary-relational models in which the relation R satisfies the condition that $Rxyz$ implies $Rxzz$, a condition which, when imposed on the models, renders unfalsifiable all instances of the earlier pair of supernumerary schemata in $\rightarrow$, $\wedge$, and $\vee$.[7] It would be worth investigating to see if further work along these lines led to a simpler and more natural axiomatization of Urquhart's logic than is currently available ([2], [8]).

Another topic worth exploring would be the modifications needed to the present semantics in order to get validity to match provability in various subsystems of R^+. The condition of pseudo-idempotence is particularly readily detached, contributing towards nothing but the validation of Contrac. Keeping everything else the same, but deleting this condition, we obtain a semantics for the result (positive $R-W$, or positive RW, as it is sometimes called) of removing this schema from our axiomatization of R^+. Another question concerns the precise effect of replacing the clause (*) for disjunction in Section 3 with something analogous to (***), with '+' in place of '$\cdot$' on the right hand side, and concomitantly deleting the Decomposition condition from the definition of an R-frame. (The role of that condition in our discussion has been simply to carry the inductive proof of the crucial Plus Lemma through the case of disjunction, which would now be secured automatically—as with (***) for fusion—by the form of the truth-definition.) The idea would be to provide a model-theoretic treatment of Ortho-R^+, axiomatized by dropping $\wedge/\vee$-Distrib from our list of schemata for R^+.[8] For this axiom is no longer valid on the semantics thus revised, though $\vee$Intr1,2 and $\vee$Elim survive unscathed. A snag, of whose surmountability I am uncertain, with this line of thought arises over the '$\vee$' case of the proof of the Fundamental Theorem, on which our completeness proof rested. For in showing that truth implied membership for disjunctive formulas we appealed to the result that the canonical frame for R^+ met the Decomposition condition, a result itself established by, *inter alia*, an appeal to the presence of $\wedge/\vee$-Distrib.

We close with some further questions arising out of the extension of Urquhart's semantics presented in this paper, reserving for Appendix B some remarks on the extension to quantifiers. What are the prospects for treating fission with at least the same elegance as fusion? Or is this connective too heavily involved with negation for any similar treatment to be expected? And finally, what of negation (in one or other form) itself? Were Urquhart's pessimistic comments in [15] on this score perhaps premature? After all, one would need to have been similarly pessimistic about disjunction in R^+ if the operational semantics in the original shape it has in [15] had to be taken as the final arbiter of intelligibility.

Appendix A: Idempotence and mingle As was mentioned in Section 3, there is a problem about extending one particularly appealing feature of the Urquhart semilattice semantics to incorporate the operation $+$ which has played so prominent a role in our discussions: the idempotence of $\cdot$. In this appendix, I should like to amplify this remark. There is the following difficulty about trying to get the product operation in the canonical frames to be idempotent, or more particularly, in trying to show that x is included in x^2 (for we already know the converse inclusion holds): suppose $A \in x$; we must find some B for which both B and $B \to A$ belong to x (to show $A \in x^2$). But this B, which may depend upon the given A, of course, for which reason let us denote it by $\Omega(A)$, will then have to be provably implied by A, as will the formula $\Omega(A) \to A$, for us to be able to conclude, when the sole information we have about A and x is that $A \in x$, that these formulas also both belong to x. Now, if the Mingle schema $A \to (A \to A)$ were available, this would allow us to take $\Omega(A)$ to be A itself. But in fact, whenever there is a formula $\Omega(A)$ available, the Mingle formula in A is a consequence of its availability. That is, from $A \to (\Omega(A) \to A)$ and $A \to \Omega(A)$ we can deduce in R^+ the formula $A \to (A \to A)$. Hence there is no question of showing that the product operation in the canonical R-frame for the system R^+ itself is idempotent.

However, this does not show that some completeness proof, organized (as for example Urquhart's was in [15]) along quite different lines, might not succeed in showing R^+ have as theorems all those formulas valid on every R-frame satisfying the idempotence condition. But we can show that there is no such completeness result to be had, at least without some alteration to the definition of truth or to the concept of a model (such as tampering with condition (C)), and the demonstration again focuses on the Mingle schema. Although in Urquhart's semantics, imposing the condition of idempotence (as he did) on the product operation did not render this schema valid, in the present setting that is indeed the effect of this condition. For suppose that $x \models A$ yet not $x \models A \to A$. Then for some y, $y \models A$ and not $xy \models A$. By the Lemma from Section 3, since A is true at each of x,y, we have $x + y \models A$; so by the assumption that $\cdot$ is idempotent, $x + y = (x + y)^2 = x^2 + xy + y^2$, i.e., something of the form $xy + \dots$, so by the Lemma again, since $x + y \models A$, we must have $xy \models A$—a contradiction. (Compare Urquhart's way of capturing the Mingle schema semantically via a condition on models, mentioned above in Section 2.) We conclude from these observations that the system axiomatized by adding the Mingle schema to the axioms of R^+ has for its theorems all and only those formulas of L^+ which are valid on every structure $\langle S,\cdot,+,1,0\rangle$ in which S is a semilattice under $\cdot$ with identity 1 and a semilattice under $+$ with identity 0, and the ring-distribution, zeroing, and decomposition conditions from Section 2 hold.

Appendix B: Universal quantifiers In this appendix, we indicate how the semantics of the present paper can be extended to quantified (positive) relevant logic. More specifically, we consider the addition of the universal quantifier. Take each atomic formula which is not a sentential constant to consist of an n-place predicate letter followed by n individual parameters (or 'individual constants'). There is assumed to be at least one such predicate letter (for some n), and countably many individual parameters in all. Complex formulas are formed by the operations of conjunction, disjunction, implication and fusion; we also presume the sentential constants t and F discussed in Section 5 are present. If A is a formula containing a parameter c then $(\forall v)A'$ is also a formula where A' is like A except that it has the individual variable v where A has c. As with individual parameters, these variables come in countably infinite supply. We interpret this language in a highly substitutional way, objectualizing the interpretation being a routine matter. Atomic formulas paired with elements of the carrier set of R-frames in the sense of Section 3 are assigned truth-values by the function V which turns such a frame into a model; the conditions (C0) and (C+) of that section remain in force. The definition of truth runs as before, with the quantifiers being dealt with by:

$$\langle S,\cdot,+,1,0,V\rangle\ x \models (\forall v)A(v) \text{ iff for each individual parameter } c\ \langle S,\cdot,+,1,0,V\rangle\ x \models A(c)$$

where $A(c)$ is the result of replacing all occurrences of v in A by c.

To the axiom-schemata and rules from the main body of this paper governing the various connectives we have assumed present, the following two governing the universal quantifier are to be added:

($\forall$Elim) $(\forall v)A(v) \to A(c)$ with the notation to be understood as above.

($\forall$Intr) $\dfrac{\vdash A \to B(c)}{\vdash A \to (\forall v)B(v)}$ provided the parameter c does not occur in A.

There is no problem verifying that all theorems of this system are valid on every R-frame when the current truth-definition is used. But there is a problem about adapting the completeness-proof method of [6], [14], or Section 4 above, to this setting. This difficulty (amongst others) is mentioned by Routley at p. 336 of [13]. We would like to prove that truth and membership coincide for the canonical model here as in Section 4. For universally quantified statements this means showing that if $(\forall v)A(v)$ belongs to one of the elements of the canonical model, so does each instance $A(c)$ – which is no problem since this follows by deductive closure and ($\forall$Elim) – and further that whenever each such $A(c)$ belongs, so does $(\forall v)A(v)$. And with this latter requirement, we do meet a problem. Calling a deductively closed set of formulas $\forall$-complete whenever it meets the condition just described, the problem is that even if we start off with $\forall$-complete sets of formulas (such as the present logic itself, which is easily seen to meet it, taking $A = t$ in $\forall$Intr), the product of two $\forall$-complete deductively closed (nonempty) sets need not itself be $\forall$-complete. For we might have $A(c) \in xy$ for each c, so that there exists $B_1 \in y$ with $B_1 \to A(c_1) \in x$, $B_2 \in y$ with $B_2 \to A(c_2) \in x$, and so on, with neither these conditionals nor their antecedents exhibiting the commonality of form we should need in order to exploit the hypothesis that x and y are $\forall$-complete and so conclude that $(\forall v)A(v) \in xy$.

There is a simple solution to this difficulty on the present semantics which is not available (unless infinitary formulas or infinitely many further sentential constants are added – cf. [13]) to those treatments requiring that only when at least one disjunct is true at a point is the disjunction true there. Such treatments have to have not just deductively closed but "saturated" or "prime" sets of formulas as the points in the canonical model at the propositional level; sets, that is, which always contain either A or B when they contain the formula $A \vee B$. The following considerations do not apply to such canonical models.

Instead of taking the canonical R-frame in Section 4 to consist of all deductively closed nonempty sets of formulas, we could equally well – had we there already introduced the sentential constant t and the fusion connective – have taken it to consist of just those sets Γ for which there is a single formula A with Γ being the deductive closure of A. (These of course include all the sets describable as the deductive closure of a finite collection $\{B_1, \ldots, B_n\}$ of formulas since one then takes A as the conjunction of the B_i. These "formula-generated" deductively closed may conveniently be represented in the notation $[A]$, where A is the generating formula; in other words, we write "$[A]$" to abbreviate '$dc(\{A\})$'. The logic (R^+ with t, F, and $\circ$) itself is, as before, the 1 of the structure; it is available to us here as $[t]$. Similarly we use $[F]$ as the 0. Where $x = [A]$ and $y = [B]$, for xy take $[A \circ B]$, and for $x + y$, $[A \vee B]$. The reader may verify that these operations are well-defined by such stipulations, and that the resulting structure is indeed a model for which the (analogue to the) Fundamental Theorem from Section 4 holds.

The interest of this variation on the old style of canonical model is that it helps us with our difficulty about inferring the $\forall$-completeness of xy from that of x and y. We take the logic now to be that described in the opening paragraphs of this Appendix, and build a canonical model out of the formula-generated deductively closed sets of formulas of its language. This sidesteps the above difficulty because all such sets are $\forall$-complete from the outset. For suppose

$B(c) \in x$ for every individual parameter c, where $x = [A]$. Then $\vdash A \to B(c)$ for each c, and so in particular for a c_k chosen so as not to be amongst the (at most finitely many) parameters occurring in A. But this particular implication then provides a premise for the application of ($\forall$Intr) and we conclude that $\vdash A \to (\forall v)B(v)$, and hence that x contains $(\forall v)B(v)$.

This simple argument contrasts strikingly with the situation in the Routley–Meyer semantics, with respect to which Kit Fine has recently shown (in work as yet unpublished) that the proof theory proposed by Routley and Meyer is incomplete with respect to the semantics they offer and can be completed only by the addition of further axioms whose very description involves considerable complexities. (Fine has also investigated modifying the semantics so as to get it to match the proposed proof theory; again the needed modifications are rather surprising.) Of course it remains the case that we have looked only at the positive connectives and at the *universal* quantifier. If existential quantification were to be treated in the same manner as disjunction in this paper, an infinitary version of the + operation would presumably be called for. The same holds for stating a generalized form of the condition of Decomposition which would validate the schema (invalid on the current semantics): $(\forall v)(A \vee B(v)) \to (A \vee (\forall v)B(v))$. An infinitary form of the product operation would also no doubt repay attention, though here the failure of its idempotence creates conceptual obstacles to be overcome, in connection with the 'fusion quantifier', 'Fv', say, where $(Fv)B(v)$ counts as true at x just in case x bears $\leq$ to some product of points $y_1, \ldots, y_n, \ldots$, these factors verifying, respectively, its instances $B(c_1), \ldots, B(c_n), \ldots$. But these further extensions of the present machinery will not occupy us here.

NOTES

1. These are of course various notions short of validity over a class of (e.g., modal) algebras, such as validity on a given member of the class, or being assigned the '1' of the algebra by a given homomorphism from formulas to algebra elements. But these correspond respectively to validity on a frame and to truth throughout a model on that frame, not to the notion of truth at a point in a model. The crudely drawn contrast in the text leaves out of account the matrix approach, which may be thought of as a generalization of algebraic semantics (allowing various designated elements) or as a generalization of two-valued model-theoretic semantics, depending on the particular application. The contrast may also be somewhat overstated since below when some semantic ideas from [10] are borrowed, the entities with respect to which truth is evaluated in models look much more like the elements of modal algebras than the points in a Kripke model.

2. The atomic formulas at the present stage of discussion (until Section 5) should be taken to be some stock of sentence letters; as long as there is at least one of these (so that the class of formulas is nonempty), the discussion to follow is unaffected by their exact cardinality. Complex formulas are formed in accordance with the usual inductive definition.

3. For the treatment of *RM*(ingle) in terms of the apparatus to be introduced here, see Appendix A.

4. When an earlier version of the present paper was delivered to the annual conference of the Australasian Association for Logic (Melbourne, November 1985), I was informed by Richard Sylvan that Urquhart has in fact taken just such a line, in work as yet unpublished.

5. Warning: what Urquhart calls the "empty piece of information" in [15] is not our 0 but our 1.

6. See Lemma 2, part (i), which appears on p. 353 of [6]; note that at line 10 from the base of that page, "$\cdot$" appears misprinted as "f".

7. Thinking of $Rxyz$ as (i) $xy = z$, we may infer (ii) $x^2y = xz$, by multiplying both sides by x. The left-hand sides of (i) and (ii) are then equal, on Urquhart's assumption of idempotence for $\cdot$, from which we conclude that so also are their right-hand sides, i.e., $Rxzz$.

8. The possibility of such a line of development was suggested to me by Chris Mortensen on the occasion mentioned in note 4.

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