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A System of Predicate Logic with Trans-Atomic Units

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Preliminary remarks The original idea of introducing trans-atomic units into systems of formal logic was presented at the World Congress of Philosophy in 1983. Formal development of this concept at the truth-functional level was subsequently investigated in this journal [1]. This paper extends the concept of trans-atomic (*TA*) units to Predicate Logic (*PL*).

Motivation for investigating TA units The concept of *TA* units allows the introduction of special connectives over and above the 16 limitation of standard two-valued logic without leaving the confines of a two-valued system.

The one particular connective introduced in this paper has interesting possibilities as regards its use as a causal connective in the formulation of lawlike generalizations. Briefly, the difficulties with the Philonian (material) conditional in the formulation of lawlike generalizations concern its properties as regards confirmation and support:

$(x)(Fx \rightarrow Gx)$ is confirmed (totally) by
(1) $(x)Gx$ (and consequently by the pair $\langle (x)Gx, (x)Fx \rangle$ as well as the pair $\langle (x)Gx, (x){\sim}Fx \rangle$)
(2) $(x){\sim}Fx$.
is supported by
(1) Ga (and consequently by the pair $\langle {\sim}Fa, Ga \rangle$
(2) ${\sim}Fa$.

There seems to be no escape from these difficulties. Even restricting evidence or support to instances of the corresponding conjunction, i.e., Fa & Ga, Fb & Gb, etc., is of no avail. Since $(x)(Fx \rightarrow Gx)$ is logically equivalent to $(x)({\sim}Gx \rightarrow {\sim}Fx)$ the latter would be supported by ${\sim}Fa$ & ${\sim}Ga$ and hence the former also. The partial connective, '—c', subsequently introduced avoids the

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above difficulties yet supports a type of modus ponens and modus tollens making it an interesting connective as regards discussions of causality as a connective.

These remarks are not meant to be conclusive, but rather present some introductory considerations for the study of partial connectives imbedded in *TA* units. The purpose of this paper, moreover, is not to prove the utility of partial connectives but to demonstrate their logical viability.

The system PLT The following system of predicate logic with trans-atomic units (*PLT*) is an extension of the Fitch-Suppes type system of predicate logic (*PL*) with predicate letters and individual constants but without function letters.

Trans-atomic units Atomic units such as sentence letters, predicate letters, and individual constants are assigned an extension in an interpretation on an arbitrary basis. Compound units, by contrast, are assigned an extension on the basis of a determination (which could be an assignment) of the extension of its component units. In this respect, *TA* units are partially compound and partially atomic. In some cases assignment of an extension is purely arbitrary. In other cases the assigned extension is dependent upon the assignment to its components.

Viewed from another perspective, *TA* units enable the introduction of a total of 81 connectives (3 to the power 4) into two-valued systems of logic. This total includes the 16 full truth-functional connectives with the remaining connectives being partially truth-functional. It is to be emphasized that partially truth-functional connectives are such that their truth-functionality is partially determinate and otherwise arbitrary. It is not that such connectives have values other than T,F as arguments.

Consider, for example, the connective '—c' (read 'connect') which is like the material conditional in the second case (T,F) but which is otherwise atomic. The atomic (arbitrary cases) can be indicated in truth-table fashion by repeating the case entry on the left:

P	Q	$(P$ —c $Q)$
T	T	T
T	T	F
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

(As defined, all cases except the (T,F) case are arbitrarily T or F in a given interpretation, as would be the case for an atomic sentence. The (T,F) case, by contrast, has a unique value and is determinate)

In order to get an intuitive grasp of '—c' several semantic consequences of its characterization are given:

Entailments which hold	Entailments which do not hold
1. P, $(P$ —c $Q) \models Q$	1. $\sim P \models (P$ —c $Q)$ (Duns Scotus)
2. P, $\sim Q \models \sim(P$ —c $Q)$	2. $Q \models (P$ —c $Q)$
3. $(P$ —c $Q)$, $\sim Q \models \sim P$	3. $(P$ —c $Q) \models (\sim Q$ —c $\sim P)$
	4. Transitivity of —c

On the approach taken here, morphologically distinct *TA* units are semantically independent much as morphologically distinct sentence letters are semantically independent. Thus $(P \& P$ —c $Q)$ does not entail $(P$ —c $Q)$. It is possible to take a different approach and allow semantic relations between morphologically distinct *TA* units; however, the penalty paid in semantic complexity is considerable and such an approach will not be pursued in the metalogical investigations which follow.

The introduction of *TA* units requires rethinking of the notion of a sentence of a system X. This seemingly straightforward notion does not extend directly to *TA* units. If X is a set of sentences, say, $\{(x)(Fx \rightarrow Gx), (\exists x)Mx, Fa\}$, then any sentence composed of the vocabulary of $X(\{F, G, M, a\})$ is a sentence of the system X, though, of course, not necessarily a thesis or theorem of X. This approach will not do for *TA* units. Because of their hybrid nature (partially compound and partially atomic) variations of *TA* units in a set of sentences X are not necessarily sentences of X. At the truth-functional level, only *TA* units in X are *TA* units of X. This total restriction must be relaxed at the *PL* level. Thus if X contains $(x)(Fx$ —c $Gx)$ and Fa, then the following would be sentences of X without necessarily being in X: $(Fa$ —c $Ga)$, $(Fy$ —c $Gy)$, $(z)(Fz$ —c $Gz)$.

TA units

1. If S, T are (open/closed) sentences of *PL*, then $(S$ —c $T)$ is a *TA* unit of *PLT*.
2. S is a *TA* unit only by (1). Remark: $(x)(Fx$ —c $Gx)$ is not a *TA* unit but a generalization of a *TA* unit.

Sentences of PLT

1. Sentences (open/closed) of *PL* are sentences of *PLT*.
2. *TA* units and universal closures of *TA* units are sentences of *PLT*.
3. If S, T are sentences of *PLT* then $(S \& T)$, $(S \rightarrow T)$. $\sim S$, $(S \vee T)$ are sentences of *PLT*. (Other connectives by definition.)
4. If S is not a *TA* unit and S is a sentence then $(x)S$, $(\exists x)S$ are sentences of *PLT*.

Examples

Sentences	Non-sentences
1. $(x)(Sx \rightarrow (Px$ —c $Dx))$	1. $(x)(Hx$ —c $(Gx$ —c $Hx))$
2. $(x)((Ey)Hxy$ —c $Gx)$	2. $(\exists x)(Sx$ —c $Rx)$
3. $\sim(x)(Fx$ —c $Gx)$	3. $(x) \sim (Sx$ —c $Rx)$

Inference system for PL

1. Affirming the Antecedent (modus ponens)
2. Denying the Consequent (modus tollens)
3. Conditional Proof: If S occurs on a line then $(Q \rightarrow S)$ may be entered on a subsequent line. The premise dependencies of the new line are the premise dependencies of the previous line with the exception of the premise dependencies of the line on which Q occurs.

4. Interchange: The following forms of sentences are / interchangeable: (1) $(R \vee S)/(\sim R \rightarrow S)$; (2) $(R \,\&\, Q)/\sim(R \rightarrow \sim Q)$; (3) $\sim\sim R/R$.
5. Universal Specification: From $(x)S$ to derive $S(t)$, where t is any term which does not contain a variable which is subject to capture.
6. Universal Generalization: From S to derive $(x)S$, provided x is not flagged. A variable free in a premise is flagged and remains flagged in any line in which it is free and depends upon the premise.

Inference system for PLT

1. Rules of *PL*
2. Connect Denial:

$$\frac{\begin{array}{c}(S \text{ —c } T)\\ \sim T\end{array}}{\vdash \ \sim S}$$

Derived Rule (Connect Affirm)

$$\frac{\begin{array}{c}(S \text{ —c } T)\\ S\end{array}}{\vdash \ T}$$

Possible extensions Alternative systems might find the following inference principles useful. The inference rules depend on the concept of derivability in *PL*.

1. $$\frac{\begin{array}{c}(S \text{ —c } T)\\ \vdash_{PL}(S \rightarrow T) \leftrightarrow (R \rightarrow Q)\end{array}}{\vdash \ (R \text{ —c } Q).}$$

2. $$\frac{\begin{array}{c}(S \text{ —c } T)\\ \vdash_{PL}(R \rightarrow S)\end{array}}{\vdash \ (R \text{ —c } T)}$$

3. $$\frac{\begin{array}{c}(S \text{ —c } T)\\ \vdash_{PL}(T \rightarrow R)\end{array}}{\vdash \ (S \text{ —c } R)}$$

A system X A system X of *PLT* is any set of closed sentences of *PLT*. In some treatments a deductive system or theory is held to be semantically closed, i.e., closed with respect to the relation of semantic entailment. This approach is not followed here.

Sentence of a system X Not every sentence of *PLT* built up from the predicates and individual constants which are components of elements of X is a sentence of X. Let $X = \{Fa, (x)(Fx \text{ —c } Gx)\}$. Even though $(x)(\sim Fx \rightarrow Gx)$ is a sentence of X, $(x)(\sim Fx \text{ —c } Gx)$ is not a sentence of X.

Define

$C(X)$ = {z: z is a constant which is a component of an element of X}
$P(X)$ = {z: z is a predicate which is a component of an element of X}
$TA(X)$ = {z: z is a *TA* unit which is a component of an element of X}
$MTA(X)$ = The master *TA* set of X
= $\bigcap\{z: (s)(s \in TA(X) \rightarrow s \in z) \,\&\, (u)(v)(u \in z \,\&\, (Cvu \vee SPvu) \rightarrow v \in z)\}$; where Cvu iff v is a universal closure of u and $SPvu$ iff v is a specification u (which, if constants are used, uses only constants of $C(X)$). Informally, $MTA(X)$ contains the *TA* units which are components of elements of X and closures and specifications of elements.
$S(X)$ = The symbol set of X
= $C(X) \cup P(X) \cup MTA(X)$

$SS(X)$ = The sentence set of X
= $\{z$: z is a sentence of *PLT* based on $S(X)\}$.

Interpretation M of a PLT system X Define $M = \langle A, D\rangle$ where D is a nonempty set and A: $S(X) \to D \cup \mathbf{P}(D) \cup \mathbf{P}(D^2) \ldots \cup \mathbf{P}(D^n)$ such that

1. If $z \in C(X)$ then $A(z) \in D$.
2. If $z \in P(X)$ and z is of degree n then $A(z) \in \mathbf{P}(D^n)$.
3. If $z \in TA(X)$ and the prototype of z ($pro(z)$) is of degree n then $A(pro(z)) \in \mathbf{P}(D^n)$.

Explanation of terms: The prototype of a *TA* S is the *TA* S' obtained from S by replacing each free variable and constant in S with a new variable from a standard sequence of variables.

Examples: Let the standard sequence be $\langle u, v, w, x, y, z\rangle$, then

1. $pro((Fx$ —c $Gxaa)) = (Fu$ —c $Gvwx)$
2. $pro((Fa$ —c $(\exists y)Gay)) = (Fu$ —c $(\exists y)Gvy)$
3. $pro((Fa$ —c $(\exists u)Gau)) = (Fv$ —c $(\exists u)Gwu)$.

The degree of a prototype *TA* is the number of free variables in the *TA*. The degree of (1) above is 4 and the degree of (2) is 2.

Remark: Not every *TA* is assigned an extension in an interpretation. Only *TA*'s which are prototypes are assigned an extension.

$M \models S(d)$: M models S at d (d satisfies S with respect to M)

Let (1) X = *a system of PLT* sentences and $S \in X$
(2) $M = \langle A, D\rangle$
(3) VC = the set of variables of *PLT* and elements of $C(X)$
(4) P^n be a predicate letter of degree n which is an element of $P(X)$
(5) t_i be any variable or constant which is an element of VC.

Define d: $VC \to D$ such that if c is a constant then $d(c) = A(c)$.
$M \models S(d)$ defined: If S is of the form

1. $P^n t_1 \ldots t_n$ then $M \models S(d)$ iff $\langle d(t_1), \ldots, d(t_n)\rangle \in A(P^n)$
2. $(Q \to R)$, $(Q \,\&\, R)$, $(Q \vee R)$, $\sim R$, standard.
3. $(\mathfrak{v})Q$ and $\mathfrak{v}$ is a variable $\in VC$ then $M \models S(d)$ iff $M \models Q(d')$ for every d' which differs at most from d only at $\mathfrak{v}$ $(d' =_{\mathfrak{v}} d)$.
4. $(R$ —c $Q)$ and $t_1, \ldots, t_n$ are the terms of S in order of occurrence then $M \models S(d)$ iff $\langle d(t_1), \ldots, d(t_n)\rangle \in A(pro(z))$.

$M \models S$: M models S (S is true in M) $M \models S$ iff $M \models S(d)$ for all d.

Observation 1: If $d(\mathfrak{v}) = d(c)$ and $\mathfrak{v}$ is the one free variable in $S(\mathfrak{v})$ then $M \models S(\mathfrak{v})(d)$ iff $M \models S(c)$.

Observation 2: If S is closed then $M \models S$ iff $(\exists d)M \models S(d)$.

$M \models X$ $\quad$ $M \models X$ iff

(1) if $S \in X$ then $M \models S$, for all S. (Distributive Requirement)
(2) if $M \models R$ —c Q then $\sim(\exists d) M \models_{PL} (R \mathbin{\&} \sim Q)(d)$. (Collective)
Comment: It is at this point that M_{PLT} appeals to M_{PL}.

It is also to be noted that if M models X distributively it does not thereby follow that M models X collectively, i.e., that $M \models X$. If $M \models (R$ —c $Q)$, R, $\sim Q$, then $\sim(M \models \{(R$ —c $Q)$, R, $\sim Q\})$. The appeal to M_{PL} is quite legitimate since every M_{PLT} defines uniquely an M_{PL} for sentences of X which do not contain *TA* units and R, Q cannot be *TA* units if they are components of a *TA* unit.

Saturation lemma (based on Lindenbaum's Lemma) If X is a consistent system of sentences of *PLT* then there is a consistent saturated extension of X. A system X of sentences is said to be saturated iff for every closed sentence S of X either $S \in X$ or $\sim S \in X$. Note that every saturated system is complete but not vice versa. The system $X = \{P, (P \rightarrow Q)\}$ of *SL* is complete but not saturated since neither Q nor $\sim Q$ are elements of X. However, if X is closed under $\vdash$ and complete then X is saturated. Let $A_1, \ldots, A_n, \ldots$ be an enumeration of all closed sentences of X. Define a sequence of systems $X_0, \ldots, X_n, \ldots$ such that $X_0 = X$ and $X_{n+1} = X_n \cup \{A_{n+1}\}$ if $\sim(X_n \vdash \sim A_{n+1})$; otherwise $X_{n+1} = X_n \cup \{\sim A_{n+1}\}$. Define $XX = \bigcup\{X_n: n \in \mathbf{N}\}$.

Consistency of XX Since $X_0 = X$, X_0 is consistent by hypothesis. By hypothesis of induction (HI) X_n is consistent. Assume $\sim(X_{n+1}$ cons). Assume $X_{n+1} = X_n \cup \{A_{n+1}\}$. Then, by the construction of X_{n+1}, $\sim(X_n \vdash \sim A_{n+1})$. But by assumption, $X_n \cup \{A_{n+1}\} \vdash R \mathbin{\&} \sim R$. In which case $X_n \vdash A_{n+1} \rightarrow (R \mathbin{\&} -R)$ and $X_n/\sim A_{n+1}$ which contradicts the construction of X_{n+1}. Assume $X_{n+1} = X_n \cup \{\sim A_{n+1}\}$. Then, by construction of X_{n+1}, $X_n \vdash \sim A_{n+1}$. But if X_n is consistent and $X_n \vdash \sim A_{n+1}$ then X_{n+1} cons, contradicting the main assumption.

The Gödel-Henkin theorem for PLT (GH)

Theorem $\quad$ *X cons $\rightarrow (\exists M) M \models X$.*

Proof: Let $A_1(v_1), \ldots, A_n(v_n), \ldots$ be an enumeration of sentences of X with one free variable. Let $B = \{b_1, \ldots, b_n, \ldots\}$ be a denumerable set of individual constants not in $C(X)$.

Definition $\quad$ $S_k = \sim(v_k)A(v_k) \rightarrow \sim A_k(b_k)$
$X_0 = X \cup \{Fc \vee \sim Fc$: $F \in P(X) \mathbin{\&} c \in B\}$. Comment: This merely extends the symbol set of X_0 such that $S(X_0) = S(X) \cup B$.
$X_{n+1} = X_n \cup \{S_{n+1}\}$
$X^* = \bigcup\{X_n: n \in \mathbf{N}\}$

Theorem $\quad$ *X^* is consistent.*

Proof: 0: X_0 is consistent since X is consistent and adding tautologies to a consistent system does not affect the system.

$n + 1$: By hypothesis of induction X_n is consistent. Assume X_{n+1} is inconsistent. Then

1. $X_n \cup \{S_{n+1}\} \vdash R \,\&\, {\sim}R$
2. $X_n \vdash {\sim}S_{n+1}$
3. $X_n \vdash (v_{n+1})A_{n+1}(v_{n+1})$ (From (2), denial of a conditional)
4. $X_n \vdash A_{n+1}(b_{n+1})$ (From (2), denial of a conditional)
5. $X_n \vdash A_{n+1}(x)$
   Since b_{n+1} occurs in no sentence of X_n, b_{n+1} does not occur in the premises X_n of the derivation D of $A_{n+1}(b_{n+1})$. Choose a variable x which does not occur in D. Construct D^* by replacing occurrences of b_{n+1} with x. Then D^* is a derivation of $A_{n+1}(x)$ from X_n.
6. $X_n \vdash (x)A_{n+1}(x)$ Since x is not a flagged variable (x does not occur free in the premises of D^*).
7. (6) contradicts (3).

Construction of XX By the Saturation Lemma, let XX be a consistent saturated extension of X^*.

Interpretation M of XX

1. $M = \langle A, D \rangle$
2. $D = C(X)$
3. $A(c) = c$
4. $A(F^n) = \{\langle c_1, \ldots, c_n \rangle : F^n c_1 \ldots c_n \in XX\}$
5. $A(pro(R \text{ —c } Q)) = \{\langle c_1, \ldots, c_n \rangle : pro(R \text{ —c } Q)c_1/v_1 \ldots c_n/v_n \in XX\}$. Where $pro(R \text{ —c } Q)c_1/v_1 \ldots c_n/v_n$ is the result of substituting c_i for v_i in $pro(R \text{ —c } Q)$ which is of degree n.

Theorem *$S \in XX$ iff $M \models S$ (distributive result).*

Proof: By induction on the length of S.

0: S has no connectives. S is of the form $F^k c_1 \ldots c_k$. By definition $M \models F^k c_1 \ldots c_k$ iff $\langle c_1, \ldots, c_k \rangle \in \{\langle c_1, \ldots, c_k \rangle : F^n c_1 \ldots c_k \in XX\}$ iff $F^k c_1 \ldots c_k \in XX$.

$n + 1$: *Cases 1-4:* S is of the form ${\sim}Q$ or $(R \vee Q)$ or $(R \,\&\, Q)$ or $(R \rightarrow Q)$: standard.

Case 5: S is of the form $(v)Q$. Let Q be $R_k(v_k)$. Assume $v = v_k$, as otherwise $R_k(v_k)$ is closed and Case 5 reduces to the previous cases. *Assume* $S \in XX$ and ${\sim}(M \models S)$. Then $(\exists d)\, {\sim}(M \models (v)R_k(v)(d))$ and for some $d' =_v d$ ${\sim}(M \models R_k(v)(d))$. Suppose $d'(v) = c$, then $d'(v) = d'(c)$, since $d(c) = c$ for all d. By observation 1, ${\sim}M \models R_k(c)$. By hypothesis of induction ${\sim}(R_k(c) \in XX)$. But from the assumption $XX \vdash R_k(c)$ by Universal Specification. *Assume* $M \models S$ and ${\sim}(S \in XX)$. By the saturation of XX, ${\sim}S \in XX$. By the construction of XX, $({\sim}S \rightarrow {\sim}R_k(b_k)) \in XX$. Hence, ${\sim}R_k(b_k) \in XX$. By the hypothesis of induction $-(M \models R_k(b_k))$. But if $M \models (v)R_k(v)(=S)$ then $M \models R_k(b_k)$. Proof of Case 5 is also standard.

Case 6: S is of the form $(R \text{ —c } Q)$, where R, Q are both closed. By definition $M \models (R \text{ —c } Q)$ iff $(\exists d)$ $M \models (R \text{ —c } Q)(d)$. Also by definition $M \models (R \text{ —c } Q)(d)$ iff $\langle d(t_1), \ldots, d(t_n) \rangle \in A(pro(R \text{ —c } Q))$, assuming $t_1, \ldots, t_n$ are the terms of $(R$ —c $Q)$. But $\langle d(t_1),\ldots,d(t_n)\rangle = \langle d(c_1),\ldots,d(c_n)\rangle = \langle c_1,\ldots,c_n\rangle$. Hence $M \models (R$ —c $Q)(d)$ iff $\langle c_1,\ldots,c_n\rangle \in A(pro(R$ —c $Q))$. Since $A(pro(R$ —c $Q)) = \{\langle c_1 \ldots c_n\rangle: pro(R$ —c $Q)c_1/v_1 \ldots c_n/v_n \in XX\}$, and $pro(R$ —c $Q)c_1/v_1 \ldots c_n/v_n = (R$ —c $Q)$, $M \models (R$ —c $Q)$ iff $(R$ —c $Q) \in XX$.

Theorem *If* $M \models (Q$ —c $R)$ *then* $\sim(\exists d)M \models_{PL} (Q \,\&\, {\sim}R)(d)$ (*collective result*).

Proof: Assume $M \models (Q$ —c $R)$ and that $(Q$ —c $R)$ is closed. Assume further $(\exists d)M \models_{PL} (Q \,\&\, {\sim}R)(d)$, i.e., $M \models Q(d)$ and $M \models {\sim}R(d)$ for some d. Since M_{PL} is a restriction of $M(M_{PLT})$, $M \models Q(d)$ and $M \models {\sim}R(d)$. Since in general, if S is closed, $M \models S$ iff $(\exists d)M \models S(d)$, $M \models Q$ and $M \models {\sim}R$. By the Distributive Result, Q, ${\sim}R$, $(Q$ —c $R)$ are elements of XX. This contradicts the consistency of XX.

Theorem *If* XX *is cons then* $(\exists M)M \models XX$.

Proof: This follows by definition from the Distributive and Collective Results.

Theorem *If* X *cons then* $(\exists M)M \models X$ (*the Gödel-Henkin theorem*).

Proof: Since there is an M for every XX such that $M \models XX$ and XX is a superset of X, there is an M for every X such that $M \models X$, given X and XX consistent.

Theorem $X \models S$ *only if* $X \vdash S$ (*completeness*).

Proof: It may be assumed that X is consistent (otherwise completeness is trivial). Assume ${\sim}(X \vdash S)$. By Lindenbaum's Lemma $X \cup \{{\sim}S\}$ is consistent. By **GH** $X \cup \{{\sim}S\}$ has a model M. M is also a model of X. From the hypothesis it follows that every model of X is a model of S. This leads to the absurdity that $M \models S, {\sim}S$.

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