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Reduced Models for Relevant Logics Without WI

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A major motivating concern of the relevant logics has been to devise a decent theory of the logical behavior of the operation of implication, expressed in the object language by '$\rightarrow$' and corresponding to the *relation* of implication which is naturally expressed in the metalanguage. That being so, it should be expected that

$$p \rightarrow q$$

be true if and only if p implies q, and more generally that the conditional be true in model m iff p implies q according to m. A conditional is true in m iff it holds at the base world (the real world) in m, and its antecedent implies its consequent according to m iff the consequent holds in m at every world at which the antecedent holds. The motivationally natural thought just sketched emerges in the Routley-Meyer semantics as the requirement that the base world O should satisfy, for all a and b,

$$a \leq b \text{ iff } ROab.$$

A frame in which this requirement is met is said to be *reduced*. Note that a reduced frame satisfies in particular

$$ROOO,$$

whence the *truth* in any model in such a frame is closed under detachment for the arrow as one might expect. It is well known that the system R is characterized by reduced frames (see [9] for example). What is less well known is the scandal that the arguments establishing reduced modelings for R and its close relatives do not extend to the weaker systems, lacking the theorem scheme WI

$$(A \rightarrow B) \mathbin{\&} A \rightarrow B,$$

supported as insightful by such as Brady [2], Priest [7], Routley [8,10], and the author. The object of the present paper is to prove a reduced modeling theorem

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for a good range of logics which previously escaped. It employs techniques previously used by Meyer in [4] to show material detachment admissible for R and by the author in [11] to show the primeness of three systems weaker than R.[1]

The sentential logic **B** has the connectives &, ∨, →, ~. It may be axiomatized with the following axiom and rule schemes:

(A1) $A \to A$
(A2) $A \,\&\, B \to A$
(A3) $A \,\&\, B \to B$
(A4) $(A \to B) \,\&\, (A \to C) \to .\ A \to B \,\&\, C$
(A5) $A \to A \vee B$
(A6) $B \to A \vee B$
(A7) $(A \to C) \,\&\, (B \to C) \to .\ A \vee B \to C$
(A8) $A \,\&\, (B \vee C) \to (A \,\&\, B) \vee C$
(A9) $\sim\sim A \to A$
(R1) $A \to B, A \Rightarrow B$
(R2) $A, B \Rightarrow A \,\&\, B$
(R3) $A \to B, C \to D \Rightarrow B \to C \to .\ A \to D$
(R4) $A \to \sim B \Rightarrow B \to \sim A.$

For much more information on **B**, including variant axiom sets, see [10]. We may consider adding postulates to **B** to produce stronger relevant and irrelevant logics. Of importance in what follows are the postulates:

(A10) $A \to \sim B \to .\ B \to \sim A$
(A11) $A \to B \to .\ C \to A \to .\ C \to B$
(A12) $A \to B \to .\ B \to C \to .\ A \to C$
(A13) $(A \to B) \,\&\, (B \to C) \to .\ A \to C$
(A14) $A \to .\ A \to A$
(A15) $A \to .\ B \to A$
(A16) $(A \to B \vee C) \,\&\, (A \,\&\, B \to C) \to .\ A \to C$
(A17) $A \to .\ A \to B \to B$
(A18) $(A \to .\ B \to C) \to .\ B \to .\ A \to C$
(A19) $(A \to B) \,\&\, A \to B$
(R5) $A \Rightarrow B \to A$
(R6) $A \Rightarrow A \to B \to B.$

Some of the logics produced in this way are:

DW	**B** + (A10)	
TW	**DW** + (A11) + (A12)	
DJ	**DW** + (A13)	
TJ	**TW** + (A13)	
EW	**TW** + (R6)	
RW	**DW** + (A18)	or **TW** + (A17)
R	**RW** + (A19)	or **RW** + (A13)
BCK	**RW** + (A15)	or **RW** + (R5)
RM	**R** + (A14).	

An **M1** *logic* is any system axiomatizable as **B** plus zero or more of (A10) through (A16) with or without (R5). An **M2** *logic* is one axiomatizable as **B** plus (R6) plus any selection from (A10), (A11), (A12), (A14), (A15), (A17), (A18), (R5). This paper will provide sensible reduced modelings for all **M1** logics and all **M2** logics. Most, though not quite all, weaker relevant logics in which there has been recent interest fall into one of the two groups.

Where L is a logic containing **B**, an L theory T is a set of sentences in the language of L satisfying, for all A and B:

(i) if $A \in T$ and $B \in T$ then $A \& B \in T$.
(ii) if $\vdash_L A \to B$ and $A \in T$ then $B \in T$.

Furthermore, T is:

regular	iff whenever $\vdash_L A$, $A \in T$
detached	iff whenever $A \to B \in T$ and $A \in T$, $B \in T$
affixed	iff whenever $A \to B \in T$ both $C \to A \to. C \to B \in T$ and $B \to C \to. A \to C \in T$
transpositive	iff whenever $A \to B \in T$, $\sim B \to \sim A \in T$
consistent	iff whenever $\sim A \in T$, $A \notin T$
prime	iff whenever $A \vee B \in T$, either $A \in T$ or $B \in T$
stable	iff detached, affixed and transpositive
ordinary	iff regular and stable
normal	iff ordinary, prime and consistent
A-consistent	iff $A \notin T$.

The main lemma to be proved below is that where L is any **M1** or **M2** logic, if A is a nontheorem of L then there is a normal A-consistent L theory. This theory is moreover "large" in an intuitively reasonable sense.

Before deriving any lemmas we need more definitions. A *frame* is a structure $\langle K, P, O, R, *\rangle$ where K is a set (of "worlds"), P is a subset of K, O is a member of P, R is a ternary relation defined on K and $*$ a unary function defined on K, such that for all members a, b, c, d of K.

(D1) $a \leq b =_{df} \exists x(x \in P \& Rxab)$
(D2) $R^2abcd =_{df} \exists x(Rabx \& Rxcd)$
(D3) $R^2a(bc)d =_{df} \exists x(Rbcx \& Raxd)$

(P1) $a \leq a$
(P2) For all x in P, if R^2xabc or $R^2a(xb)c$ or $R^2x(ab)c$ then $Rabc$
(P3) $a^{**} = a$
(P4) If $a \leq b$ then $b^* \leq a^*$.

Routley and Brady give $(p2)$ in the form

For all x in P, *if* R^2xabc then $Rabc$

and claim that its other parts (which do not affect the stock of theorems) are "optional extras". I agree that they are extras.

Let SL be the set of sentential letters. A *model* is a pair $\langle F, v\rangle$ where F is a frame and v is a total function from $SL \times K$ into $\{\top, \bot\}$ such that for all p in SL and all a, b in K if $a \leq b$ and $v(p, a) = \top$ then $v(p, b) = \top$. Each model determines an interpretation I in accordance with the following conditions:

(I1) $I(p, a) = v(p, a)$
(I2) $I(A \& B, a) = \top$ if both $I(A, a) = \top$ and $I(B, a) = \top$
$I(A \& B, a) = \bot$ otherwise

(I3) $I(A \vee B, a) = \top$ if either $I(A, a) = \top$ or $I(B, a) = \top$
$I(A \vee B, a) = \bot$ otherwise
(I4) $I(\sim A, a) = \top$ if $I(A, a^*) = \bot$
$I(\sim A, a) = \bot$ otherwise
(I5) $I(A \rightarrow B, a) = \top$ if for all b, c such that $Rabc$,
if $I(A, b) = \top$ then $I(B, c) = \top$
$I(A \rightarrow B, a) = \bot$ otherwise.

A is *true* in $\langle F, v\rangle$ iff $I(A, O) = \top$. That is, to be true is to hold at the real world, which is unsurprising. A set X of sentences *implies* A in $\langle F, v\rangle$ iff there is no a in K such that $I(B, a) = \top$ for all B in X but $I(A, a) = \bot$. X *entails* A on a set S of frames iff X implies A in every model in every frame in S. Finally, A *holds strongly* in a model iff its interpretation is $\top$ at every member of P in that model.

Where L is a logic as before, an L *frame* is a frame F such that in every model in F

(i) every theorem of L holds strongly;
(ii) every primitive rule of L preserves strong holding.

Often, and usually in logics of the type specified above and encountered in the relevant logical literature, L frames are defined by stipulating conditions, mainly on the relation R, which they meet only if they yield strong L matrices.[2] The reader is referred to [10], Chapter 4, for details.

A frame $\langle K, P, O, R, *\rangle$ is *reduced* iff $P = \{0\}$. In reduced frames strong holding amounts simply to truth, and it is easy to show that A implies B in a reduced model iff $A \rightarrow B$ is true therein. As suggested in the opening remarks, it is motivationally desirable that relevant logics should be characterized by their reduced frames, so that they may deliver on the claim that the arrow not merely indicates but actually expresses implication.

We now approach the main theorems by way of some observations for which let L be a supersystem of **B** and let T be an ordinary prime L theory. Let Pt_T be the set of prime L theories U closed under T-implication: whenever $A \rightarrow B \in T$ and $A \in U$, $B \in U$. Let the relation R_T be defined on Pt_T:

$$R_T(U, V, W) \text{ iff for all } A, B \text{ if } A \rightarrow B \in U \text{ and } A \in V \text{ then } B \in W.$$

Define the operation $*_T$ on Pt_T:

$$(U)*_T = \{A\colon \sim A \notin U\}.$$

Then

Observation 1 *$\langle Pt_T, \{T\}, T, R_T, *_T\rangle$ is an L frame.*

Proof: See [10], Chapter 4. It is necessary that T be detached, to ensure that T is in Pt_T, and that it be transpositive for (P4).

Observation 2 *There is an interpretation in the frame just defined such that for all A and U, $I(A, U) = \top$ iff $A \in U$.*

Proof: As for Observation 1. That T be affixed is necessary here to ensure that (I5) is satisfied.

Observation 3 *Let T be an A-consistent ordinary prime L theory. Then there is a reduced L model falsifying A.*

Proof: Immediate from Observations 1 and 2.

Observation 4 *Let A be a nontheorem of L. Then there is a regular prime A-consistent L theory.*

Proof: By a version of Lindenbaum's construction.

Let $\langle B_1 \ldots B_n \ldots \rangle$ be an enumeration of the formulas of L. Define the sets T_i of formulas:

$$T_0 = \{B \colon \vdash_L B\}.$$

If for some conjunction C of members of T_i, $\vdash_L B_{i+1}$ & $C \to A$ then $T_{i+1} = T_i$. Otherwise $T_{i+1} = T_i \cup \{B_{i+1}\}$. $T = \bigcup_i (T_i)$.

Four things are to be shown: T is (a) regular, (b) prime, (c) an L theory, (d) A-consistent. Of these, (a) and (d) are trivial, and (c) is straightforward by familiar moves. For (b) note that if neither C nor D is in T, then there is some E in T such that both E & $C \to A$ and E & $D \to A$ are L theorems. But then quite easily $(C \vee D)$ & $E \to A$ is an L theorem, so $C \vee D$ is not in T.

Observations 3 and 4 are true of *all* logics containing **B**, but they imply reduced modeling only for systems all of whose regular theories are stable and therefore ordinary. Clearly a sufficient condition for the stability of all L theories is that L have as theorems all instances of (A10), (A11), (A12), and (A19). Many of the standard logics in the relevant family, including *T*, *E*, *R*, *RM*, and classical logic, are thus characterized by reduced models.[3] Where L is an **M1** or an **M2** logic, lacking in particular WI, there may be *extraordinary* regular prime L theories produced by the Lindenbaum construction. In such a case, we can still define Pt_T, of course, but there will be no guarantee that T is itself one of the "worlds" in the resultant putative frame.

The first step toward improving matters is to rework the Lindenbaum lemma using a fairly orthodox notion of 'derivation'. There are five cases of *immediate consequence*:

A & B is an immediate consequence of A and B.
B is an immediate consequence of A and $A \to B$.
$\sim B \to \sim A$ is an immediate consequence of $A \to B$.
$C \to A \to . \ C \to B$ is an immediate consequence of $A \to B$.
$B \to C \to . \ A \to C$ is an immediate consequence of $A \to B$.

A *derivation* of formula A from set X in logic L is a finite sequence of formulas, the last of which is A and each of which is either a member of X, a theorem of L, or an immediate consequence of earlier ones. Now define some more sets:

$T'_0 = \varnothing$.
If there is a derivation in L of A from $T'_i \cup \{B_{i+1}\}$ then $T'_{i+1} = T'_i$. Otherwise $T'_{i+1} = T'_i \cup \{B_{i+1}\}$.
$T' = \bigcup_i (T'_i)$.

It is easy enough to establish that T' is an L theory, that it is regular and stable, and that A is not in it. Thus

Observation 5 *Let L be a superlogic of* **B** *and let A be a nontheorem of L. Then there is an ordinary A-consistent L theory.*

Proof: As usual.

Even Observation 5, however, does not yet give a reduced modeling theorem, for the theory T' may fail to be prime. To see how this might happen, note that without WI and its cognates we may have derivations of A from B and from C without having a derivation of A from $B \vee C$, despite the availability of (A7). The reason is that in the derivations of A from B and from C it may have been necessary to assume one of the premises more than once for detachment purposes. The reader who sensed familiar ground underfoot when derivations were defined in the last paragraph is permitted a double take at this point. Why is the number of uses of a premise of any significance? The answer is bound up with the delicate forms taken by the deduction theorem in logics without WI and was explored somewhat in [6]. Briefly, where there are L theorems

$$B \to D$$
$$B \to . D \to A,$$

there are also

$$B \to . D \& (D \to A)$$
$$B \to . B \to A;$$

but without WI there will not in general be a theorem

$$B \to A.$$

Uses of detachment in the derivation are reflected in the repetitions of the antecedent of the L-provable conditional corresponding to it. Even if the numbers of detachments in the two derivations were the same it would not in general be possible to amalgamate them, for logics without WI do not usually admit the rule

$$B \to . B \to A,\ C \to . C \to A \Rightarrow B \vee C \to . B \vee C \to A.$$

The problem is ineradicable.

If the problem of nonprimeness cannot be eradicated, perhaps it can be circumvented. We may suitably begin by considering a justly famous solution to an analogous problem besetting more familiar relevant logics such as **R**. We have seen that **R** is characterized by its reduced frames; but is it characterized by its *normal* frames—i.e., by reduced ones in which $0 \leq 0^*$? The key theorem entailing a positive answer is the

Normalization Theorem for R *Let A be a nontheorem of* **R**. *Then there is a normal A-consistent* **R** *theory.*

Proof (Meyer): By metavaluation. By observation 3 there is an ordinary prime A-consistent **R** theory T. Define a set MT:

For p in SL, $p \in MT$ iff $p \in T$.
$B \& C \in MT$ iff $B \in MT$ and $C \in MT$.

$B \vee C \in MT$ iff $B \in MT$ or $C \in MT$.
$\sim B \in MT$ iff $\sim B \in T$ and $B \notin MT$.
$B \rightarrow C \in MT$ iff $B \rightarrow C \in T$ and if $B \in MT$ then $C \in MT$.

$MT \subseteq T$ as shown by easy induction on the complexity of formulas. Therefore MT is A-consistent. Obviously MT is prime, consistent, and closed under adjunction and detachment. To show that MT is the required normal theory, then, it suffices that we show every axiom of **R** is in MT. This is done in two stages. First (by induction on complexity again) if $B \notin MT$ then $\sim B \in T$. The effect of this lemma is to ensure that $\sim B \in MT$ iff $B \notin MT$, so that the double negation Axiom (A9) presents no problem. Then the proof is completed by going through the axiom schemes case by case. Details are omitted here as the result is well known.

It is worth pausing to examine the strategy of Meyer's proof. The Lindenbaum construction yields too large a theory—in particular it is inconsistent—because the closure condition, based on the logic, is weak enough to be tolerant of theories with such excess content. A metavaluation (a kind of forcing) is therefore used to trim the edges of T, cutting it back to normality. Much the same problem faces the attempt to find reduced frames for **M1** and **M2** logics: because of the weakness of the logics the Lindenbaum theories tend to be too big, often allowing in disjunctions without their disjuncts. Can we therefore rectify matters by applying a metavaluation to T'? It turns out that we can, but to do so requires an extension of the technique. The simple metavaluation used by Meyer in the normalization theorem for **R** will not work, for the lemma that if $B \notin MT'$ then $\sim B \in T'$ is false (e.g., let the nontheorem be $p \vee \sim p$ and let B be p).

What does work is a pair of metavaluations intended to correspond to the worlds 0 and 0* is the frame they determine. The definitions for the **M1** and the **M2** cases are slightly different.

Let T be a set of sentences. Define m_1T, m_2T, m_1^*T and m_2^*T as follows:

For p in SL, $p \in m_1T$ iff $p \in T$	$p \in m_2T$ iff $p \in T$
$A \& B \in m_1T$ iff $A \in m_1T$ and $B \in m_1T$	$A \& B \in m_2T$ iff $A \in m_2T$ and $B \in m_2T$
$A \vee B \in m_1T$ iff $A \in m_1T$ or $B \in m_1T$	$A \vee B \in m_2T$ iff $A \in m_2T$ or $B \in m_2T$
$\sim A \in m_1T$ iff $A \notin m_1T$ and $A \notin m_1^*T$ and $\sim A \in T$	$\sim A \in m_2T$ iff $A \notin m_2T$ and $A \notin m_2^*T$ and $\sim A \in T$
$A \rightarrow B \in m_1T$ iff $A \rightarrow B \in T$ and if $A \in m_1T$ then $B \in m_1T$ and if $A \in m_1^*T$ then $B \in m_1^*T$	$A \rightarrow B \in m_2T$ iff $A \rightarrow B \in T$ and if $A \in m_2T$ then $B \in m_2T$ and if $A \in m_2^*T$ then $B \in m_2^*T$
$p \in m_1^*T$ iff $p \in T$ or $\sim p \notin T$	$p \in m_2^*T$ iff $p \in T$ or $\sim p \notin T$
$A \& B \in m_1^*T$ iff $A \in m_1^{*T}$ and $B \in m_1^*T$	$A \& B \in m_2^*T$ iff $A \in m_2^*T$ and $B \in m_2^*T$
$A \vee B \in m_1^*T$ iff $A \in m_1^*T$ or $B \in m_1^*T$	$A \vee B \in m_2^*T$ iff $A \in m_2^*T$ or $B \in m_2^*T$

$\sim A \in m_1^* T$ iff $A \notin m_1 T$	$\sim A \in m_2^* T$ iff $A \notin m_2 T$
$A \to B \in m_1^* T$ whatever happens	$A \to B \in m_2^* T$ iff if $A \in m_2 T$, $B \in m_2^* T$

Note that the only difference between the sets of definitions in the two cases is in the last line: $A \to B$ is in $m_1^* T$ if anything is the case, while it is in $m_2^* T$ only on a specific condition. Note also how the negation clauses relate the starred definitions to the plain ones. As in Myer's normalization proof we begin by establishing some inclusions between various of the sets.

Lemma 1 *Where T is a* **B** *theory, $m_1 T \subseteq T$ and $m_2 T \subseteq T$.*

Proof: Induction on the complexity of A, showing that if $A \in m_1 T$ or $A \in m_2 T$ then $A \in T$. Base case and *Cases* $A = \sim B$ and $A = B \to C$ are trivial.

Case $A = B \,\&\, C$. If $A \in m_1 T$ then $B \in m_1 T$ and $C \in m_1 T$, so by induction hypothesis $B \in T$ and $C \in T$ whence $B \,\&\, C \in T$. Similarly for $A \in m_2 T$.

Case $A = B \vee C$. If $A \in m_1 T$ (respectively $m_2 T$) then either $B \in m_1 T$ ($m_2 T$) or $C \in m_1 T$ ($m_2 T$), so by induction hypothesis either $B \in T$ or $C \in T$. So $A \in T$ by (A5) or (A6) and a little **B** logic.

Lemma 2 *For any set T of sentences, $m_1 T \subseteq m_1^* T$ and $m_2 T \subseteq m_2^* T$.*

Proof: Induction on complexity again. Omitted as too trivial.

Lemma 3 *Where T is a* **B** *theory, if $A \notin m_1^* T$ then $\sim A \in T$.*

Proof: By induction again. Base case trivial, as is *Case* $A = B \to C$.

Case $A = B \,\&\, C$. Suppose $A \notin m_1^* T$. Then either $B \notin m_1^* T$ or $C \notin m_1^* T$. By hypothesis, then, either $\sim B \in T$ or $\sim C \in T$, whence $\sim (B \,\&\, C) \in T$ by DeMorgan lattice logic and the closure of T.

Case $A = B \vee C$. Dual of last case.

Case $A = \sim B$. Suppose $A \notin m_1^* T$. Then $B \in m_1 T$, so $B \in T$ by Lemma 1, so $\sim\sim B \in T$ by easy moves in the logic **B.**

Corollary to Lemmas 2 and 3 *Let T be any* **B** *theory and let $m_1 T$ be as above. Then $\sim A \in m_1 T$ iff $A \notin m_1^* T$.*

Lemma 4 *Where T is a stable* **B** *theory closed under* (R6), *if $A \notin m_2^* T$ then $\sim A \in T$.*

Proof: Exactly as for Lemma 3 except for the case $A = B \to C$. Suppose $B \to C \in m_2^* T$. Then $B \in m_2 T$ and[4] $C \notin m_2^* T$. So $B \in T$ by Lemma 1, and $\sim C \in T$ by the induction hypothesis. But then $B \to C \to C \in T$ by the closure of T under (R6), so $\sim A \in T$ by transposition and detachment moves licensed by the stability of T.

Corollary to Lemmas 2 and 4 *Where T is a stable* **B** *theory closed under* (R6), *$\sim A \in m_2 T$ iff $A \notin m_2^* T$.*

Lemma 5 *Where L is an* **M1** *logic and T is an ordinary L theory, $m_1 T$ is a normal L theory.*

Proof: Trivially, $m_1 T$ is prime, consistent, detached, and closed under adjunction. It remains to show it transpositive, affixed, regular, and an L theory. For transposition, suppose $A \to B \in m_1T$. Then $A \to B \in T$ by Lemma 1, so $\sim B \to \sim A \in T$ by the ordinariness of T. Moreover, if $B \notin m_1T$ (respectively m_1^*T) then $A \notin m_1T$ (m_1^*T), which is to say if $\sim B \in m_1^*T$ (m_1T) then $\sim A \in m_1^*T$ (m_1T). In sum, $\sim B \to \sim A \in m_1T$. Closure under affixing is shown similarly. Suppose $A \to B \in m_1T$. We want to show that $C \to A \to. C \to B \in m_1T$; the argument for suffixing is analogous. Well, $A \to B \in T$ by Lemma 1, whence by T's ordinariness $C \to A \to. C \to B \in T$; and if $A \in m_1T$ (respectively m_1^*T) then $B \in m_1T$ (respectively m_1^*T). Further suppose $C \to A \in m_1T$. Then $C \to A \in T$, so $C \to B \in T$; and if $C \in m_1T$ (m_1^*T) then $A \in m_1T$ (m_1^*T), whence, by affixing in the metalogic, if $C \in m_1T$ (m_1^*T) then $B \in m_1T$ (m_1^*T). Thus $C \to B \in m_1T$. Discharging the further assumption, then, if $C \to A \in m_1T$ then $C \to B \in m_1T$. Trivially if $C \to A \in m_1^*T$ then $C \to B \in m_1^*T$. Hence $C \to A \to. C \to B \in m_1T$ as required.

To show m_1T regular and an L theory it now suffices that we show every axiom of L to be in m_1T, since we already have closure under all the rules of inference. Each axiom is of the form $A \to B$, so three things are to be shown in each case: that $A \to B \in T$ (which is trivial and will be left unstated), that if $A \in m_1T$ then $B \in m_1T$, and that if $A \in m_1^*T$ then $B \in m_1^*T$. In cases where B is of the form $C \to D$ the last of these is trivial and will again be left unstated.

For (A1) If $A \in m_1T$ then $A \in m_1T$, and if $A \in m_1^*T$ then $A \in m_1^*T$. Therefore $A \to A \in m_1T$. Notice that the metalogical reasoning uses exactly the principle involved.

For (A2) If A & $B \in m_1T$ (m_1^*T) then $A \in m_1T$ (m_1^*T) and $B \in m_1T$ m_1^*T), so $A \in m_1T$ (m_1^*T) in that case by metalogical (A2).

For (A3) Similar.

For (A4) Suppose $(A \to B)$ & $(A \to C) \in m_1T$. Then $A \to B \in m_1T$ and $A \to C \in m_1T$, so if $A \in m_1T$ (m_1^*T) then both $B \in m_1T$ (m_1^*T) and $C \in m_1T$ (m_1^*T), which is to say B & $C \in m_1T$ (m_1^*T).

For (A5) Dual of case for (A2).

For (A6) Similar.

For (A7) Dual of case for (A4).

For (A8) Suppose A & $(B \vee C) \in m_1T$ (m_1^*T). Then $A \in m_1T$ (m_1^*T) and either $B \in m_1T$ (m_1^*T) or $C \in m_1T$ (m_1^*T). By meta-(A8) then either $A \in m_1T$ (m_1^*T) and $B \in m_1T$ (m_1^*T) or $C \in m_1T$ (m_1^*T), which is to say $(A$ & $B) \vee C \in m_1T$ (m_1^*T).

For (A9) Suppose $\sim\sim A \in m_1T$. Then $\sim A \notin m_1^*T$, so $A \in m_1T$. Suppose $\sim\sim A \in m_1^*T$. Then $\sim A \notin m_1T$, so by the corollary to Lemmas 2 and 3, $A \in m_1^*T$.

For (A10) Suppose $A \to \sim B \in m_1T$. Then if $\sim B \notin m_1T$ (m_1^*T) then $A \notin m_1T$ (m_1^*T), which is to say if $B \in m_1^*T$ (m_1T) then $\sim A \in m_1^*T$ (m_1T).

For (A11) Trivial because m_1T is affixed.

For (A12) Similar.

For (A13) Suppose $(A \to B)$ & $(B \to C) \in m_1T$. Then if $A \in m_1T$ (m_1^*T) then $B \in m_1T$ (m_1^*T), and if so then $C \in m_1T$ (m_1^*T).

For (A14) Suppose $A \in m_1T$. Then (by meta-(A14)) if $A \in m_1T$ then $A \in m_1T$; and since $m_1T \subseteq m_1^*T$ by Lemma 2, if $A \in m_1^*T$ then $A \in$

$m_1^* T$ (by meta-(A14) again). This finesse is just in case any mingle-loving relevantist should worry; the rest of you have a nice day.

For (A15) Similar, using (A15) in the metalogic.

For (A16) Suppose $(A \to B \vee C)$ & $(A$ & $B \to C) \in m_1 T$. Then if $A \in m_1 T$ $(m_1^* T)$ then either $B \in m_1 T$ $(m_1^* T)$ or $C \in m_1 T$ $(m_1^* T)$, and if both $A \in m_1 T$ $(m_1^* T)$ and $B \in m_1 T$ $(m_1^* T)$ then $C \in m_1 T$ $(m_1^* T)$. So if $A \in m_1 T$ $(m_1^* T)$ then $C \in m_1 T$ $(m_1^* T)$. So $A \to C \in m_1 T$.

For (R5) As for (A15) but without the need for (A15) to be in T.

M1 Normalization Theorem *Let L be any* **M1** *logic and let A be a non-theorem of L. Then there is a normal A-consistent L theory.*

Proof: Immediate from Observation 5, Lemma 1, and Lemma 5.

M1 Reduced Frame Theroem *Every* **M1** *logic is characterized by its reduced frames.*

Proof: Observation 3 and the **M1** normalization theorem.

Lemma 6 *Where L is an* **M2** *logic and T is an ordinary L theory $m_2 T$ is a normal L theory.*

Proof: Much of the proof of Lemma 5 carries over, *mutatis mutandis*. For closure under affixing we need to show that if $A \to B \in m_2 T$ then if $C \to A \in m_2^* T$ then $C \to B \in m_2^* T$. Suppose the antecedents. Then if $A \in m_2^* T$ then $B \in m_2^* T$, and if $C \in m_2 T$ then $A \in m_2^* T$, whence if $C \in m_2 T$ then $B \in m_2^* T$, i.e, $C \to B \in m_2^* T$. The other significant changes are to various cases of the axioms. The proofs for (A1), (A2), (A3), (A5), (A6), (A8), (A9), and (R5) are much as before.

For (A4) First part as for Lemma 5.
Suppose $(A \to B)$ & $(A \to C) \in m_2^* T$. Then if $A \in m_2 T$ then both $B \in m_2^* T$ and $C \in m_2^* T$; so $A \to B$ & $C \in m_2^* T$.

For (A7) Dual of case for (A4).

For (A10) First supposition as for Lemma 5.
Suppose $A \to {\sim}B \in m_2^* T$. That is, if $A \in m_2 T$ then $B \notin m_2 T$. By (A10) in the metalogic then if $B \in m_2 T$ then $A \notin m_2 T$ which is to say $B \to {\sim}A \in m_2^* T$.

For (A11) We need to show that $m_2^* T$ is affixed. Suppose $A \to B \in m_2^* T$. That is, if $A \in m_2 T$ then $B \in m_2^* T$. Further suppose $C \to A \in m_2 T$. Then if $C \in m_2 T$, $A \in m_2 T$. So if $C \in m_2 T$ then $B \in m_2^* T$, which is to say $C \to B \in m_2^* T$. Discharging the further assumption, if $C \to A \in m_2 T$ then $C \to B \in m_2^* T$; i.e., $C \to A \to. C \to B \in m_2^* T$ as required.

For (A12) Similar.

For (A14) As for Lemma 5, except that to show if $A \in m_2^* T$ then if $A \in m_2 T$ then $A \in m_2^* T$ seems to require a second appeal to Lemma 2.

For (A15) Similar.

For (A17) Suppose $A \in m_2T$. Then if $A \to B \in m_2T$ then $B \in m_2T$, and if $A \to B \in m_2^*T$ then $B \in m_2^*T$, both by meta-(A17); so $A \to B \to B \in m_2T$.
Suppose $A \in m_2^*T$. Then if $A \to B \in m_2T$ then $B \in m_2^*T$, which is to say $A \to B \to B \in m_2^*T$.

For (A18) Suppose $A \to. B \to C \in m_2T$. Then

(i)	if $A \in m_2T$ then $B \to C \in m_2T$.	
(ii)	if $A \in m_2^*T$ then $B \to C \in m_2^*T$.	
(iii)	if $A \in m_2T$ then if $B \in m_2T$ then $C \in m_2T$	(from (i))
(iv)	if $B \in m_2T$ then if $A \in m_2T$ then $C \in m_2T$	(from (iii))
(v)	if $A \in m_2^*T$ then if $B \in m_2T$ then $C \in m_2^*T$	(from (ii))
(vi)	if $B \in m_2T$ then if $A \in m_2^*T$ then $C \in m_2^*T$	(from (v))
(vii)	if $B \in m_2T$ then $A \to C \in T$	(Lemma 1, etc.)
(viii)	if $B \in m_2T$ then $A \to C \in m_2T$	(from (iv), (vi), (vii))
(ix)	if $A \in m_2T$ then if $B \in m_2^*T$ then $C \in m_2^*T$	(from (i))
(x)	if $B \in m_2^*T$ then if $A \in m_2T$ then $C \in m_2^*T$	(from (ix))
(xi)	if $B \in m_2^*T$ then $A \to C \in m_2^*T$	(from (x))
(xii)	$A \to. B \to C \in T$	(Lemma 1)
(xiii)	$B \to. A \to C \in T$	(closure of T)
(xiv)	$B \to. A \to C \in m_2T$	(from viii, ix, xiii).

Suppose $A \to. B \to C\ m_2^*T$. That is, if $A \in m_2T$ then $B \to C \in m_2^*T$, which is to say if $A \in m_2T$ then if $B \in m_2T$ then $C \in m_2^*T$. Permuting, if $B \in m_2T$ then if $A \in m_2T$ then $C \in m_2^*T$, i.e., $B \to. A \to C\ m_2^*T$.

For (R6) As for (A17), without the need for (A17) to be in T.

M2 Normalization Theorem *Let L be an* **M2** *logic and let A be a nontheorem of L. Then there is a normal A-consistent L theory.*

Proof: Immediate from Observation 5, Lemma 1, and Lemma 6.

M2 Reduced Frame Theorem *Every* **M2** *logic is characterized by its reduced frames.*

Proof: Observation 3 and the **M2** normalization theorem.

Corollary 1 to the Normalization Theorems *Every* **M1** *logic and every* **M2** *logic admits the rule* (γ) *of material detachment.*

Corollary 2 to the Normalization Theorems *Every* **M1** *logic and every* **M2** *logic is prime. That is, the set of theorems of L is a normal (ordinary, prime, consistent) theory.*

Corollary 3 to the M1 Normalization Theorem *No* **M1** *logic has any theorem of the form* $\sim(A \to B)$.

In the light of Corollary 2 it may be wondered what was the point of proving Lemmas 5 and 6 in such generality. After all, if logic itself is a normal theory in which all nontheorems fail together, what need of any more elaborate constructions? Well, one point was to provide normal theories which somewhat resemble the real world. The normality of the canonical frame, based on the Lindenbaum algebra of the logic, is a sort of happy accident, good and useful but rather silly. Even with due allowance made for logicians' hubris, no one really expects the logical truths to be the *only* truths. It is splendid that the main contractionless relevant systems allow the possibility of models in which there are no contingent truths, for logic is in a way impure when it claims there are truths about matters which are not its business; but it is also desirable that there be models looking like reality, in which there are many truths beyond what is given by Pure Reason. It is accordingly gratifying, and reassuring, to find logics without WI characterized by sets of models in which the truth is large. The concept of largeness here is a little obscure, though presumably lightly trimmed relatives of maximal A-consistent theories will count as large by any reckoning. Note at least that selection of suitable sets of irrelevant sentential letters to go into T can assure uncountably many normal A-consistent L theories, so some of them must be large enough to allow a fair degree of freedom in the construction.

Finally, lest it be thought that we have proved everything, note (what was already known to the authors of [10]) that there is no reduced modeling theorem for any system between $\mathbf{B} + X$ and $\mathbf{TJ}$ + (R6) + X where X is the "excluded middle" scheme

(A20) $A \vee {\sim}A$.

The reason is that all reduced frames for such systems validate

$$(A \rightarrow B) \mathbin{\&} A \supset B$$

which is invalid in the following $\mathbf{TJ}$ + (R6) + X frame:

$K = \{0, 1\}$. $P = \{0, 1\}$. $0^* = 0$. $1^* = 1$.
$R = \{\langle 1, 0, 0\rangle, \langle 1, 1, 1\rangle\}$.

NOTES

1. These techniques have a longer history, going back at least to Harrop's [3] of 1956. Rasiowa, Fine, Dwyer and Meyer seem to be among those who have independently rediscovered the concept of metavaluation. See [5] for some more comments.

2. 'Only if' is deliberate here, as the converse 'if' is not always true. The general definition of the present paper is more liberal than some of the system-specific ones to be found elsewhere. For example, the Anderson-Belnap system E is usually modeled by a set of frames ("E model structures") excluding many important E frames. See Note 3.

3. Note that the reduced frames needed here may not in general be of the canonical kinds usually specified as "T model structures", "E model structures", etc. E, in particular, is known to resist reduced modeling on the usual definition of what counts as an E model. So much the worse for that definition.

4. Note for the deeply initiated: this 'and' and the next are fusions if the metalogic is to be formalized relevantly. Innocent readers are *not* to worry about it.

   Metavaluational arguments do seem to be extremely sensitive to the choice of metalogic, and while the arguments of this paper are in stilted English, not in any formal system, they are so couched as to suggest the resources needed for their formal reconstruction. Far more than pure logic is needed for any such reconstruction project, of course—a theory of sets or other collections being a start. No strong claims on such matters are being entered here, but the issue of the "relevant validity" of relevant metatheory is in the air just now, so that those of us who have lost our innocence should try to be aware of the problem in our informal proofs. The point to be noted even by the innocent is that the metalogical reasoning closely mirrors, without bad circularity, the properties being established for the object systems.

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