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# A Simplified Natural Deduction Approach to Certain Modal Systems

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The natural deduction approach to standard logic has a number of virtues, among them being ease of application. However, attempts in the literature to combine natural deduction with modality have generally resulted in cumbersome systems little more flexible than straightforward axiomatics.[1] This paper presents a natural deduction approach to the *T*-*B*-*S*4-*S*5 ring which, if I am not mistaken, renders them as simple as the ordinary first-order logic taught in introductory courses. One of its most important advantages is semantic transparency: it virtually wears its interpretation on its sleeve. It will perhaps become obvious that this approach has it roots in the metaphor of modal logic as quantification over possible worlds, though this relationship will not be spelled out in any detail.

***1 Intuitive groundwork*** Think of possible worlds as flagged or represented by distinct numerals, with 1 representing the actual world and other numerals representing others. In modal logic, truth-predicates are not applied absolutely, but only relative to particular possible worlds. We may observe this graphically by indexing each sentence to a world at which it is (asserted to be) true, by attaching that world's numeral to the propositional expression as a subscript. $P_1$ will mean that $P$ (is true) at world 1; $(Q \& R)_2$ will mean that $(Q \& R)$ at world 2, and so on.

Modal systems differ in their construal of the access relation between worlds. Let possible-world indices be arranged in strings or *chains*. Then the access relationships between worlds represented by those indices may be mapped or represented by spatial relationships between numerals in the chain.

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The basic chain-convention is that each numeral has access to its *immediate* follower in the chain. Additionally, in the system $T$ where (only) reflexivity of access is guaranteed, each numeral will have access to itself. In $S4$, where transitivity is also guaranteed, each numeral will have access to the immediate follower of an accessible numeral, hence to all of its followers as well as to itself. In *Brouwersche*, where access is symmetric as well as reflexive, each numeral will have access to its immediate predecessor as well as its immediate follower and itself.

In $S5$, where access is ubiquitous, there is no need to use numeral-chains to represent the relationship: we can get by with single numerals, together with the convention that each of them has access to all others.

When a sentence or formula is subscripted by a chain of numerals, it is indexed by the final number in the chain. For example, $P_{1,3,5}$ says that $P$ (is true) at world 5, and also conveys information about access between 1, 3, and 5, depending on the particular system.

So much for the representation of possible worlds. Consider next four intuitively plausible principles of modal inference.

(a) *Truth-Functional Principle:* Truth-functional inferences are valid only within a given possible world.

Given that $(P \& Q)$ is true at world $n$, we may infer that $P$ is true at $n$ but not that it is true at any other world. Similarly, given that $(P \supset Q)$ and $P$ are both true at $n$ we may infer that $Q$ at $n$, but not at any other world. And given only that $(P \supset Q)$ at $n$ and $P$ at a distinct world $m$, nothing may be inferred about the truth of $Q$ at any world.

(b) *Modal Negation Principle:* Whatever is not necessary (at a given world) is possibly false (at that world), and whatever is not possible (at a given world) is necessarily false (at that world).
(c) *Necessity/Truth Principle:* Whatever is necessary at a given world is true at every world accessible from that one.
(d) *Impossibility Principle:* Whatever is demonstrably false at an arbitrarily chosen world accessible from $n$ is false at every world accessible from $n$, hence is impossible at $n$.

These four principles of inference, together with the way of representing possible worlds, provide the intuitive basis for the formal business coming next.

*2 The formal business* A subscript chain is a string of zero or more numerals separated by commas. Henceforth, mid-alphabet letters $n$ and $m$ serve as variables for chains of any length, including the null chain. Front-alphabet letters $a$, $b$ and $c$ serve as variables for individual subscript numerals.

Let $S$ be some natural-deduction system of propositional logic, with mechanisms for taking on and discharging assumptions and having RAA as a primitive or derived rule (hence having Conditional Proof as a primitive or derived rule). To form a natural-deduction version of the modal system $T$, augment the formation rules of $S$ to include unary operators for necessity $\Box$ and possibility $\Diamond$, and add the following inference rules.

($\alpha$) *Subscript Insertion:* Attach a chain of one or more numerals as a subscript to each premise, and to each assumption as it is introduced.

(1) *S-Restriction:* A (truth-functional) rule of $S$ may employ two or more lines in the derivation only if they bear the same subscript; and under all inferences via such rules, the subscript carries over unchanged to the derivate.

(1) is a formal counterpart of principle (a) in the preceding section.

(2) *Modal Negation Elimination* (MN):
From $\sim\Box P_n$ infer $\Diamond\sim P_n$
From $\sim\Diamond P_n$ infer $\Box\sim P_n$.

(2) is the formal counterpart of principle (b).

(3) *Box Elimination* (T-descent):
From $\Box P_n$ infer $P_n$ and $P_{n,b}$ for any chain $n,b$ in the derivation.

(3) is the formal counterpart of principle (c).

(4) *Impossibility Introduction* (Impossibilitation):
From $(Q \mathbin{\&} \sim Q)_m$, if the latest undischarged assumption $P_{n,b}$ contains the first occurrence of numeral $b$ in the derivation, discharge that assumption and infer $\sim\Diamond P_n$. Schematically,

$$\begin{array}{l} \to P_{n,b} \quad \text{(no earlier occurrence of } b \text{ in the derivation)} \\ \quad \vdots \\ (Q \mathbin{\&} \sim Q)_m \\ \hline \sim\Diamond P_n. \end{array}$$

(4) is the formal counterpart of principle (d).

($\beta$) *Subscript Deletion:* From $P_n$ not within the scope of any undischarged assumption, infer $P$.

($\alpha$) and ($\beta$) are the housekeeping rules. Subscript chains form no part of a wff. Rather, they are part of the proof-structural notation, cousins of assumption and discharge indicators and the like. Rule ($\beta$) allows the final line of a closed derivation (e.g., a theorem) to be a pure wff, without residual parts of the proof clinging to it.

The only rule of the system allowing the introduction of new subscript numerals is rule ($\alpha$): new numerals enter the derivation only via assumptions. An essential ingredient in using this system is the maneuver of "Advancing" a possibility-wff:

*Advancement:* From a line $\Diamond P_n$, *assume* $P_{n,b}$ where $b$ is a numeral new to the derivation.

Intuitively, $\Diamond P_n$ says that $P$ is possible at (the final numeral of) $n$, meaning that $P$ is true at some world accessible from $n$. Assuming $P_{n,b}$ with fresh $b$ is equivalent to stipulating "let $b$ be that world".

Here, to illustrate the system in action, is a proof of one of the standard theorems of $T$: the inheritance of necessity across strict implication.

| | |
|---|---|
| 1. $\Box(A \supset B)_1$ | / ∴ $(\Box A \supset \Box B)$ |
| 2. $\Box A_1$ | / ∴ $\Box B$ |
| 3. $\sim\Box B_1$ | (RAA) |
| 4. $\Diamond\sim B_1$ | 3, MN |
| 5. $\sim B_{1,2}$ | (Advance, 4) |
| 6. $(A \supset B)_{1,2}$ | 1, $T$-descent |
| 7. $A_{1,2}$ | 2, $T$-descent |
| 8. $B_{1,2}$ | 6, 7 modus ponens |
| 9. $(B \,\&\, \sim B)_{1,2}$ | 8, 5, conjunction |
| 10. $\sim\Diamond\sim B_1$ | 5–9, impossibilitation |
| 11. $(\Diamond\sim B \,\&\, \sim\Diamond\sim B)_1$ | 4, 10, conjunction |
| 12. $\Box B_1$ | 3–11, RAA, double negation |
| 13. $(\Box A \supset \Box B)_1$ | 2–12, conditional proof |
| 14. $(\Box(A \supset B) \supset (\Box A \supset \Box B))$ | 1–13, conditional proof, subscr. del. |

To form a natural-deduction version of $S4$, replace $T$-descent with

*S4-Descent:* From $\Box P_n$ infer $P_n$ and $P_{n,m}$ for any chain $n, m$, in the derivation.

Since $m$ represents a chain rather than a single numeral, this says that $\Box P_n$ may $S4$-descend to any chain with $n$ as initial segment. In $T$, descent proceeds only one step from home base; in $S4$ it is permitted all the way down the chain. As an illustration, here is a proof of the characteristic $S4$ theorem $(\Diamond\Diamond A \supset \Diamond A)$.

| | |
|---|---|
| 1. $\Diamond\Diamond A_1$ | / ∴ $\Diamond A$ |
| 2. $\sim\Diamond A_1$ | (RAA) |
| 3. $\Box\sim A_1$ | 1, MN |
| 4. $\Diamond A_{1,2}$ | (Advance, 1) |
| 5. $A_{1,2,3}$ | (Advance, 4) |
| 6. $\sim A_{1,2,3}$ | 3, $S4$-descent |
| 7. $(A \,\&\, \sim A)_{1,2,3}$ | 5, 6, conjunction |
| 8. $\sim\Diamond A_{1,2}$ | 5–7, impossibilitation |
| 9. $(\Diamond A \,\&\, \sim\Diamond A)_{1,2}$ | 4, 8, conjunction |
| 10. $\sim\Diamond\Diamond A_1$ | 4–9, impossibilitation |
| 11. $(\Diamond\Diamond A \,\&\, \sim\Diamond\Diamond A)_1$ | 1, 10, conjunction |
| 12. $\Diamond A_1$ | 2–11, RAA, double negation |
| 13. $(\Diamond\Diamond A \supset \Diamond A)$ | 1–13, conditional proof, subscript deletion. |

To form a natural-deduction version of *B*, replace *T*-descent with

*B-Descent:* From $\Box P_{n,b}$ infer $P_n^*$, $P_{n,b}$, and $P_{n,b,c}$ for any chain $n, b, c$ in the derivation   *Provided $n$ is not null.

In *B*, descent also proceeds only one step from home base, but it may be either direction along the chain. The above proof of the characteristic *S*4 theorem cannot be conducted in *T* or *B* since neither *T*-descent nor *B*-descent allows the move at step (6).

Proof of the characteristic *B* theorem ($P \supset \Box\Diamond P$) is an elementary matter, involving descent from $\Box{\sim}P_{1,2}$ to ${\sim}P_1$.

A natural-deduction version of *S*5 can be formed by replacing *T*-descent with

From $\Box P_n$ infer $P_m$ for any $m$ in the derivation which has a numeral in common with $n$.

However, it is simpler in practice to drop chains altogether and go with single-numeral subscripts. (Underlying assumption: each numeral has access to every other. Formal requirement: if there are premises, they all get the same subscript.) The natural-deduction version of *S*5 may then be formed by replacing *T*-descent and impossibilitation with:

*S5-Descent:* From $\Box P_b$ infer $P_c$ for any numeral $c$ in the derivation.

*S5-Imposs:*
$$\begin{array}{|l} \to P_b \quad \text{(no earlier occurrence of } b \text{ in the derivation)} \\ \quad \vdots \\ (Q \,\&\, {\sim}Q)_a \\ \hline \end{array}$$
$${\sim}\Diamond P_c \quad (c \text{ is any numeral in the derivation}).$$

In *S*5, Advancement is carried out by replacing the old subscript numeral with one fresh to the derivation.

These systems can be made to resemble the familiar IntElim natural deduction structure a little more closely by adding the following two derived rules (the derivations are straightforward).

*Box Introduction* (Necessitation): Where $b$ does not appear in any premise or undischarged assumption:
For *T*, *B*, *S*4: From $P_{n,b}$ infer $\Box P_n$
For *S*5: From $P_b$ infer $\Box P_c$ for any $c$ in the derivation.

*Diamond Introduction* (Possibilitation):
For *T*, *B*, *S*4: From $P_n$ or $P_{n,b}$ infer $\Diamond P_n$
For *S*5: From $P_b$ infer $\Diamond P_c$ for any $c$ in the derivation.

***3 Decision procedures*** Modal systems developed in this way admit of a decision procedure (I call it Precipitation) which is simpler and more direct than any I am aware of. It involves no diagrams or tableaux, and is carried out within the natural deduction format, by conducting a derivation in accordance with a specified routine guaranteed to lead either to closure or to termination. If the derivation closes, it constitutes a proof of validity. If it terminates, a counter-model may be read off from the lines of the derivation. I describe it first for the truth-functional system *S*, and then extend it to the modal systems.

**Precipitation for *S*** Given the standard definitions of $\supset$ and $\equiv$, together with DeMorgan's Laws, every wff of *S* which is not basic (an atomic wff or the negation of one) will be a double negation or else equivalent to either a conjunction or a disjunction. Thus only three types of nonbasic formulas need be considered in the precipitation rules.

*Precipitation Rules for S*:

(1) *Infer:* From a disjunction, together with the negation of its left disjunct, add its right disjunct to the derivation.
(2) *Positivize:* From a double negation, add the double negate to the derivation.
(3) *Split:* From a conjunction, add both conjuncts to the derivation.
(4) *Branch:* From a disjunction, add its left disjunct to the derivation as an *assumption*.

These rules are given in their order of priority. The precipitation routine begins by assuming the negation of the formula to be evaluated. All discharges are by RAA. Treatable lines are treated by the appropriate precipitation rule, and the results are added to the derivation, omitting those which would duplicate lines already present. If contradicting lines appear, conjoin them and discharge the latest assumption. At each step, apply the highest-priority precipitation rule that can be applied. In the event of multiple opportunities, apply it to the earliest line (or the pair with the earliest member).

A derivation conducted according to this routine must either close or terminate. If it terminates, assign the value *T* to each letter (atomic wff) standing alone as a line in the derivation, assign *F* to all other letters, and assign values to nonatomic wffs in standard fashion. Under this assignment, every wff constituting a line in the derivation will receive the value *T*; thus the assignment satisfies the initial assumption, and countersatisfies its negate (see [2], [3], [11]).

**Precipitation for *T*, *B*, and *S5*** Precipitation for the three modal systems *T*, *B*, and *S5* is as for *S*, with the addition of three further precipitation rules to the bottom of the priority list.

(5) *Shift:* From $\sim\Box P_n(\sim\Diamond P_n)$, add $\Diamond\sim P_n(\Box\sim P_n)$ to the derivation.
(6) *Execute:* From $\Box P_n$, add $P_m$ to the derivation for each *m* accessible from *n* in the derivation.
(7) *Advance:* From $\Diamond P_n$, if the derivation does not contain $P_m$ for some *m* accessible from *n*, *assume* $P_{n,b}$ for fresh *b*.

Giving *Advance* bottom priority helps keep lines of like subscript clustered in blocks within the derivation. Advances are discharged by Impossibilitation, all other assumptions by RAA. A line inferred by *S5*-Impossibilitation receives the subscript of the $\Diamond$-wff advanced to produce the now-discharged assumption. $\Box$-wffs are re-Executed to new accessible chains (numerals) as they show up in the derivation. If lines earlier than a discharged assumption have precipitates within the scope of that assumption, they must be reprecipitated following the discharge.

A precipitation derivation for any of these three modal systems must either close or terminate. In case of termination, a Kripke model may be read off by the following recipe. Let $W$ = the set of subscript numerals $c$ in the derivation. Determine $R$ from the subscript chains in the derivation, according to their interpretation for the system in question. For single letters $p$, let $V(p;c) = T$ iff $p_{n,c}$ is a line in the derivation, otherwise let $V(p;c) = F$; and assign values to larger wffs in standard fashion. Or, if one prefers, a Hintikka model-set may be read off by letting all lines of like subscript constitute one of the sets in the model. $R$ between the sets is again determined by the subscript chains generating the sets.

As an illustration, here is a $T$ counterproof of the characteristic $S5$ theorem $(\Diamond P \supset \Box\Diamond P)$, followed by a "mechanical" proof of it in $S5$.

*Derivation in T:*

| | | |
|---|---|---|
| → | 1. $\sim(\Diamond P \supset \Box\Diamond P)_1$ | initial assumption |
| | 2. $\Diamond P_1$ | 1, split |
| | 3. $\sim\Box\Diamond P_1$ | 1, split |
| | 4. $\Diamond\sim\Diamond P_1$ | 3, shift |
| → | 5. $P_{1,2}$ | 2, advance |
| → | 6. $\sim\Diamond P_{1,3}$ | 4, advance |
| | 7. $\Box\sim P_{1,3}$ | 6, shift |
| | 8. $\sim P_{1,3}$ | 7, execute. |

At this point the derivation terminates, yielding the following model:

$W = \{1, 2, 3\}$
R = Reflexive + $\{\langle 1, 2\rangle, \langle 1, 3\rangle\}$
$V(P;1) = F \quad V(P;2) = T \quad V(P;3) = F.$

Inspection will show that this countersatisfies the negate of the initial assumption.

*Derivation in S5:*

| | | |
|---|---|---|
| → | 1. $\sim(\Diamond P \supset \Box\Diamond P)_1$ | initial assumption |
| | 2. $\Diamond P_1$ | 1, split |
| | 3. $\sim\Box\Diamond P_1$ | 1, split |
| | 4. $\Diamond\sim\Diamond P_1$ | 3, shift |
| → | 5. $P_2$ | 2, advance |
| → | 6. $\sim\Diamond P_3$ | 4, advance |
| | 7. $\Box\sim P_3$ | 6, shift |
| | 8. $\sim P_1$ | 7, execute |
| | 9. $\sim P_2$ | 7, execute |
| | 10. $\sim P_3$ | 7, execute |
| | 11. $(P \,\&\, \sim P)_2$ | 5, 9, conjunction |
| | 12. $\sim\Diamond\sim\Diamond P_1$ | 6–11, impossibilitation |
| | 13. $(\Diamond\sim\Diamond P \,\&\, \sim\Diamond\sim\Diamond P)_1$ | 4, 12, conjunction |

| | |
|---|---|
| 14. $\sim\Diamond P_1$ | 5–13, impossibilitation |
| 15. $(\Diamond P \,\&\, \sim\Diamond P)_1$ | 2, 14, conjunction |
| 16. $(\Diamond P) \supset \Box\Diamond P)$ | 1–15, RAA, double negation. |

**Precipitation for $S4$** The precipitation routine is essentially a treewalk carried out in the natural deduction format. Since, as Kripke proved long ago, $S4$ is not decidable in a finite tree, unadorned precipitation is not a decision procedure for $S4$. Relatively simple formulas, such as $(\Box\Diamond P \,\&\, \Box\Diamond Q)$, yield precipitation derivations in $S4$ which neither close nor terminate but proceed eternally by generating, through Execution, new $\Diamond$-wffs to be Advanced to further subscripts, calling for still further Executions, and so on. A device is needed to shut off these endless repetitive paths.

Each subscript chain in a derivation represents an access path through the tree. The set of lines bearing the same subscript constitute a "block", and in S4 precipitation the lines of a block are always contiguously grouped together. The mark of a potentially endless path is the presence of two identical blocks on that common path: blocks containing all and only the same formulas, where the earlier of the two subscript chains $n,c$ is an initial segment of the later one $n,m$. When this occurs, the solution is to treat all $\Diamond$-wffs$\Diamond P_{n,m}$ not by Advancement, but by "Foster"-advancement: the final numeral $c$ in the chain of the anticipating block $n$, $c$ is carried down and attached to the chain of $\Diamond P_{n,m}$ preceded by a slash, yielding $\Diamond P_{n,m/c}$.

A line with a slash in its chain is indexed by the numeral before the slash, but the entire chain is interpreted according to the $S4$ convention, in which each numeral has access to all of its followers.

The effect of Foster-Advancement is to give $m$ access to $c$, which already had access to $m$, thereby looping the access path back onto itself. When an $S4$ precipitation derivation terminates, a model is read off in the usual fashion, except that the chains will determine a subordinate quasi-access relation $r$, and R will be defined as the ancestral of $\boldsymbol{r}$. If the derivation contains no slash-numerals, $\boldsymbol{r}$ will be its own ancestral.

To illustrate these matters, here is a proof of the $S4$-satisfiability of $(\Box\Diamond p \,\&\, \Box\Diamond q)$ – a formula which, left to its own devices, would generate an infinite tree.

| | | |
|---|---|---|
| ⟶ 1. | $(\Box\Diamond p \,\&\, \Box\Diamond q)$ | initial assumption |
| 2. | $\Box\Diamond p_1$ | 1, split |
| 3. | $\Box\Diamond q_1$ | 1, split |
| 4. | $\Diamond p_1$ | 2, execute |
| 5. | $\Diamond q_1$ | 3, execute |
| ⟶ 6. | $p_{1,2}$ | 4, advance |
| 7. | $\Diamond p_{1,2}$ | 2, execute |
| 8. | $\Diamond q_{1,2}$ | 3, execute |
| ⟶ 9. | $q_{1,3}$ | 5, advance |
| 10. | $\Diamond \mathrm{p}_{1,3}$ | 2, execute |
| 11. | $\Diamond \mathrm{q}_{1,3}$ | 3, execute |
| ⟶ 12. | $q_{1,2,4}$ | 8, advance |

| | |
|---|---|
| 13. $\Diamond p_{1,2,4}$ | 2, execute |
| 14. $\Diamond q_{1,2,4}$ | 3, execute |
| → 15. $p_{1,3,5}$ | 10, advance |
| 16. $\Diamond p_{1,3,5}$ | 2, execute |
| 17. $\Diamond q_{1,3,5}$ | 3, execute |
| → 18. $p_{1,2,4,6}$ | 13, advance |
| 19. $\Diamond p_{1,2,4,6}$ | 2, execute |
| 20. $\Diamond q_{1,2,4,6/2}$ | 3, execute |
| → 21. $q_{1,3,5,7}$ | 16, advance |
| 22. $\Diamond p_{1,3,5,7/3}$ | 2, execute |
| 23. $\Diamond q_{1,3,5,7}$ | 3, execute. |

At this point the derivation terminates. $W = \{1,2,3,4,5,6,7\}$; $R$ = Reflexive + $\{\langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 1,5\rangle, \langle 1,6\rangle, \langle 1,7\rangle, \langle 2,4\rangle, \langle 2,6\rangle, \langle 3,5\rangle, \langle 3,7\rangle, \langle 4,2\rangle, \langle 4,6\rangle, \langle 5,3\rangle\ \langle 5,7\rangle\ \langle 6,2\rangle, \langle 6,4\rangle, \langle 7,3\rangle, \langle 7,5\rangle\}$; $V(p,1) = F$, $V(q,1) = F$, $V(p,2) = T$, $V(q,2) = F$, $V(p,3) = F$, $V(q,3) = T$, $V(p,4) = F$, $V(q,4) = T$, $V(p,5) = T$, $V(q,5) = F$, $V(p,6) = T$, $V(q,6) = F$, $V(p,7) = F$, $V(q,7) = T$. Observe that lines 20 and 22 were treated by Foster-Advancement, rather than by regular Advancement.

***4 Some gory details*** It is obvious that the systems here called ***T***, ***S***4, ***S***5, and *Brouwersche* contain their axiomatic counterparts. Proof of containment in the other direction is available, but too cumbersome to give here in full detail. A sketch will perhaps suffice. We shall concentrate just on *T*.

Let the axiom system $T^a$ be formed on some axiom system of propositional logic containing modus ponens, by adding $\Box$ as primitive, $\Diamond$ defined as $\sim\Box\sim$, and the following axioms and rule:

$T^a$-1 $(\Box P \supset P)$.
$T^a$-2 $(\Box(P \supset Q) \supset (\Box P \supset \Box Q))$.
$T^aR$ From $\vdash P$, infer $\vdash \Box P$.

$T^a$ is a standard axiomatization of the modal system *T*.

Next, let a natural deduction system $T^*$ be formed on the propositional system *S* by adding $\Box$ as primitive, $\Diamond$ defined as $\sim\Box\sim$, and the following rules:

$T^*$-1 From $\Box P$, infer *P*.
$T^*$-2 From $\Box(P \supset Q)$, infer $(\Box P \supset \Box Q)$
$T^*$-3 From *P* which is neither a premise nor assumption, and which depends on no premises and no undischarged assumptions, infer $\Box P$.

It is obvious, I hope, that the axiom system $T^a$ and the natural deduction system $T^*$ are equivalent, in the sense of yielding the same theorems. What needs to be shown is that $T^*$ contains our system *T*.

The notion of *certification* for $T^*$ is defined as follows:

1. A line *P* is certified in a derivation *D* if $\Box P$ is also a line of *D*.
2. If *P* and *Q* are both certified in *D*, so are their (individual or joint) derivates in *D*, if any.
3. Nothing else is certified in *D*.

If a line $P$ depends on a certified line $R$, say that its (transitive) dependency on the certifying line $\Box R$ is *incidental*. The following derived rule of $T^*$ may then be stated:

$T^*$-4 From $(Q \,\&\, {\sim}Q)$ which depends on no uncertified lines earlier than the latest undischarged assumption $P$ (incidental dependence doesn't count), discharge $P$ and infer ${\sim}\Diamond P$, citing all certifying lines earlier than $P$ upon which $(Q \,\&\, {\sim}Q)$ depends.

The derivation of $T^*$-4 is shown schematically as follows: Suppose the following derivation sequence in $T^*$:

$\Box R$
⋮
$\Box S$
⋮
$R$
⋮
$S$
⋮
→ $P$
⋮
$(Q \& {\sim}Q)$; from $R$, $S$, $P$

that is, the contradiction depends only on $P$ and certified lines. Then the following alternative derivation sequence can be constructed:

$\Box R$
⋮
$\Box S$
⋮
→ $R$
→ $S$
→ $P$
⋮
$(Q \,\&\, {\sim}Q)$ as before, from $R$, $S$, $P$
${\sim}P$ RAA
$(S \supset {\sim}P)$ CP
$(R \supset (S \supset {\sim}P))$ CP
$\Box(R \supset (S \supset {\sim}P))$ $T^*$-3

| | |
|---|---|
| $(\Box R \supset \Box(S \supset {\sim}P))$ | $T^*$-2 |
| $\Box(S \supset {\sim}P)$ | modus ponens |
| $(\Box S \supset \Box{\sim}P)$ | $T^*$-2 |
| $\Box{\sim}p$ | modus ponens |
| ${\sim}\Diamond P.$ | definition of $\Diamond$. |

$T^*$-4 is $T^*$'s version of the "Impossibilitation" rule of Section 2 above. $T^*$ can now be shown to contain our $T$ as follows. Bearing in mind that subscripts form no part of a wff, any inference justified by $T$-descent will be justified by $T^*$-1, and any justified by MN will be justified by the definition of $\Diamond$.

As for Impossibilitation, let $D$ be a $T$ derivation whose latest open assumption is $P_{n,c}$ with a later line $(Q \,\&\, {\sim}Q)_m$. If $m = n,c$ then $(Q \,\&\, {\sim}Q)$ comes just from $P$, or from $P$ together with $T$-descent derivates of earlier lines in $D$. It cannot depend on earlier lines in any other way, since only $T$-descent can crank out lines of subscript $n,c$ from lines without that subscript. Those $T$-descendants are thus certified, hence the dependence on the certifying lines is incidental, and the inference to ${\sim}\Diamond P$ is justified under $T^*$-4. On the other hand, if $m \neq n,c$ then since no new subscript has been introduced since $P_{n,c}$ it follows that $(Q \,\&\, {\sim}Q)$ is independent of $P_{n,c}$ and hence comes entirely from earlier lines. Earlier lines which yield $(Q \,\&\, {\sim}Q)$ yield any wff whatsoever, including ${\sim}\Diamond P$. Hence, the inference is again justified in $T^*$. It follows that $T^*$ contains $T$, hence $T^a$ contains $T$: our system $T$ is equivalent to its axiomatic counterpart. Similar proofs may be given for the other three modal systems.

Because modal wffs are finitely complex, and precipitation yields shorter wffs from longer ones, precipitation can generate only a finite number of distinct wffs from a finite number of initial assumptions. Since precipitation omits duplicates of lines already in the derivation, a nonclosing derivation could fail to terminate only by generating eternally ascending subscripts attached to duplicates of earlier wffs, after the manner of unadorned $S4$-precipitation of $(\Box\Diamond P \,\&\, \Box\Diamond Q)$. This cannot happen in $S5$, since $\Diamond P_b$ is not Advanced if the derivation already contains $P_c$ for any $c$. It cannot happen in $T$ or $B$ since Execution proceeds only one step from home base. If those Executions yield new $\Diamond$-wffs and consequent Advances, the resulting new chains will not be accessible for further Execution. It cannot happen in $S4$ when Foster-Advancement is employed. A $\Box$-wff is executed only down its own access path. Repeated generation of $\Diamond$-wffs on a given access path, and their subsequent Advancement, may yield further $\Box$-wffs to be executed, but these will eventually all be duplicates of $\Box$-wffs earlier on, hence already executed down, the path. Thus there must eventually arise a subscript-block identical, save for subscript chain, with an earlier block on the path. At that point, the $\Diamond$-wffs in that block will be Foster-Advanced and that access path will proceed no further.

Arguments paralleling those of Bennett ([2], [3]) show that the model read off from a terminated derivation satisfies every line of the derivation. A completeness theorem follows immediately: a wff unprovable by precipitation is invalid, hence every valid wff can be proven.

***5 Doing it with quantifiers*** The modal rules of Section 2 combine easily with first-order logic to produce familiar systems of quantified modal logic. In what follows below, the notation $\Phi\alpha/\beta$ represents the formula resulting when every occurrence of $\alpha$ in $\Phi\alpha$ is changed to an occurrence of $\beta$; $\Phi\alpha!\beta$ represents the formula resulting when every occurrence of $\alpha$ in $\Phi\alpha$ is changed to one of $\beta$, and every occurrence of $\beta$ comes from such a change; $\Phi\alpha//\beta$ represents any formula resulting when one or more occurrences of $\alpha$ in $\Phi\alpha$ is changed to an occurrence of $\beta$—it represents a range of formulas rather than a single formula.

Let *S* be extended to a natural-deduction system *F* of first-order logic having Universal Instantiation and Universal Generalization as its primitive quantificational rules. Quantificational inferences, like propositional inferences, are valid only within a given possible world. Thus when the apparatus of Section 2 is added to *F*, subscripts will carry over unchanged under all quantificational inferences. This may be assured by phrasing the quantifier rules as:

*Universal Instantiation* From $(\forall\alpha)\Phi\alpha_n$, infer $\Phi\alpha/\beta_n$.

*Universal Generalization* From $\Phi\beta_n$, infer $(\forall\alpha)\Phi\beta!\alpha_n$, provided $\beta$ does not occur free in any premise or undischarged assumption.

Augmenting *F* with the modal rules from Section 2 yields the quantified modal systems *QB*, *Q*5, *QT* + *BF*, and *Q*4 + *BF*. To obtain *QT* and *Q*4 without the Barcan Formula, the quantifier rules *for those systems* must be replaced by

*UI (restricted)* From $(\forall\alpha)\Phi\alpha_n$, infer $\Phi\alpha/\beta_n$, provided that $\beta$ has its first occurrence in the derivation in a line whose subscript is a (proper or improper) initial segment of $n$.

*UG (restricted)* From $\Phi\beta_n$, infer $(\forall\alpha)\Phi\beta!\alpha_n$, provided $\beta$ does not occur free in any premise or undischarged assumption, and has its first free occurrence in the derivation in a line whose subscript has $n$ as a (proper or improper) initial segment.

The semantics for standard quantified modal systems requires that the domain of each world be included in the domain of every accessible world. Additionally, the validity of the Barcan Formula $((\forall x)\Box Fx \supset \Box(\forall x)Fx)$ requires that the domain of each world include that of every accessible world, effectively requiring a common domain for all worlds.

Intuitively, we may think of the primary domain to which a free letter belongs as being fixed by the subscript of the line in which that letter *first* occurs free: it belongs to the domain of the final numeral in that subscript chain. Since containment is transitive, and each numeral in a chain has access to its immediate successor, it follows that the domain of each numeral is contained in that of every successor, and contains that of every predecessor. The effect of UI (restricted) is to confine instantiation at $n$ to letters certified to belong to the domain of $n$, or to fresh letters which are thereby stipulated to be in $n$'s domain. Conversely, UG (restricted) allows generalization at $n$ only when it would be valid at a world whose domain contains that of $n$. For systems containing the Barcan Formula as a theorem, hence requiring a common domain for all worlds, no such restrictions are necessary.

Here, as an illustration, is a proof in *QB* of the Barcan Formula:

| | | |
|---|---|---|
| → | 1. $(\forall x)\Box Fx_1$ | |
| → | 2. $\sim\Box(\forall x)Fx_1$ | |
| | 3. $\Diamond\sim(\forall x)Fx_1$ | 2, MN |
| → | 4. $\sim(\forall x)Fx_{1,2}$ | (3, advance) |
| | 5. $\Box Fa_1$ | 1, UI |
| | 6. $Fa_{1,2}$ | 5, B-descent |
| | 7. $(\forall x)Fx_{1,2}$ | 6, UG |
| | 8. $((\forall x)Fx \mathbin{\&} \sim(\forall x)Fx)_{1,2}$ | 4, 7, conjunction |
| | 9. $\sim\Diamond\sim(\forall x)Fx_1$ | 4–8, impossibilitation |
| | conjoin 3 and 9; close by RAA then CP. | |

Observe that this derivation cannot be carried out in *QT* or *Q*4, since the step at line (7) is not permitted by UG (restricted).

For the systems employing unrestricted UI and UG, the traditional rules of Existential Generalization and Existential Instantiation may be derived and employed in the usual fashion. For the systems employing the restricted versions, these derived rules will have the following formulation.

*EG (restricted):* From $\Phi\beta_n$, infer $(\exists\alpha)\Phi\beta//\alpha_n$, provided $\beta$ has its first free occurrence in the derivation in a line whose subscript is a (proper or improper) initial segment of $n$.

*EI (restricted):*

$$(\exists\alpha)\Phi\alpha_n$$

$$\to \Phi\alpha!\beta_n$$

$$\vdots$$

$$\Psi_n$$

$$\Psi_n$$

provided that $\beta$ does not occur free in $\Psi$, nor in any premise or undischarged assumption, and has its first free occurrence in a line whose subscript has $n$ as a (proper or improper) initial segment.

## NOTES

1. Anderson and Johnstone [1] give a natural-deduction version of *S*4 only, taking strict implication as primitive. Canty [4] gives versions of *T*, *S*4, and *S*5, also taking strict implication as primitive (he mistakenly claims that it is not a modal operator). Ohnishi and Matsumoto [8, 9] give versions of *T*, *S*4, *S*5, and a couple of other Lewis systems, taking necessity and possibility as primitive and using techniques that closely follow the Gentzen canons. Fitch [5] gives versions of *T*, *S*4, and *S*5, taking necessity as primitive. In my opinion, his comes closest to being a "usable" approach.
2. An implementation of the precipitation routine for *S*, *T*, *B*, *S*4, and *S*5 has been programmed in FORTRAN 77 on the University of Utah Computer Center Sperry-UNIVAC 1100, by the author.

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