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Quantifier Responsiveness

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This paper is a logical defense of the essential rightness of the Scholastic theory of distribution, the theory of distributed and undistributed terms. This theory has been severely attacked by Peter Geach,[1] who regards it as a blot on the history of logic. I, in fact, agree with Geach's criticisms themselves, but I wish to overturn his ultimate verdict. The theory of distribution is not a blot on the history of logic, I believe, but rather an imperfect carrying out of a viable and exciting idea.

Recently I have been writing a book on Greek philosophy, and in one chapter I engage in some speculations about how Aristotle came to invent syllogistic logic, and hence logic itself. According to my speculations, Aristotle had noticed that the problem of the One and the Many showed that the logic of categorical statements was not exactly the same as the logic of simple identity statements. So Aristotle thought that the 'is' of identity was different from the categorical 'is'. But if this were so then there ought to be only accidental similarities between identity logic and syllogistic logic. Instead there are rather massive similarities and parallelisms. I speculate that Aristotle developed syllogistic logic hoping to find some enlightenment that would remove this puzzlement. But he never found it.

I go on to find the answer to Aristotle's puzzlement in the work of Medieval logicians: the supposition theorists and the distribution theorists.

From this perspective, distribution theory is part of the answer to the historically fundamental problem of logic and cannot be a blot on the history of logic. I therefore set out to defend distribution theory.

From a purely logical point of view, which is the view of the present paper, Geach's important criticisms can be rolled into one: the traditional rules about distributed terms are, it seems, a special—even almost accidental—feature of syllogistic logic and have no *general* significance or meaning.

But in this paper I present a method I call "quantifier responsiveness analysis". This is a method of validating quantificational arguments and is a generalization of the traditional rules for validating syllogisms, i.e., the traditional

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doctrine of distribution. I show in effect that these traditional ideas can, in a way, be extended to all of quantificational logic.

Consider the syllogism

$$\begin{array}{c} \text{All } AB \\ \text{All } BC \\ \hline \text{All } AC. \end{array}$$

If a variable 'a' ranges over As, 'b' over Bs, and 'c' over Cs—each class assumed nonempty—then the syllogism can be symbolized (following the Medieval supposition theorists[2]) as:

$$\begin{array}{c} (a)(\exists b)(a = b) \\ (b)(\exists c)(b = c) \\ \hline (a)(\exists c)(a = c)\ . \end{array}$$

Here the syllogistic 'is' is taken to be exactly the same as the 'is' of identity.

Now the traditional rules of quality and distribution have the effect of dividing the above argument into two parts.

$$\boxed{\begin{array}{c} (a)(\exists b) \\ (b)(\exists c) \\ \hline (a)(\exists c) \end{array}} \qquad\qquad \boxed{\begin{array}{c} a = b \\ b = c \\ \hline a = c \end{array}}$$

These are: a valid nonquantificational argument involving identity; and a prefixed quantifier pattern. The validity of the matrix argument

$$\begin{array}{c} a = b \\ b = c \\ \hline a = c \end{array}$$

is then transmitted by the prefixed quantifier pattern

$$\begin{array}{c} (a)(\exists b) \\ (b)(\exists c) \\ \hline (a)(\exists c) \end{array}$$

to the whole argument (the syllogism). And so a parallelism between syllogistic and identity logic is explained: the identity argument is part of the syllogism, and the latter derives its validity from the former.

The traditional rules of quality are in fact the rules for the validity of identity arguments which are matrices of syllogisms, provided the categorical statements are symbolized as:

$$\begin{tikzcd} (a)(\exists b)(a = b) \arrow[d] \arrow[dr, <->] & (a)(b)(a \neq b) \arrow[d] \arrow[dl, <->] \\ (\exists a)(\exists b)(a = b) & (\exists a)(b)(a \neq b) \end{tikzcd}$$

The traditional rules of distribution are intended to be rules for what I call quantifier-pattern responsiveness; however, the traditional rules are defective.

To understand what I mean by responsiveness, consider the quantifier pattern before us in our syllogism. Let

$$\frac{\begin{array}{c} R\ ab \\ S\ bc \end{array}}{T\ ac}$$

be any valid and conformable matrix. That is, let $(x)(y)(z)(Rxy \;\&\; Syz \supset Txz)$ be taken as necessarily true. Then,

$$\frac{\begin{array}{c} (a)(\exists b)Rab \\ (b)(\exists c)Sbc \end{array}}{(a)(\exists c)Tac}$$

will also be valid.

For, pick any arbitrary a, say a_1. Then there will be some b, say b_1, such that Ra_1b_1. Then, by the second premise for b_1, there will be some c, say c_1, such that Sb_1c_1. Then, by the matrix argument, since Ra_1b_1 and Sb_1c_1, we have Ta_1c_1. But a_1 was arbitrary. Therefore, $(a)(\exists c)Tac$.

This reasoning shows that the quantifier pattern

$$\frac{\begin{array}{c} (a)(\exists b) \\ (b)(\exists c) \end{array}}{(a)(\exists c)}$$

is a responsive quantifier pattern; it will transmit the validity of *any* valid and conformable matrix argument to the whole quantified argument.

The traditional rules correctly require that a syllogistic quantifier pattern cannot be responsive unless: (1) a term distributed—that is, universally quantified—in the conclusion must also be distributed in its premise. Thus the term 'a' is distributed in the conclusion and in its premise. And they require (2) that the middle term (namely 'b') must be distributed at least once.

These rules need to be generalized to be applicable to arguments other than the syllogistic ones. They become: (1) any term distributed in the conclusion must be distributed in every premise (if any) in which it occurs; and (2) no term may be undistributed more than once in the premises.

Even so, the generalized rules are not sufficient for responsiveness. Consider the syllogism:

$$\frac{\begin{array}{l} \text{All } A\ B \\ \text{All } B\ C \end{array}}{\text{Some } C \text{ is not } A.}$$

In symbols:

$$\frac{\begin{array}{c} (a)(\exists b)(a = b) \\ (b)(\exists c)(b = c) \end{array}}{(\exists c)(a)(c \neq a)\ .}$$

The traditional rules (of quality) correctly reject the matrix as invalid, and hence reject the syllogism. However, the rules of distribution do not distinguish $(\exists c)(a)$ from $(a)(\exists c)$ and so incorrectly see the quantifier pattern as responsive, which it is not.[3] It may be noted, however, that in 256 syllogisms, this is the *only* one where the traditional rules go wrong.

Before looking to complete a correct criterion for responsiveness, we must look again at the connection between responsiveness and validity. By definition of 'responsive', if an argument has *both* a valid matrix *and* a responsive quantifier pattern, then the argument is valid.

The converse is true for the 256 syllogisms but is not true in general.

First, an argument with quantifiers and with all statements in prenex form may be valid and yet not have a valid matrix.

This happens, for instance, in syllogistic-with-negative-terms. Thus

$$\frac{\text{No } A \text{ is } B}{\text{Every } A \text{ is non-}B}$$

is symbolized as

$$\frac{(a)(b)(a \neq b)}{(a)(\exists\bar{b})(a = \bar{b})}$$

and the matrix

$$\frac{a \neq b}{a = \bar{b}}$$

is either invalid or uninterpretable.[4]

In standard logic we have also

$$\frac{(x)(Px)}{(y)(Py),}$$

though $\dfrac{Px}{Py}$ does not look valid, and we have

$$\frac{\begin{array}{l}(x)Px\\(y)Qy\end{array}}{(z)(Pz \,\&\, Qz)}$$

though $\dfrac{\begin{array}{l}Px\\Qy\end{array}}{Pz \,\&\, Qz}$ doesn't look valid.

In these examples, the problem is that the variables do not have independently assigned ranges. In the traditional syllogism, *a*, *b*, and *c* are each *independently* assigned to a nonempty range. But in the negative syllogistic, *b* is assigned to some range, and then $\bar{b}$ is *dependently* assigned to the complement of *b*'s range.

In modern logic, *x*, *y*, and *z* are all automatically assigned to the *same* range, and so not to independently chosen ranges.

If we reinterpret all variables as independently assigned, then the problem cases are no longer valid. Thus,

$$\frac{(a)(b)(a \neq b)}{(a)(\exists\bar{b})(a = \bar{b})}$$

would be invalid because no man (a) is a tree (b) and yet not every man (a) is a camel ($\bar{b}$ is no longer the complement of b).

Also $\dfrac{(x)Px}{(y)Py}$ is invalid because if every stone (x) is P, it doesn't follow that every tree (y) is P. And the last argument is similar.

The *validity* of the three arguments can be recaptured with independently ranged variables if the required dependencies are *stated explicitly* as premises.

Thus if we want $\bar{b}$ to range over the complement of b's range, we add the premises $(b)(\bar{b})(b \neq \bar{b})$ and $(a)(\exists b)(\exists \bar{b})(a \neq b \supset a = \bar{b})$. This last is the prenex of $(a)(\sim\exists b(a = b) \supset \exists \bar{b}(a = \bar{b}))$.[5]

Then the argument in b becomes:

$$\begin{array}{c} (a)(\exists b)(\exists \bar{b})(a \neq b \supset a = \bar{b}) \\ (b)(\bar{b})(b \neq \bar{b}) \\ (a)(b)(a \neq b) \\ \hline (a)(\exists \bar{b})(a = \bar{b}). \end{array}$$

And it now has a valid matrix (and also responsive quantifiers).

To revalidate the argument $\dfrac{(x)Px}{(y)Py,}$ it suffices to require that every Y is X. This can be expressed in the premise $(y)(\exists x)(y = x)$. If this premise is added, then the matrix will be valid (and the quantifier pattern responsive).

However, if we further assure that every X is Y by adding also $(x)(\exists y)(x = y)$, then the matrix will still be valid, but the quantifier pattern will no longer be responsive. Of this, more later.

And, finally, if we add to the last example the needed premises $(z)(\exists x)(z = x)$ and $(z)(\exists y)(z = y)$, it will have a valid matrix (and again, with *only* these additions, a responsive quantifier pattern).

I hereafter mean by a "quantified argument" that all statements are prenex and that all variables are independently ranged.

It should be clear that in principle the question of the validity or invalidity of quantified arguments in this sense is equivalent to that question for ordinary arguments in standard quantification theory, given that these latter are in prenex form.

Take any argument $\dfrac{\alpha}{\beta}$ in prenex form in standard quantification theory. Suppose $x, y, \ldots, w, v$ are all the variables involved. Reinterpret $\dfrac{\alpha}{\beta}$ as involving independently ranged variables and add the premises: $(x)(\exists y)(x = y)$, $(y)(\exists z)(y = z), \ldots, (w)(\exists v)(w = v)$, $(v)(\exists x)(v = x)$. That is, just *say* that every X is Y, every Y is Z, etc., every W is V, and every V is X, so that all variables cover the same entities. In this way we get the "quantified argument" corresponding to the original argument.

The original argument is standardly valid *iff* the corresponding quantified argument is valid in the semantics with independently ranged variables. This is obvious from the very meaning of "validity in a semantics".

Conversely, suppose we have a quantified argument in variables a, b, $c, \ldots, f$, g. To find a corresponding ordinary argument, take first the premises $(\exists x)Ax$, $(\exists x)Bx, \ldots, (\exists x)Gx$, where A, B, $C, \ldots, G$ are new predicates not already occurring in the matrix of the argument. Then, reinterpreting each variable $a, b, c, \ldots, g$ as a standard variable, rewrite all the original premises from the inside out by replacing $(\exists a)\phi a$ by $(\exists a)(Aa \,\&\, \phi a)$ and $(a)\phi a$ by $(a)(Aa \supset \phi a)$ and similarly for b and B, c and $C, \ldots, g$ and G. (These rewrites can then be made prenex if desired.)

An analysis of "quantified arguments" in my sense is thus equivalent to doing ordinary quantificational logic.

Theorem *Any quantified argument which is valid has also a valid matrix.*

Proof: Suppose for example that

$$\begin{array}{c} (a)(\exists b)Rab \\ (b)(\exists c)Sbc \\ \hline (a)(\exists c)Tac \end{array}$$

were valid but its matrix was not. Then choose x, y, z such that Rxy and Syz are true but Txz is not. Let a range over the unit set of x, so that $(a)\phi a = \phi x = (\exists a)\phi a$, and let b range over that of y, and c over that of z. So the quantified argument is also invalid, contrary to our supposition. QED

We can now state the necessary and sufficient conditions for the responsiveness of a quantifier pattern in a quantified argument.

Theorem *A quantifier pattern is responsive iff it satisfies the following three conditions:*
(a) *if a term is distributed in the conclusion, it is distributed in every premise (if any) in which it occurs.*
(b) *no term is undistributed more than once in the premises, i.e., it is undistributed in at most one premise.*
(c) *the terms (i.e., variables) are strictly orderable, with no circles in the ordering, so that if* $\ldots (a) \ldots \exists b \ldots$ *occurs in any premise, then a is before b in the ordering, and if* $\ldots \exists d \ldots (e) \ldots$ *occurs in the conclusion, then d is before e in the ordering.*

Proof:
Necessity:

(1) Suppose the pattern is of the type

$$\begin{array}{c} \ldots\ldots\ldots \\ \ldots \exists a \ldots \\ \ldots\ldots\ldots \\ \hline \ldots (a) \ldots . \end{array}$$

Then for every quantifier not shown, supply trivial matrix: $\ldots (d) \ldots (\ldots \,\&\, d = d \,\&\, \ldots)$. For the shown quantifiers supply Pa. The argument reduces to $\dfrac{(\exists a)Pa}{(a)Pa.}$ The matrix is valid but the argument is not. Therefore, the pattern is unresponsive.

(2) After similar reduction, $\begin{array}{c} (\exists a)Pa \\ (\exists a)\sim Pa \\ \hline Q \end{array}$ has valid matrix but is itself invalid.

(3) After reduction to eliminate by trivialization the irrelevant quantifiers, we have a circular pattern.

$$
\begin{array}{c}
(a)(\exists b)\\
(b)(\exists c)\\
(c)(\exists d)\\
\cdots\\
(e)(\exists f)\\
\underline{(f)(\exists a)}
\end{array}
\qquad \text{or} \qquad
\begin{array}{c}
(a)(\exists b)\\
(b)(\exists c)\\
\text{xxx}\\
\cdots\\
(e)(\exists f)\\
(f)(\exists a)\\
\hline
(\exists c)(d)
\end{array}\ .
$$

Let all variables range over all the natural numbers and take the matrix:

$$
\begin{array}{c}
(a)(\exists b)(a < b)\\
(b)(\exists c)(b < c)\\
(c)(\exists d)(c < d)\\
\cdots\\
(e)(\exists f)(e < f)\\
(f)(\exists a)(f < a)\\
\hline
Q \,\&\, {\sim}Q
\end{array}
\qquad
\begin{array}{c}
(a)(\exists b)(a < b)\\
(b)(\exists c)(b < c)\\
\text{xxx}\\
\cdots\cdot\\
(e)(\exists f(e < f)\\
(f)(\exists a)(f < a)\\
\hline
(\exists c)(d)(c \not< d)
\end{array}\ .
$$

In the first case the matrix is valid (has contradictory premises) but the argument is not. Its premises are all trivial (and equivalent). The second case is the same as the first, but the conclusion has been transposed with a premise and then omitted as tautological.

Sufficiency: Suppose the matrix is valid and the three conditions satisfied. We wish then to show that the whole argument is valid.

Take the last variable, say l, in the ordering. Then there is no $(l)(\exists v)$ in any premise, and no $(\exists l)(v)$ in the conclusion. So if (l) is in any premise, it is before only universal quantifiers and may be put last in the quantifier prenex.

If $(\exists l)$ is in any premise, it may also be moved to last place, possibly weakening but not strengthening the premise. This will happen (by condition 2) in at most one premise.

If l does not occur at all in a premise, add a vacuous (l) at the end of the quantifier prenex.

If $(\exists l)$ occurs in the conclusion, it may be moved to the end of the prenex.

If (l) occurs in the conclusion, it may be moved to the end, possibly strengthening the conclusion. This happens only if $(\exists l)$ was in no premise.

Now the original matrix was, say,

$$
\begin{array}{c}
Pl\\
Ql\\
Rl\\
\hline
Sl
\end{array}
$$

and was valid. We now take as new matrix

$$
\begin{array}{c}
(l)Pl\\
(l)Ql\\
(l)Rl\\
\hline
(l)Sl
\end{array}
\qquad \text{or} \qquad
\begin{array}{c}
(l)Pl\\
(\exists l)Ql\\
(l)Rl\\
\hline
(\exists l)Sl
\end{array}
$$

and this new matrix is valid also since the original one was (as can be seen by instantiating to l, deriving according to matrix, and regeneralizing).

Now do the next to the last variable. Etc. QED

In trying to extend the classical method of validating syllogisms to quantificational logic generally, our first problem was that a valid "argument with quantifiers" might have an invalid matrix. But this no longer holds for "quantifier arguments".

Still, it may happen that a quantified argument—even in my sense—may be valid and yet not have responsive quantifiers. This is our second problem.

First we may find examples if we consider arguments with vacuous quantifiers.

Thus $\dfrac{(\exists a)(\exists b)Pb}{(a)(\exists b)Pb}$ is valid but has unresponsive $\dfrac{(\exists a)}{(a)}$ in its pattern. Interestingly, however, if we differentiate a into two variables, we may see this example as a special case of $\dfrac{(\exists f)(\exists b)Pb}{(g)(\exists b)Pb}$ which *does* have responsive quantifiers.

To get more exciting cases, we may violate various conditions which Aristotle imposed in his notion of 'syllogism'.

Aristotle said[6] a syllogism must have no unneeded premises. An example I mentioned earlier:

$$\begin{array}{c}(x)\exists y(x = y)\\ (y)\exists x(y = x)\\ (x)Px\\ \hline (y)Py\end{array}$$

has an unneeded first premise and the unresponsive circle pattern

$$\begin{array}{c}(x)(\exists y)\\ \underline{(y)(\exists x)}\\ \hline\end{array}$$

as well as the pattern $\dfrac{\exists y}{(y).}$ However, again, we may differentiate variables and see this as a special case of

$$\begin{array}{c}(u)(\exists v)(u = v)\\ (y)(\exists x)(y = x)\\ (x)Px\\ \hline (y)Py,\end{array}$$

which is valid and *does* have responsive quantifiers.

Next Aristotle said that a syllogism could not have contradictory premises. Sure enough, $\dfrac{(\exists a)(Pa \,\&\, {\sim}Pa)}{(a)Qa}$ is valid and has nonresponsive quantifiers. But it is an instance of $\dfrac{(\exists b)(Pb \,\&\, {\sim}Pb)}{(c)Qc}$ which does have responsive quantifiers.

Again Aristotle said that a syllogism should not be breakable into shorter arguments. If we violate this condition, consider the argument

$$\begin{array}{ll} (a)(\exists b)(Pa \supset Qb) & \\ (\exists a)Pa & (/\therefore \text{ a sub-conclusion } (\exists b)Qb) \\ (\exists a)Sa & \\ \hline (\exists a)(\exists b)(Qb \,\&\, Sa). & \end{array}$$

This argument has three premises but can be broken into two arguments with two premises each. Sure enough, it has unresponsive quantifiers, since we have $\begin{array}{l}\exists a\\ \underline{\exists a}.\end{array}$ But this argument is an instance of

$$\begin{array}{lll} & (d)(\exists b)(Pd \supset Qb) & \\ & (\exists d)Pd & (/\therefore\ (\exists b)Qb) \\ \text{of} & (\exists f)Sf & \\ \cline{2-2} & (\exists f)(\exists b)(Qb \,\&\, Sf). & \end{array}$$

So far it looks as if any example of a valid quantified argument with unresponsive quantifiers can be shown to be a special case of a more general form of argument which is reached by differentiation of the troublesome variables and which has responsive quantifiers.

However, things are not quite so simple. A variation of our last example shows this:

$$\begin{array}{rl} 1. & (\exists a)(Pa \,\&\, Qa) \\ 2. & (\exists a)(\sim Pa \,\&\, Qa) \\ 3. & (a)(Qa \supset Ra) \\ 4. & (a)(\exists b)(a = b) \\ \hline \therefore & (\exists a)(\exists b)(Pa \,\&\, Ra \,\&\, \sim Pb \,\&\, Rb). \end{array}$$

Here $Pa \,\&\, Ra$ comes from premises 1 and 3, while $\sim Pb \,\&\, Rb$ comes from 2, 3, and 4. So 3 is used for both parts of the conclusion and the a in 3 must be used both with the a in 1 and with the a in 2 and 4. If we try to replace the a in 1 and the conclusion by d, then the a in 3 will have to be d. But if we try to replace the a in 2 and 4 by f, the a in 3 will have to be f.

So we have to double the a in 3 in some way. One way would be to use premise 3 twice and then differentiate to get

$$\begin{array}{rl} 1. & (\exists d)(Pd \,\&\, Qd) \\ 2. & (\exists f)(\sim Pf \,\&\, Qf) \\ 3.1. & (d)(Qd \supset Rd) \\ 3.2. & (f)(Qf \supset Rf) \\ 4. & (f)(\exists b)(f = b) \\ \hline \therefore & (\exists d)(\exists b)(Pd \,\&\, Rd \,\&\, \sim Pb \,\&\, Rb) \end{array}$$

A less simple way, but one that turns out to be more general in its utility, is to imagine that $A = D \cup F$ and to rewrite 3 as $(d)(f)((Qd \supset Rd) \,\&\, (Qf \supset Rf))$.

The original argument is then regarded as a special case of:

$$\begin{array}{rl} 1. & (\exists d)(Pd \,\&\, Qd) \\ 2. & (\exists f)(\sim Pf \,\&\, Qf) \\ 3. & (d)(f)((Qd \supset Rd) \,\&\, (Qf \supset Rf)) \\ 4. & (f)(\exists b)(f = b) \\ \hline \therefore & (\exists d)(\exists b)(Pd \,\&\, Rd \,\&\, \sim Pb \,\&\, Rb). \end{array}$$

Before proceeding further, it is useful to go to a simpler formulation of the conditions for responsiveness.

If $\begin{array}{c} p \\ \underline{q} \\ r \end{array}$ is an argument, then $\begin{array}{c} p \\ q \\ \underline{\sim r} \end{array}$ is the antilogism.[7] The argument is valid iff the antilogism is inconsistent.

In writing a quantified antilogism, we will suppose that the negation in $\sim r$ has been driven inwards by $\sim\exists = \forall\sim$ and $\sim\forall = \exists\sim$, so that the negation is part of the matrix. If the antilogism is inconsistent, so will be the matrix. The argument has responsive quantifiers (transmitting validity) iff the antilogism has responsive quantifiers (transmitting inconsistency).

Now $\begin{array}{c} p \\ q \\ \sim r \end{array}$ is inconsistent iff $\begin{array}{c} p \\ q \\ \underline{\sim r} \\ Q \end{array}$ is valid for arbitrary unquantified Q. The effect of switching to the antilogism is thus to simplify the conditions for responsiveness by omitting reference to a conclusion.

Theorem *The quantifiers of an antilogism are responsive iff*
(a) *no term is undistributed more than once*
(b) *the terms are orderable so that, if... (a) ... $(\exists b)$ ... is in any statement, then a comes before b in the ordering (or as I shall say '$a > b$') and there are no circles.*

Proof: Clear from above remarks. Alternatively, the new condition 1 is the old 1 plus the old 2, and the new 2 is the old 3. QED

Let us now return to the problem of trying to see valid arguments (or really inconsistent antilogisms) with nonresponsive quantifiers as "special cases" of other arguments (antilogisms) with responsive quantifiers. Suppose

$$\begin{array}{r} (\exists a)Pa \\ (\exists a)Qa \\ \underline{(a)Ra} \\ (\exists a)Sa \end{array}$$

is valid, violating condition 1: Then take the antilogism

$$\begin{array}{r} (\exists a)Pa \\ (\exists a)Qa \\ (a)Ra \\ \underline{(a)\sim Sa} \end{array}$$

and this is inconsistent.

Consider then

$$\begin{array}{l} (\exists b)Pb \\ (\exists c)Qc \\ (b)(c)(Rb \;\&\; Rc) \\ \underline{(b)(c)(\sim Sb \;\&\; \sim Sc).} \end{array}$$

This second antilogism is, as a general form (given fixed P, Q, R, S), equivalent to the original, but now has responsive quantifiers.

For if the first antilogism is inconsistent, suppose the second is consistent for some B and C. Take $A = B \cup C$. Then the first would be true in the interpretation that makes the second true. So the second is not consistent.

Conversely, suppose the second is inconsistent but the first is true for some A. Then take $B = C = A$ and the second is true. So the first is inconsistent.

So the original argument can be regarded as a special case, and equivalent to, the argument

$$\begin{array}{l} (\exists b)Pb \\ (\exists c)Qc \\ \underline{(b)(c)(Rb \mathbin{\&} Rc)} \\ (\exists b)(\exists c)(Sb \vee Sc), \end{array}$$

namely the case where both B and C are the same class A.

We may then validate the second of these arguments by the Medieval method[8]—that is, by direct application of the fact that the matrix is valid and the quantifiers are responsive—and then validate the original as a special case of the second.

If every valid quantified argument is a special case of some other argument validatable by the Medieval method, then the Medieval method acquires complete generality. I shall show exactly this for arguments that do not violate condition 2 (for antilogisms—the noncircularity condition), and shall show something close to this for all quantified arguments.

To approach the general method for cases satisfying the noncircularity condition, let us look at an example.

$$\begin{array}{llll} (\exists a)(b)(\exists c)\ (d)Pabcd & & & \text{[Note } (b)(\exists c)] \\ (\exists a)(b)\ \ (c)(d)Qabcd & & & \text{[Note double } (\exists a)] \\ \underline{\ (a)\qquad (c)\qquad Rac\ } & & & \\ (\exists a)(b)(\exists c)(\exists d)\sim Sabcd. & & & \text{[Note } (\exists a)(b)] \end{array}$$

The antilogism is

$$\begin{array}{l} (\exists a)\ \ (b)(\exists c)(d)Pabcd \\ (\exists a)\ \ (b)\ \ (c)(d)Qabcd \\ \ (a)\qquad\quad (c)\qquad Rac \\ \underline{\ (a)(\exists b)\ \ (c)(d)Sabcd.} \end{array}$$

Condition 1 is violated by $\exists a$ in statement 1 and 2.

In the ordering for condition 2, we must have a before b because of $(a)(\exists b)$ in statement 4, and we must have b before c because of $(b)(\exists c)$ in 1. A possible ordering is $a > b > c > d$. When we split a to satisfy condition 1, b will be split also, and then c. If noncircularity were violated, our method would lead to an infinite expansion.

Specifically let $A = A^1 \cup A^2$, splitting a into a^1 and a^2.

$$\begin{array}{ll} (\exists a^1)(b)(\exists c)(d)Pa^1bcd & \\ (\exists a^2)(b)\ \ (c)(d)Qa^2bcd & \\ \ \ (a^1)(a^2) & [(c)Ra^1c \mathbin{\&} (c)Ra^2c] \\ \ \ (a^1)(a^2) & [(\exists b)(c)(d)Sa^1bcd \mathbin{\&} (\exists b)(c)(d)Sa^2bcd]. \end{array}$$

The third statement can now be prenexed by pulling out (c), but in the fourth statement b needs to be split. So let $B = B^1 \cup B^2$.

Then b is split, first in the fourth statement, and then working up to the first and second statements. The result is

$$\begin{aligned}
&(\exists a^1)(b^1)(b^2)\,[(\exists c)(d)Pa^1b^1cd \,\&\, (\exists c)(d)Pa^1b^2cd]\\
&(\exists a^2)(b^1)(b^2)\,[(c)(d)Qa^2b^1cd \,\&\, (c)(d)Qa^2b^2cd]\\
&(a^1)(a^2)(c)(Ra^1c \,\&\, Ra^2c)\\
&(a^1)(a^2)(\exists b^1)(\exists b^2)(c)(d)\,[Sa^1b^1cd \,\&\, Sa^2b^2cd].
\end{aligned}$$

And now the second statement can be prenexed by pulling out (c) and (d), but c has to be split in statement 1. We get

$$\begin{aligned}
&(\exists a^1)(b^1)(b^2)(\exists c^1)(\exists c^2)(d)\,[Pa^1b^1c^1d \,\&\, Pa^1b^2c^2d]\\
&(\exists a^2)(b^1)(b^2)(c^1)(c^2)(d)\\
&\qquad [Qa^2b^1c^1d \,\&\, Qa^2b^2c^1d \,\&\, Qa^2b^1c^2d \,\&\, Qa^2b^2c^2d]\\
&(a^1)(a^2)(c^1)(c^2)\,[Ra^1c^1 \,\&\, Ra^2c^1 \,\&\, Ra^1c^2 \,\&\, Ra^2c^2]\\
&(a^1)(a^2)(\exists b^1)(\exists b^2)(c^1)(c^2)(d)\\
&\qquad [Sa^1b^1c^1d \,\&\, Sa^2b^2c^1d \,\&\, Sa^1b^1c^2d \,\&\, Sa^2b^2c^2d].
\end{aligned}$$

And now we have responsive quantifiers. So the validity or invalidity of the original argument corresponds to the inconsistency or consistency of the *matrix* of this expanded version. Note also that the original ordering was $a > b > c > d$. An ordering for our final result is $a^1 > a^2 > b^1 > b^2 > c^1 > c^2 > d$.

Notice also that the original argument (or more obviously the antilogism) had in effect the *arbitrary* conformable matrix, since P, Q, R, S could be any formulas with the right number of places. So the expansion is really intrinsic to the quantifier pattern itself and does not depend on the particular matrix.

This result is completely general for antilogisms violating condition 1 but satisfying condition 2.

Theorem *Any quantified antilogism satisfying 2 but violating 1 can be mechanically expanded in a way depending only on the quantifier pattern into an equivalent antilogism with responsive quantifiers.*

Proof: The variables may be ordered $a > b > c \ldots > w$. Take the *first* of these which occurs more than once existentially. Say it is b. Replace $\exists b\phi_1 b$, $\exists b\phi_2 b, \ldots, \exists b\phi_n b$ by $\exists b_1\phi_1 b_1$, $\exists b_2\phi_2 b_2, \ldots, \exists b_n\phi_n b_n$. Also replace any $(b)\phi b$ by $(b_1)(b_2)\ldots,(b_n)\,[\phi b_1 \,\&\, \phi b_2 \,\&\ldots\&\, \phi b_n]$. This last is no longer prenex, which we worry about later. Moreover, by replicating ϕ, we multiply any $\exists v$ in ϕ. But this v is *after* b in the order, since $(b)\phi = (b)\ldots\exists v\ldots$ if $\exists v$ is in ϕ; so b no longer is a multiple $\exists$. The new order may be taken to be $a > b_1 > b_2 > b_3 > b_4 \ldots > b_n > c > d > \ldots > w$, and all multiple $\exists$ variables are now after b_n. Take the *next* multiple $\exists$'d variable, whether it was multiple in original antilogism or has been made multiple by treatment of b. Suppose it is d. Replace each $\exists d\phi_1 d$, $\exists d\phi_2 d$, etc. by $\exists d_1\phi_1 d_1$, $\exists d_2\phi_2 d_2$, etc., and each $(d)\phi d$ by $(d_1)(d_2)\ldots[\phi d_1 \,\&\, \phi d_2 \,\&\ldots]$ and take the new order as $a > b_1 > \ldots > b_n > c > d_1 > d_2 \ldots > \ldots > w$. Now all multiple $\exists$'d variables are after the ds. Etc. Eventually there are no multiple existentials.

Now we have to consider the re-prenexing of our antilogism. Prenexing itself is not always possible in logic with independently ranged quantifiers. A statement such as $(a)Pa \supset (\exists a)Qa$ or $(\exists a)Pa \ \& \ (\exists a)Qa$ cannot be prenexed, for instance, since there is no other variable logically equivalent in range to a. However $(a)Pa \ \& \ (a)Qa$ can be prenexed as $(a)(Pa \ \& \ Qa)$ and $R \ \& \ (\exists a)Qa$ can be prenexed as $(\exists a)(R \ \& \ Qa)$ if there is no a in R.

We also wish to extract the various quantifiers into prenex position so as not to violate the restrictions imposed by the $>$ ordering. The placement of quantifiers need not actually be in the order of the $>$ ordering but must conform to its requirements. For instance, if $a > b$, then we might have $(b)(a)$, $(\exists b)(\exists a)$, or $(\exists b)(a)$, but *not* $(b)(\exists a)$.

The original placements in the original antilogism conformed to the condition of the original ordering, and we have defined the new $>$ ordering in terms of the original one and the splitting process. So we need to be sure that the new placements are conformable to that new $>$ ordering.

In the splitting process, scope relations among quantifiers were in a sense preserved. When $\exists d\phi d$ is replaced by $\exists d_i \phi d_i$, the descendent d_i takes up the scope relations of its ancestor d. When $(d)\phi d$ is replaced by $(d_1) \ldots (d_n) [\phi d_1 \ \& \ \phi d_2 \ \& \ldots \& \ \phi d_n]$, the results are as follows. Among themselves, the d_i's are scoped in the determinate order $d_1, d_2, \ldots, d_n$. Quantifiers in one ϕ replica have no scope relations with those in other ϕ replicas. The various d_i all take up the scope relations to quantifiers in the ϕ's which their ancestor d had.

In the original antilogism, no quantifier was scoped under any propositional operator. In the splitting process, quantifiers are scoped only under conjunction.

Take any statement in our new antilogism. We now wish to re-prenex it so that the placement of quantifiers is the same as in the original statement, except that split variables are replaced by their descendants or the sequence of their descendants.

Suppose this done for the first n quantifiers of the original statement—taken in the order in which they occurred in the prenex—but not yet for the $n + 1^{\text{th}}$. The $n + 1^{\text{th}}$ quantifier or its descendants, or various copies thereof, are still in the matrix of the so far extracted prenex. The matrix either begins with a quantifier or is a conjunction of statements that do. The versions or the descendants of the $n + 1^{\text{th}}$ quantifier do not occur under the scope of those of the $(n + 2)^{\text{th}}$ or later quantifiers, so they occur at the beginnings of (some of) the conjuncts. Moreover, the descendants are presented in the order determined by the split. If the original variable d occurred universally, all its descendants are universal and each may be extracted by $[(d_i)\phi d_i \ \& \ (d_i)\psi d_i \ \& \ (d_i)\pi d \ \& \ m] = (d_i)[\phi d_i \ \& \ \psi d_i \ \& \ \pi d_i \ \& \ m]$. If d was existential, so are its descendants, and each occurs only once if at all and may be extracted when it occurs by $[\phi \ \& \ \psi \ \& \ \exists d_i \pi d_i \ \& \ m] = \exists d_i [\phi \ \& \ \psi \ \& \ \pi d_i \ \& \ m]$.

Therefore prenexing may be performed in the order indicated.

But the order of prenexing which results is conformable to the new $>$ order. If $(a_i) \ldots (\exists b_j)$ occurs then $a_i > b_j$. For since a_i is universal and b_j is existential, the original a is different from the original b. But if $a \neq b$, the relative placements of (a_i) and $(\exists b_j)$ are the same as those of the original (a) and $(\exists b)$ and so conform to the original and hence the new $>$ order.

The original antilogism has thus been expanded into a new prenexed antilogism satisfying conditions 1 and 2. QED

This method applies only when noncircularity is satisfied. When this method applies it allows us to validate a quantified argument after expansion by inspection, as with the traditional way of validating syllogisms. It also allows invalidation, and gives us a decision procedure for arguments satisfying noncircularity.

When condition 2 is violated things are not so nice. Consider the argument

$$\begin{array}{ll} 1. & (a)(\exists b)aRb \\ 2. & \underline{(b)(\exists a)bRa} \\ 3. & \quad (a)Qa. \end{array}$$

Now interpret Q as follows. If there is an infinite sequence, possibly repeating, such that $x_1Rx_2Rx_3Rx_4\ldots$ and if x is a member of such a sequence, then Qx.

Now the matrix argument $\begin{array}{c} aRb \\ \underline{bRa} \\ Qa \end{array}$ is valid, since if the premises are true then $aRbRaRbRaRb\ldots$ is such a sequence for a.

Further the quantified argument is valid. For pick any a_1. Then (by 1) find b_1 such that a_1Rb_1. Then (by 2) find a_2 such that $a_1Rb_1Ra_2$. Etc. And we get $a_1Rb_1Ra_2Rb_2Ra_3Rb_3\ldots$, and so Qa_1.

If we take the antilogism

$$\begin{array}{c} (a)(\exists b)aRb \\ (b)(\exists a)bRa \\ (\exists a)\sim Qa \end{array}$$

we see that both conditions for responsiveness are violated. $(a)\exists b$ and $(b)\exists a$ is a circle. And also $\exists a$ occurs twice, though this point is insignificant; the conclusion $\exists aQa$ could be taken instead. The violation of noncircularity is the important point.

Now it is possible to expand our argument into one with responsive quantifiers—provided we allow an infinite premise set.

$$\begin{array}{ll} 1.1 & (a_1)(\exists b_1)(a_1Rb_1) \\ 2.1 & (b_1)(\exists a_2)(b_1Ra_2) \\ 1.2 & (a_2)(\exists b_2)(a_2Rb_2) \\ 2.2 & (b_2)(\exists a_3)(b_2Ra_3) \\ & \quad\ldots \\ 1.n & (a_n)(\exists b_n)(a_nRb_n) \\ 2.n & (b_n)(\exists a_{n+1})(b_nRa_{n+1}) \\ & \quad\ldots \\ \hline & (a_1)Qa_1. \end{array}$$

Here a_1 is distributed in its only premise (1.1) and circles have been eliminated. Unfortunately however, the conclusion now follows from *all* the premises, but not from any finite subset of them. Indeed the matrix conclusion Qa_1 follows only from *all* the matrix premises.

This example shows that we cannot automatically expand an unresponsive quantifier pattern violating condition 2 into an equivalent *finite* responsive quantifier pattern (with, of course, a different matrix reached by expansion). Nevertheless, in this example we are dealing with a *nonformal* validity, resting on the interpretation of Q. Because of this, the compactness theorem for formal validity does not come into play. This suggests that if the same quantifier pattern occurred in a *formally* valid argument, some finite subset of the premises of an infinite expansion would suffice for the conclusion, though *which* finite subset would depend on the particular matrix, rather than being intrinsic to the quantifier pattern itself. The same suggestion also emerges from the expansion process used for arguments violating only condition 1. If condition 2 is violated, that process leads to infinite regress, so perhaps compactness could be used to limit that regress.

In fact, I shall not take exactly the approach suggested by these considerations but shall instead take an approach with similar results but based on instantiation. I did at first try the approach suggested above and found that the variables tended to get snarled up in each other and things got too messy for me. (The problem was that one does not know in what order to extract the new variables in re-prenexing.)

Since the expansion process for nonresponsive quantifiers in general will be guided by considerations of instantiation, let me briefly consider instantiation.

In standard quantificational natural-deduction systems, we are familiar with instantiation and generalization, that is, EI, UI, EG, and UG. In logic with independently sorted variables, these processes are just the same as usual except that we have different variables a, b, c, etc. Since these variables are nonequivalent, we must instantiate a to corresponding dummy-variables a_1, a_2, a_3, and b to b_1, b_2, b_3, and each different variable-of-quantification to its own dummy variables. Then when generalizing, we generalize a_1 or a_2 or a_3 to a, and b_1 or b_2 or b_3 to b, and so forth.

All instantiation and generalization will be to dummy-variables and never to proper names. An individual, Socrates, may be named 's'. He may fall in the range of h (human), x (thing), m (man or male human), or d (mortal). But none of these variables will be instantiated to s. Rather they will be instantiated to h_1, x_1, m_1, or d_1, and Leibniz's Law will be used to bring s in.[9]

Take the example: Socrates is human: all humans are mortal; therefore Socrates is mortal. Using the general-thing variable x, and predicates H and D, the two premises are Hs, $(x)(Hx \supset Dx)$. And we need for validity the extra premise $\exists x(s = x)$. Then instantiating to x_1, we get $s = x_1$, $Hx_1 \supset Dx_1$, and so by Leibniz's Law $Hs \supset Ds$. But Hs, and so Ds.

Or we may take as premises $(h)Dh$ and $\exists h(s = h)$. Instantiation to h_1 gives $s = h_1$ first and then Dh_1. From which, by Leibniz's Law, Ds. If we replace D by $\exists d(\ldots = d)$ throughout, the result is similar.

So instantiation to names is not needed and all instantiation and generalization is between a given variable and corresponding dummy-variables.

We are going to approach the problem of expanding nonresponsive quantifiers into responsive ones by way of instantiation. That there is indeed a real relation between the process of instantiation and the idea of responsiveness has probably already been noticed by some readers, or maybe all readers.

Suppose we have an antilogism and we regard the statements therein as premises for instantiation. I shall say that these premises are *uniformly* instantiated if all quantifiers are eliminated by instantiation and each variable a is replaced throughout the whole antilogism matrix by a single corresponding term-of-instantiation a_1.

Theorem *A quantifier pattern of an antilogism is responsive iff the antilogism can be uniformly instantiated.*

Proof:

I. *If the pattern is responsive, uniform instantiation is possible.* This is almost self-evident from the very notion of responsiveness, and rigorously provable from the two conditions for responsiveness.

From the very notion of responsiveness, we think as follows. If uniform instantiation were not possible, terms identical in the matrix would be differentiated in every instantiation and so some inconsistent matrix would lose its inconsistency, and so the quantifiers would not be transmitting or preserving inconsistency in general and so would not be responsive.

But since this reasoning, though probably correct, is not very tight, let us instead use the two conditions for responsiveness. We assume responsiveness. So the two conditions are fulfilled. No $\exists a$ occurs twice and there are no $(a)(\exists b)$ circles.

All the statements are prenex and the front quantifier of each prenex is initially *presented* for instantiation. Whenever we instantiate, a quantifier vanishes and the next quantifier in that prenex is *presented.*

We wish to instantiate each variable, a or b or c or . . . , to its single corresponding term of instantiation, a_1 or b_1 or c_1 or. . . .

We will never instantiate a universal quantifier in a given variable until *after* we have instantiated any occurring existential quantifier in that variable.

If any *presented* quantifier is existential, instantiate at once.

If any presented quantifier is universal, and no existential quantifiers in that variable occur, instantiate at once.

If at some point a universal quantifier is presented and its existential counterpart has already been instantiated away, instantiate.

In this way either total uniform instantiation will be achieved or else we will come to a stage in which all presented quantifiers are (1) universal and (2) have existential counterparts still buried in the unpresented part of the prenexes.

But this is impossible, for, by condition 2, some presented variable v is the presented variable that is first in the $>$ ordering, and its existential quantification cannot be buried after some other presented variable's (say u's) universal quantification. For if... (u) ... $(\exists v)$, then v is not first in the ordering. This proves I.

II. *If the pattern is unresponsive, uniform instantiation is not possible.* This is rigorously provable either from the notion of responsiveness or from the two conditions.

From the notion: If uniform instantiation were possible, then if the matrix is inconsistent, an inconsistent instantiation could be derived from the quantified antilogism, which would therefore be itself inconsistent. So, contrary to hypothesis, the quantifiers would be responsive.

From the conditions: Since the quantifier pattern is unresponsive, one of the conditions must be violated. If there are two copies of $\exists a$, each must be instantiated to a different term and so nonuniformly. If there is a circle pattern, say

$$(a)(\exists b)$$
$$(b)(\exists c)$$
$$(c)(\exists a),$$

then we must first instantiate a universal quantifier somewhere along the circle. Since it is a circle it doesn't matter where we break in. So, from the top, take a_1, b_1, then c_1, and now we need to instantiate $\exists a$, and need a new term of instantiation. This completes our proof. QED

In the above proof, I have brought in the very notion of responsiveness as well as the two conditions. I have done this to indicate that the *idea* of uniform instantiability is not *merely equivalent* to the idea of responsiveness (as these ideas *are* merely equivalent to the idea of satisfying the two conditions). Rather the idea of uniform instantiability and the idea of responsiveness are to a large extent the same idea. Historically one could say, as I have, that the distribution theorists were trying to work out conditions for responsiveness, but one could equally well say they were trying to work out conditions for uniform instantiation of an antilogism (or uniform instantiation-plus-generalization in an argument).

The fact that responsiveness means uniform instantiation and nonresponsiveness means nonuniform instantiation suggests a way of expanding inconsistent antilogisms with unresponsive quantifiers into ones with responsive quantifiers. Since the antilogism is inconsistent, instantiate nonuniformly until reaching an inconsistent set of quantifier-free instances. A given matrix statement will thus be instantiated into a set or in effect a conjunction of instances, and each variable-of-quantification will be differentiated into terms-of-instantiation. This is a one-many correspondence between the variable and the terms, because of nonuniformity. The obvious thought then is to expand each matrix by $M = M \,\&\, M \,\&\, M \ldots \&\, M$ into a conjunction corresponding to the conjunction of instances, carrying the quantifiers along ($QM = QM \,\&\, QM \,\&\, QM \,\&\ldots \&\, QM$), and then differentiate the variables into the terms of instantiation, taking these as one's new variables. If then the quantifiers Q can be re-prenexed suitably, the inconsistent set of instances will be deducible by a uniform, one-to-one, instantiation. I now go on to show that this suggested procedure actually works.

Definition Given an inconsistent quantified antilogism, suppose by successive instantiations (UI and EI), we derive a quantifier-free set of statements and this set is inconsistent (by propositional logic plus identity). Then the derivation is called *proof by instantiation* of the inconsistency of the original antilogism.

Idea Proof by instantiation is a kind of reductio. Since the original antilogism leads to an inconsistent set of instances, it is itself therefore inconsistent.

In an actual reductio, one derives an obvious inconsistency from a not obviously inconsistent premise set. In proof by instantiation we, as it were, adopt the sometimes laughable assumption that the inconsistency of a quantifier-free set is always "obvious" and needs no further proof.

Theorem *For any formally inconsistent quantified antilogism, there is a proof by instantiation.*

Proof: The following construction[10] is designed to produce the proof by instantiation and also an ordered list of the terms of instantiation, ordered by their first introduction.

Step 1: Put the antilogism statements into the *statement set.*

Then instantiate each variable v which occurs in the *initial universal* quantifier of any statement. Instantiate such v to v_1.

Then put the terms v_1 in the order used in the *term set.*

Then instantiate each statement $\exists v\phi v$ in the statement set (beginning with a $\exists v$) to a term $v_{\exists v\phi v}$.

Then put the terms $v_{\exists v\phi v}$ into the term set in the order used.

Then put the statements resulting from these instantiations into the statement set.

Step $n+1$: For any $(v)(\phi v)$ in the statement set and any term v_x in the term set, such that v_x is a term of type v, and $(v)\phi v$ has not yet been instantiated to v_x, now so instantiate.

If any $(v)\phi v$ is still not instantiated and is in the statement set, instantiate it now to a new term v_1. Put such new v_1's into the term set.

If any $\exists v\phi v$ in the statement set has not been instantiated, instantiate to $v_{\exists v\phi v}$. And put these terms (in the order used) into the term set.

In this process we note: every EI involves a *new* term of instantiation.

Further, the quantifier-free instances generated form a model of the quantified statements of the original antilogism, in the sense that, for instance, if a ranges over the a_x terms, b over the b_x terms, etc., then if $(a)(\exists b)(c)\phi abc$ is in the antilogism, then for every a_x term, there is some b_x term, such that for every c_x term, '$\phi a_x b_x c_x$' is in the quantifier-free part of the statement set. For $(a)(\exists b)(c)\phi abc$ will eventually be instantiated to any a_x term. If a_x goes in at stage n, $(a)(\exists b)(c)\phi abc$ is instantiated to yield $(\exists b)(c)\phi a_x bc$ either at stage n or at stage $n+1$. Then $\exists b(c)\phi a_x bc$ is instantiated at the next stage, with $b_x = b_{\exists b(c)\phi a_x bc}$, to yield $(c)\phi a_x b_x c$. Then at the next stage c is instantiated here to any c_x terms already available, and to any other later.

Now we apply the compactness theorem. Since the possibly infinite set of instance-statements forms a model of the original inconsistent antilogism, the set of instance-statements is also inconsistent. If the antilogism is *formally* inconsistent, so is the set of instance-statements. And so, by compactness, some finite subset of them is also formally inconsistent.

Therefore we can pick out of the whole possibly infinite process those parts relevant to achieving that finite inconsistent set of instances. This part will be finite and will constitute a proof by instantiation.

As a bonus we get the terms of instantiation in the order in which they were introduced in the infinite process. Eliminate all those terms not used in the finite process, and reconsider the order of the remaining terms by considering their first introduction into the finite process. QED

Definitions Some more, given by example.

Suppose $(a)(\exists b)(c)(\exists d)\phi abcd$ is a statement in the antilogism. Suppose it is instantiated to $a_1 b_1 c_1 d_1$ and also from $a_1 b_1$ to $c_2 d_2$ and $c_3 d_3$. Suppose also it is instantiated to $a_2 b_2 c_4 d_4$ and from $a_2 b_2$ to $c_1 d_5$ and $c_3 d_6$. That is, these are the instantiations retained in the proof by instantiation.

Then the *instantiation tree* for the statement is:

$$
\begin{array}{lllllll}
 & & & & c_1 & \text{———} & d_1 \\
 & a_1 & \text{———} & b_1 & c_2 & \text{———} & d_2 \\
 & & & & c_3 & \text{———} & d_3 \\
s & & & & & & \\
 & & & & c_4 & \text{———} & d_4 \\
 & a_2 & \text{———} & b_2 & c_1 & \text{———} & d_5 \\
 & & & & c_3 & \text{———} & d_6.
\end{array}
$$

The *instantiation-expansion-relettering* is then

$$
\begin{aligned}
&(a_1)(\exists b_1)\,[(c_1)(\exists d_1)\phi a_1 b_1 c_1 d_1 \;\&\; (c_2)(\exists d_2)\phi a_1 b_1 c_2 d_2 \;\&\; (c_3)(\exists d_3)\phi a_1 b_1 c_3 d_3] \\
&\quad \&\; (a_2)(\exists b_2)\,[(c_4)(\exists d_4)\phi a_2 b_2 c_4 d_4 \;\&\; (c_1)(\exists d_5)\phi a_2 b_2 c_1 d_5 \\
&\qquad\qquad \&\; (c_3)(\exists d_6)\phi a_2 b_2 c_3 d_6].
\end{aligned}
$$

This latter is the following conjunction, but written in standard linear notation:

$$
\begin{array}{lll}
 & & (c_1)(\exists d_1)\phi a_1 b_1 c_1 d_1 \\
 & (a_1)(\exists b_1)\ \& & (c_2)(\exists d_2)\phi a_1 b_1 c_2 d_2 \\
 & & (c_3)(\exists d_3)\phi a_1 b_1 c_3 d_3 \\
\& & & \\
 & & (c_4)(\exists d_4)\phi a_2 b_2 c_4 d_4 \\
 & (a_2)(\exists b_2)\ \& & (c_1)(\exists d_5)\phi a_2 b_2 c_1 d_5 \\
 & & (c_3)(\exists d_6)\phi a_2 b_2 c_3 d_6.
\end{array}
$$

The expansion-instantiation-*relettering* is so called because it is a relettering of the instantion-expansion, which is just like the instantiation-expansion (i-e)-relettering but without the subscripts. Namely:

$$
\begin{gathered}
(a)(\exists b)\,[(c)(\exists d)\phi abcd \;\&\; (c)(\exists d)\phi abcd \;\&\; (c)(\exists d)\phi abcd] \\
\&\; (a)(\exists b)\,[(c)(\exists d)\phi abcd \;\&\; (c)(\exists d)\phi abcd \;\&\; (c)(\exists d)\phi abcd].
\end{gathered}
$$

In terms of this same example, let me now develop the basic ideas of the next theorem.

Given the proof by instantiation of a given antilogism, one can find the instantiation tree for each statement of the antilogism, and hence the i-e-relettering, and hence the i-expansion.

Given any statement in the antilogism, its i-expansion is *derivable* by repeated use of $P \equiv (P \,\&\, P)$, applied to parts of the statement. The original antilogism and the antilogism of i-expansions are equivalent.

Given next the antilogism of i-expansions and adding new names of the ranges A, B, C, D, namely $A = A_1 = A_2$, $B = B_1 = B_2$, $C = C_1 = C_2 = C_3 = C_4$, $D = D_1 = D_2 = D_3 = D_4 = D_5 = D_6$, we may (in our example) derive the i-e-reletterings as a mere rewrite of the i-expansions.

If the expansion antilogism is given, the renamings are trivial and the relettered antilogism follows. However if, instead, the "relettered" antilogisms were given, the equations identifying its variables would not be trivial, so the unrelettered expansion antilogism would not follow.

In other words, assuming the consistency of the expanded antilogism, that of the relettered antilogism follows, but the reverse is not in general so.

Suppose we wanted to prove the inconsistency of the original antilogism including our example statement and had a proof-by-instantiation to *guide* us in constructing another kind of proof not involving instantiation. From the assumed-for-reductio consistency of the original antilogism, we deduce the equivalent consistency of the expanded antilogism (using only the equivalence transform $P \equiv (P \,\&\, P)$) and then (using the relettering argument only) we deduce the consistency of the relettered antilogism. We would complete our proof by one further reduction to a prenexed antilogism and this prenexed antilogism would be shown inconsistent by the Medieval method.

In the present example, assuming the order of first instantiations in the whole antilogism to have been $a_1 b_1 c_1 d_1 a_2 b_2 c_2 c_3 d_2 d_3 c_4 d_4 d_5 d_6$, the prenexed version of our statement will be:

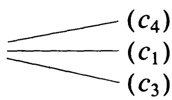
$$(a_1)(\exists b_1)(c_1)(\exists d_1)(a_2)(\exists b_2)(c_2)(c_3)(\exists d_2)(\exists d_3)(c_4)(\exists d_4)(\exists d_5)(\exists d_6)$$
$$[\phi a_1 b_1 c_1 d_1 \,\&\, \phi a_1 b_1 c_2 d_2 \,\&\, \phi a_1 b_1 c_3 d_3 \,\&\, \phi a_2 b_2 c_4 b_4 \,\&\, \phi a_2 b_2 c_1 d_5$$
$$\&\, \phi a_2 b_2 c_3 d_6].$$

In this case, the quantifiers are extracted in the order in which their variables ($a_1 b_1$, etc.) were *first* introduced by instantiation in the proof-by-instantiation.

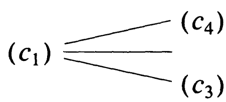
Consulting the tree formulation of the relettered statement needing prenexing, we see the methods used in the prenexing process.

$(a_1)(\exists b_1)$ because of extraction from "one side only". That is, $(x)\phi x \,\&\, \psi = (x)(\phi x \,\&\, \psi)$ and $\psi \,\&\, (x)\phi = (x)(\psi \,\&\, \phi x)$, and similarly for $\exists x$.

(c_1) by first extracting c_1 by "one side only" from the middle of



to



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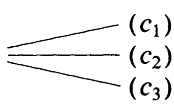
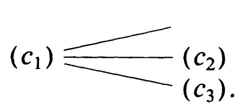
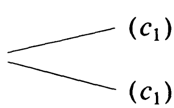


to



And then $(a_1)(\exists b_1)$ is already gone, but by "universal forward" $(a_2)(\exists b_2)(c_1) \supset (c_1)(a_2)(\exists b_2)$ brings (c_1) to the front and finally



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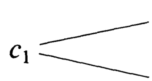

by "both sides" and we are done with c_1.

In doing c_1 we used:

(1) from one side only
(2) from both sides $[(x)\phi x \,\&\, (x)\psi x = (x)(\phi x \,\&\, \psi x)]$
(3) universal forward[11] $[(\exists y)(\underline{x}) \supset (\underline{x})\exists y; (y)(\underline{x}) \supset (\underline{x})(y)]$.

Next $\exists d_1$ by one-side-only, then ditto a_2, $\exists b_2$, and then c_2. Next c_3 is moved to each & by one-side-only and then to front by both-sides. The rest are trivial (all by one-side-only).

Now let us turn from our example to the general case.

Theorem *Given any formally inconsistent quantified antilogism, and also a proof-by-instantiation to guide us, it is possible to expand to prenexed relettered antilogism and to certify the inconsistency of the latter by the Medieval method.*

Proof: The original statements may obviously be replaced by the expanded then relettered statements. We now wish to get each statement of the relettered antilogism into prenex form, possibly weakened by universal forward. We wish in each statement to extract the quantifiers in accord with the order in which their terms were *first* introduced in the proof-by-instantiation.

Suppose we have extracted in order all terms up to v_i which is the next to be extracted. If v_i does not occur in some statements, we ignore those and extract it in the ones in which it occurs.

If $\exists v_i$ occurs in a given statement, no other occurrence of v_1 as (v_i) or $(\exists v_i)$ is found in that statement.

Further, any $\exists v_i$ represents the first instantiation of v_i, since each $\exists$ gets a new term in the instantiation process.

By the instantiation tree, any term in whose scope $\exists v_i$ lies must be instantiated before v_i is first instantiated. So when it is time to extract $\exists v_i$, all other quantifiers in whose scope $\exists v_i$ lies have already been extracted into the main prenex, and $\exists v_i$ may itself be extracted by repeated uses of "one-side-only".

Suppose then that v_i occurs only universally in a given statement. No (v_i) occurs in the scope of another copy of the very same (v_i). This is obvious because the *original* statement was prenex and every quantifier involved a different variable. So no chain in the instantiation tree contains both (v_i) and (v_j), much less (v_i) and (v_i).

So it is time to extract v_i. If a copy of (v_i) is *inside* the prenex of some part of our statement, bring it to the front of that prenex by "universal forward". If it is at the front of one conjunct of a conjunction, bring it before the conjunction by "one-side-only". If it is before both conjuncts, bring it to the front by "both sides". Continue until it is at the end of the main prenex.

In this way the desired prenex of each relettered statement is achieved and so the prenexed relettered antilogism is achieved.

It remains then to see that:

(a) The quantifier pattern of the prenexed relettered antilogism is responsive, and
(b) Its matrix antilogism is inconsistent.

Of these, (b) is obvious, since the matrix antilogism is simply the set of quantifier-free instances, with those corresponding to each original statement conjoined.

And (a) is also obvious. Since the variables are ordered in each statement consistently with the order of original instantiation, there can be no circles. And since each $\exists$ has its distinctive variable v_i, responsiveness is guaranteed. QED

The theorem just proven completes my rehabilitation of the Medieval method. Every valid quantified argument can be certified by that method, and it therefore achieves a kind of complete generality.

However, certain further questions are bound to occur to anyone with any curiosity, and I should like to explore some of those further questions, though I shall do so without any particular effort at rigor or exactness.

Quantification theory validates many complex arguments. But it does so from a basis of basic principles. If I want to show the complete generality of the Medieval method, I ought not to apply it only to complex arguments, but rather—it will be suggested—I ought to apply it to the basic principles of quantification theory.

Now this idea does not, perhaps, turn out to be as interesting as one might hope. The basic principles in question often have free variables, which my Medieval method presupposes to be bound, and these principles sometimes concern prenexing, which my method presupposes as already done.

However, let us look into the matter anyway.

In Kleene [8], p. 82, a rather formal system for standard quantification theory is given. Kleene is thinking of quantification theory with function terms, but I shall ignore these and suppose all terms are names or variables.

His system first postulates all tautologies and the rule of modus ponens. He then gives four basic principles concerning quantification.

The first principle is the rule $\dfrac{C \supset Ax}{C \supset (x)Ax}$, where C does not contain x free. I shall instead take the universal closure of the premise. Prenexing the conclusion would then give us a trivial rule, while we cannot apply our method to an unprenexed statement. However, a conditional can be derived by conditional proof. So we consider the argument

$$\begin{array}{l} (x)(C \supset Ax) \\ C \\ \hline (x)Ax. \end{array}$$

And *this* argument is certifiable by the Medieval method. But in saying that, I am pretending that there are no variables except those shown, which is not really what Kleene has in mind.

The second principle is the axiom schema $At \supset \exists x Ax$ with t free for x in A and another restriction having to do with t being possibly a function. I shall suppose t is a name.

With independently sorted variables, the principle is invalid without the extra premise $\exists x(t = x)$. We may either prenex and get

$$\frac{\exists x(t = x)}{\exists x(At \supset Ax)},$$

or we may use the conditional proof idea and get

$$\begin{array}{l} \exists x(t = x) \\ At \\ \hline \exists x Ax \end{array}.$$

Either of these is Medievally certifiable. But again I am treating Kleene's schema as if it were itself an explicitly given argument rather than a schema.

Kleene's third principle is the axiom schema $(x)Ax \supset At$, with t free for x in A. This becomes either

$$\frac{\exists x(t = x)}{\exists x(Ax \supset At)} \quad \text{or} \quad \frac{\begin{array}{c}\exists x(t = x)\\ (x)Ax\end{array}}{At},$$

and either is Medievally certifiable, with the usual proviso.

His fourth and last principle is

$$\frac{Ax \supset C}{\exists xAx \supset C} \text{ (no } x \text{ in } C).$$

This becomes

$$\frac{\begin{array}{c}(x)(Ax \supset C)\\ (\exists x)Ax\end{array}}{C}.$$

And this is Medievally certifiable, with the usual proviso.

Perhaps more interesting is to look at prenexing rules as such. Since my method presupposes prenexing, it would be interesting to see whether my method can also be used to justify prenexing rules.

Prenexing is most interesting with the material conditional, so let us restrict ourselves to this case.

The four prenexing rules for the conditional are:

$$(1)\ \frac{P \supset \exists xQx}{\exists x(P \supset Qx)} \quad (2)\ \frac{P \supset (x)Qx}{(x)(P \supset Qx)} \quad (3)\ \frac{\exists xQx \supset P}{(x)(Qx \supset P)} \quad (4)\ \frac{(x)Qx \supset P}{\exists x(Qx \supset P)}.$$

And their converses are:

$$(5)\ \frac{\exists x(P \supset Qx)}{P \supset \exists xQx} \quad (6)\ \frac{(x)(P \supset Qx)}{P \supset (x)Qx} \quad (7)\ \frac{(x)(Qx \supset P)}{\exists xQx \supset P} \quad (8)\ \frac{\exists x(Qx \supset P)}{(x)Qx \supset P.}$$

To count as rules, these eight should also be general schema, but I consider them here as explicitly given arguments.

The converses can be Medievally certified in the forms:

$$(5')\ \frac{\begin{array}{c}\exists x(P \supset Qx)\\ P\end{array}}{\exists xQx} \quad (6')\ \frac{\begin{array}{c}(x)(P \supset Qx)\\ P\end{array}}{(x)Qx} \quad (7')\ \frac{\begin{array}{c}(x)(Qx \supset P)\\ \exists xQx\end{array}}{P} \quad (8')\ \frac{\begin{array}{c}\exists x(Qx \supset P)\\ (x)Qx\end{array}}{P.}$$

The prenexing rules themselves can be transposed.

$$(1')\ \frac{(x)(P \,\&\, {\sim}Qx)}{P \,\&\, (x){\sim}Qx} \quad (2')\ \frac{\exists x(P \,\&\, {\sim}Qx)}{P \,\&\, \exists x{\sim}Qx} \quad (3')\ \frac{\exists x(Qx \,\&\, {\sim}P)}{\exists xQx \,\&\, {\sim}P} \quad (4')\ \frac{(x)(Qx \,\&\, {\sim}P)}{(x)Qx \,\&\, {\sim}P}.$$

Here I have also used quantifier conversion (${\sim}\forall = \exists{\sim}$, ${\sim}\exists = \forall{\sim}$).

Now (1′) can be broken into

$$\frac{(x)(P \;\&\; {\sim}Qx)}{P} \quad \text{and} \quad \frac{(x)(P \;\&\; {\sim}Qx)}{(x)({\sim}Qx)},$$

each of which is certifiable, and the remaining cases may be handled similarly.

Let us now turn to questions of a more practical sort. By "practical" I mean questions having less to do with what one would find in a meta-logic book and more to do with introductory logic classes.

It is natural to wonder what would happen if a student in an introductory logic class knew my method and tried to apply it to exercises he found in his logic text. How difficult would such application be?

In Copi [2], p. 128,[12] is an appropriate set of exercises. There are ten exercises, but the last seems to be misprinted, so let us count ourselves wrong there and try for 90%.

Exercise 1 is

$$\begin{array}{l} (\exists x)(y)[(\exists z)Ayz \supset Ayx] \\ \quad (y)\ (\exists z)Ayz \\ \qquad\qquad /\therefore (\exists x)(y)Ayx. \end{array}$$

Prenexing and antiloging gives

$$\begin{array}{r|l} (\exists x)(y)(z) & (Ayz \supset Ayx) \\ (y)(\exists z) & Ayz \\ (x)(\exists y) & {\sim}Ayx. \end{array}$$

And we are done, with $x > y > z$.

Exercises 3, 6, and 7 are just as swiftly done. There is no need to add extra revalidating identity premises nor to split any variable. Our so-far rather lazy student gets 40% already.

Exercise 2 is

$$(x)[(\exists y)Byx \supset (z)Bxz] /\therefore (y)(z)(Byz \supset Bzy).$$

Prenexing and antiloging we get

$$\begin{array}{r|l} (x)(y)(z) & (Byx \supset Bxz) \\ (\exists y)(\exists z) & (Byz \;\&\; {\sim}Bzy). \end{array}$$

We need to add $(z)(\exists x)(x = z)$ and $(y)(\exists z)(y = z)$ and take $y > z > x$. But there are two $\exists z$, so splitting gives

$$\begin{array}{r|l} (z_1)\exists x & x = z_1 \\ (y)\exists z_2 & y = z_2 \\ (x)(y)(z_2) & Byx \supset Bxz_2 \\ (\exists y)(\exists z_1) & Byz_1 \;\&\; {\sim}Bz_1y \end{array}$$

Exercise 4 is similar. Exercise 5 needs an identity premise but no splitting. We have already done 6 and 7. We have 70%.

Exercise 8 requires us to add an identity premise and then to split a variable in order to break a circle.

But 9 is the hardest exercise in this set.

$$
\begin{array}{l}
(x)(Wx \supset Xx) \\
(x)((Yx \,\&\, Xx) \supset Zx) \\
(x)(\exists y)(Yy \,\&\, Ayx) \\
(x)(y)((Ayx \,\&\, Zy) \supset Zx) \ / \therefore (x)((y)(Ayx \supset Wy) \supset Zx).
\end{array}
$$

After prenexing and antiloging and doing the instantiation work, we find we need to add a premise and split a variable.

$$
\begin{array}{r|l}
(y)(\exists x_2) & x_2 = y \\
(x_2) & Wx_2 \supset Xx_2 \\
(x_2) & (Yx_2 \,\&\, Xx_2) \supset Zx_2 \\
(x_1)\exists y & \underline{Yy \,\&\, Ayx_1} \\
(x_1)(y) & (Ayx_1 \,\&\, \underline{Zy}) \supset Zx_1 \\
(\exists x_1)(y) & (\underline{Ayx_1} \supset \underline{Wy}) \,\&\, {\sim}Zx_1.
\end{array}
$$

The difficulty here is seeing the inconsistency of the matrix. Roughly: $Ayx_1 \therefore Wy$, but $y = x_2 \therefore Xy \,\&\, Yy \therefore Zy$, so Zx_1 (since Ayx_1), but ${\sim}Zx_1$.

We now have 90%. In no case was it necessary to split a universal quantifier into two quantifiers. In other words, it was never necessary to expand the matrix of any statement.

A glance at Pollack [15], p. 156, set B, shows 12 exercises even simpler than Copi's. In Kretzmann [9], a quick look did not reveal an exercise set of relevant format.

In Leblanc and Wisdom [12], p. 235, we come to somewhat more exciting exercises. There are 7 exercises called *m* through *r*. Exercise *n*, after we add two identity premises, actually requires us to split two variables rather than only one.

But immediately after (p. 236) we find two further exercises *s* and *t*. These are similar to each other, so let us look at *s*. Here for the first time we see a really challenging exercise. It is

$$
\frac{(x)(\exists y)(Fx \supset Gy)}{(\exists y)(x)(Fx \supset Gy).}
$$

It is obviously valid by un-prenexing and re-prenexing the premises in a different way. The antilogism is

1. $(x)(\exists y)(Fx \supset Gy)$
2. $(y)(\exists x)(Fx \,\&\, {\sim}Gy)$.

An instantiation proof is:

$$
\begin{array}{ll}
\text{from 1,} & (\exists y)(Fx_1 \supset Gy) \\
& \quad Fx_1 \supset Gy_1 \\
\text{from 2,} & (\exists x)(Fx \,\&\, {\sim}Gy_1) \\
& \quad Fx_2 \,\&\, {\sim}Gy_1 \\
\text{from 1,} & (\exists y)(Fx_2 \supset Gy) \\
& \quad Fx_2 \supset Gy_2 \\
\text{from 2,} & (\exists x)(Fx \,\&\, {\sim}Gy_2) \\
& \quad Fx_3 \,\&\, {\sim}Gy_2.
\end{array}
$$

The order of instantiation is $x_1 y_1 x_2 y_2 x_3$.
The inconsistency is Fx_2, Gy_2, $\sim Gy_2$.
The expansion is

$$\begin{array}{r|l} (x_1)(\exists y_1)(x_2)(\exists y_2) & [(Fx_1 \supset Gy_1) \mathbin{\&} (Fx_2 \supset Gy_2)] \\ (y_1)(\exists x_2)(y_2)(\exists x_3) & (Fx_2 \mathbin{\&} \sim Gy_1 \mathbin{\&} Fx_3 \mathbin{\&} \sim Gy_2). \end{array}$$

However, the first instantiations from 1 are not really needed, so a simpler expansion is

$$\begin{array}{r|l} (x_2)(\exists y_2) & Fx_2 \supset Gy_2 \\ (y_1)(\exists x_2)(y_2)(\exists x_3) & (Fx_2 \mathbin{\&} \sim Gy_1 \mathbin{\&} Fx_3 \mathbin{\&} \sim Gy_2). \end{array}$$

As a final step in our survey of introductory logic texts, we turn to Kalish and Montague and Mar ([7], p. 172, group II). Here I expected to find tougher exercises and I did. There are 19, numbered 83 to 101. I only did the first sixteen. There were more split universal quantifiers here than in previous sets.

The hardest problems were numbers 93 and 96. In 96 two premises are added, two variables are split, including universal quantifiers. But the greatest difficulty is that the matrix involves complex statements containing biconditionals and negated biconditionals, and its inconsistency is hard to see. Let us by all means spare ourselves the details!

The hardest problem is number 93. This is a theorem.

$$(x)\,[(Fx \mathbin{\&} (\sim \exists x Fx \vee (x)Gx)) \supset (x)(Fx \vee Gx)]$$

$$\text{Prenexed, } (x)(\exists y)(\exists z)(w)\,[(Fx \mathbin{\&} (\sim Fy \vee Gz)) \supset (Fw \vee Gw)]$$

$$\text{Negated, } (\exists x)(y)(z)(\exists w)(Fx \mathbin{\&} (Fy \supset Gz) \mathbin{\&} \sim Fw \mathbin{\&} \sim Gw).$$

After proof-by-instantiation, we find we need two added premises and we need to split variables to get:

$$\begin{array}{r|l} (\exists x_1)(y_1)(z_1)(\exists w_1)(z_2)(\exists w_2) & [\underline{Fx_1} \mathbin{\&} (\underline{Fy_1} \supset Gz_1) \mathbin{\&} \sim Fw_1 \mathbin{\&} \underline{\sim Gw_1} \mathbin{\&} \\ & \;\overline{Fx_1} \mathbin{\&} (\underline{Fy_1 \supset Gz_2}) \mathbin{\&} \sim Fw_2 \mathbin{\&} \overline{\sim Gw_2}] \\ (x_1)(\exists y_1) & x_1 = y_1 \\ (w_1)(\exists z_2) & w_1 = z_2 \end{array}$$

where I have underlined the parts of the first statement relevant to inconsistency, and where the order of variables is as in the quantifiers of the first statement.

In general, my survey of exercises in various logic texts confirms what my theorem already announced, namely that valid arguments can be validated by the Medieval method. And it also seems to bring out that such validation is usually easier than my theorem gave us any right to expect. At worst we may actually have to perform the proof-by-instantiation which is no harder than what we usually do in natural deduction.

One final "practical" topic remains. There are some obvious conceptual difficulties involved in the teaching of standard natural deduction systems for quantification theory, and these difficulties all have to do with the suspiciousness of the instantiation and generalization rules.

A statement with a free variable, as reached by instantiation, has no truth value and so seems to be no statement at all, and an argument composed largely of such nonstatements, such as a standard natural deduction, seems to be a nonargument.

Logic ought, it would seem, to bring out the *basis* of valid reasoning. And so the basic rules of logic ought to be self-evident, like the rules that *Aristotle* used in syllogistic, and unlike, say, the rules of distribution.[13] The standard rules of UI, EI, UG, and EG are haunted around by too many technical restrictions to be regarded as self-evident.

No doubt the real basis of these rules is given in the meta-logic books that introductory students never see. But in practice we find ourselves talking nonsense in explaining these rules. Few of us truly believe when we UI with later anticipation of UG, that there really are any arbitrary objects, for instance. And given the premise that *some woman will win the beauty context*, it seems a bit premature to pick her out (in EI) and name her "y_1". Perhaps picking her out ought to be left to the judges! And UG looks like inferring a generalization from a single case.

Why then do we instantiate at all and thus get into all this trouble? Obviously, we instantiate in order to put the quantifiers temporarily aside so that propositional logic can get a grip on the matrices.

Since, however, the idea of quantifier responsiveness allows us to do matrix deductions without bothering to eliminate the quantifiers, the suggestion will occur to the reader that perhaps we could revamp our natural deduction systems in some way, so that instantiation and generalization would go away and be replaced by quantifier responsiveness.

Let me say at once that we should not expect too much from this suggestion. The very idea of quantifier responsiveness involves the idea of a "valid matrix". But a matrix by itself is full of free variables, and if a matrix argument is not really an argument, then the hoped-for conceptual virtue is at best an appearance of virtue. Assessing the matrix's validity amounts, one might say, to a hidden uniform instantiation.

Still, even the appearance of virtue is sometimes important in high places, and who is to say that logic is not a high place? So let us proceed to explore.

But nor can we anticipate that all the complex restrictions needed in standard systems will be smoothed away. As we shall see, there will still be the usual kinds of bugs and complications, though often somewhat relocated.

It might be thought that the artificiality of the conditions for responsiveness will be a problem. However, in the type of system I am envisaging, the *conditions* for responsiveness will not be used, though the concept will be. The student will be asked to know only that if all terms are *universally* bound, the argument is automatically responsive.

Let me say also that I am not going to actually try to construct an actual system. I am simply going to present some examples of what derivations might look like in such a system, and I am going to explore what kind of complications might be involved.

The basic idea of such a system would be that if we had a premise $\exists x\phi x$, we would define a new range $X_1 = X \cap \phi$ and, using the premise to permit the supposition that this new range was nonempty, we would introduce a variable x_1, and so deduce by "pseudo existential instantiation" or "EIp" that $(x_1)\phi x_1$.[14] Later, since $X_1 \subset X$, if we deduce $(x_1)\psi x_1$, then by EGp we would conclude that $\exists x \psi x$. Further, if we had a premise $(x)\pi x$, then by UIp we would deduce $(x_1)\pi x_1$.

Further, if we had a step $(x_1)\phi x_1$ and a step $(x_1)\psi x_1$, we would deduce $(x_1)(\phi x_1 \;\&\; \psi x_1)$ and write "Conj, qr", where the "qr" means the quantifier pattern

$$\frac{\begin{array}{c}(x_1)\\(x_1)\end{array}}{(x_1)}$$

is responsive.

Let's do an example, pretending that we know what rules we are following. In this example, I suppose, as is customary, that the initial variables x, y have the same range $X = Y$.

I do the derivation by the usual and the new method in parallel.

1. $\exists x Fx$
2. $(x)(Fx \supset (y)(Gy \supset Qy))$
3. $\exists x Gx \;/\therefore\; \exists x(Fx \vee Sx) \;\&\; \exists x(Gx \;\&\; Qx)$

OLD WAY		NEW WAY Let $X_1 = X \cap F$, $X_2 = X \cap G$.	
4. Fx_1	1, EI	4. $(x_1)(Fx_1)$	1, EIp
5. $Fx_1 \supset (y)(Gy \supset Qy)$	2, UI	5. $(x_1)(Fx_1 \supset (y)(Gy \supset Qy))$	2, UIp
6. $(y)(Gy \supset Qy)$	4, 5 MP	6. $(y)(Gy \supset Qy)$	4, 5 MP, qr
		Here (y) is part of the matrix of the inference; the responsive quantifiers are $\dfrac{\begin{array}{c}(x_1)\\(x_1)\end{array}}{\dots}$	
		6″. $(x)(Gx \supset Qx)$	from 6, since $X = Y$
7. Gx_2	3, EI	7. $(x_2)Gx_2$	3, EIp
8. $Gx_2 \supset Qx_2$	6, UI	8. $(x_2)(Gx_2 \supset Qx_2)$	6″, UIp
9. Qx_2	7,8 MP	9. $(x_2)Qx_2$	7, 8 MP, qr
10. $Gx_2 \;\&\; Qx_2$	7, 9 Conj	10. $(x_2)(Gx_2 \;\&\; Qx_2)$	7, 9 Conj. qr
11. $\exists x(Gx \;\&\; Qx)$	10, EG	11. $\exists x(Gx \;\&\; Qx)$	10, EGp
12. $Fx_1 \vee Sx_1$	4, ad	12. $(x_1)(Fx_1 \vee Sx_1)$	4, ad, qr
13. $\exists x(Fx \vee Sx)$	12, EG	13. $\exists x(Fx \vee Sx)$	12,EGp
14. 11 & 13	conj.	14. 11 & 13	conj.
		No 'qr' is required here.	

In this particular example, everything looks pretty easy. In the move from 6 to 6″, we reletter using $X = Y$. Some restriction will have to be put on this kind of move, for though

$$\begin{array}{c} (x)Ax \\ \underline{X = Y} \\ \hline \therefore (y)Ay \end{array}$$

looks good, we do not have so much liking for

$$\begin{array}{c} (\exists x)(\exists y)(x \neq y) \\ X = Y \\ \hline \therefore (\exists y)(\exists y)(y \neq y) \end{array} \quad \text{or for} \quad \begin{array}{c} (\exists x)(\exists y)(x \neq y) \\ X = Y \\ \hline \therefore (\exists x)(\exists x)(x \neq x) \end{array} \; !$$

But otherwise things look good.

Before looking at one more example where things are more complicated, let us first consider the various fallacies which the usual restrictions in quantification theory are intended to prevent.

One fallacy is that of what I shall call incomplete instantiation. It runs, in the usual kind of system, as follows:

1.	$(x)(x = x)$	Given
2.	$y = x$	UI to y—WRONG
3.	$(y)(y = x)$	UG
4.	$(x)(y)(y = x)$	UG.

In our new kind of system this might be tried as

1.	$(x)(x = x)$	
	Let $Y = X$	
2.	$(y)(y = x)$	Relettering, or perhaps UIp to an arbitrary y.
3.	$(x)(y)(y = x)$	UG.

But here the fallacy is not really natural. The third step is incompetent since UG is not in our system (only UGp is). But, more importantly, step 2 contains a free x, and so is, in our system, regarded as ill-formed.

Another fallacy is that of using an old variable in doing EI. This can involve two existentials. I do the fallacy in parallel.

OLD WAY		NEW WAY	
1. $(\exists x)Px$		1. $(\exists x)Px$	
2. $(\exists x){\sim}Px$		2. $(\exists x){\sim}Px$	
		Let $X_1 = X \cap P$	
3. Px_1	1, EI	3. $(x_1)Px_1$	1, EIp
		Let $X_1 = X \cap {\sim}P$	
4. ${\sim}Px_1$	2,EI—WRONG	4. $(x_1){\sim}Px_1$	2, EIp—WRONG
5. Px_1 & ${\sim}Px_1$	3, 4 conj.	5. $(x_1)(Px_1$ & ${\sim}Px_1)$	3, 4 conj, qr
6. $\exists x(Px$ & ${\sim}Px)$	5, EG	6. $\exists x(Px$ & ${\sim}Px)$	5, EGp.

Here again our new system acquits itself well; the fallacy requires us to define the new range X_1 by both $X_1 = X \cap P$ and $X_1 = X \cap {\sim}P$. Thus we have to define the same term twice. Obviously this is wrong.

Doing an EI to an old variable may also involve a variable representing an arbitrary x. In parallel:

OLD WAY		NEW WAY	
1. $(x)Px$		1. $(x)Px$	
2. $\exists xQx$		2. $\exists xQx$	
$/\therefore (x)(Px \,\&\, Qx)$		$/\therefore (x)(Px \,\&\, Qx)$	
		Let $Y = X$	
3. Py	1, UI to arbitrary y	3. $(y)Py$	1, UIp (or is this just relettering?)
		Let us define $Y = X \cap Q$	
4. Qy	2, EI – WRONG	4. $(y)Qy$	2, EIp – WRONG
5. $Py \,\&\, Qy$	3, 4 conj.	5. $(y)(Py \,\&\, Qy)$	3, 4 conj., qr
6. $(x)(Px \,\&\, Qx)$	5, UG	6. $(x)(Px \,\&\, Qx)$	5, UGp, or just relettering.

Here again the difficulty is that inconsistent conditions are put on Y. If step 3 were simply $(x)Px$, then we would have said "Let $X = X \cap Q$" and the problem would have been one of defining a term in terms of itself.

So far our envisaged system is acquitting itself very well. It seems to bring considerations usually left up in meta-logic down into the natural deduction derivations themselves and seems to make the fallacious moves more implausible than ever.

Unfortunately when we turn to the most important fallacy – the one corresponding to the noncircularity condition – the cracks in our system begin to emerge.

The fallacy is usually one of generalizing in the wrong order and allows us to argue from $(x)\exists y$ to $(\exists y)(x)$. It standardly looks like this:

1. $(x)(\exists y)Rxy \;/\therefore (\exists y)(x)Rxy$	
2. $(\exists y)Rxy$	1, UI to arbitrary x
3. Rxy	2, EI
4. $(x)Rxy$	3, UG WRONG
5. $(\exists y)(x)Rxy$	4, EG

In our system it might look like this:

1. $(x)(\exists y)Rxy \;/\therefore (\exists y)(x)Rxy$
   Let $X_1 = X$
2. $(x_1)(\exists y)Rx_1y$ — 1, UIp (a needless step, really)
   For each x_1 in X_1, let $Yx_1 = Y \cap R(x_1 \ldots)$
   (Here our notation is starting to crack!)
3. $(x_1)(y_{x_1})Rx_1y_{x_1}$ — 2, EIp, qr
   Here the EIp is done under the responsive pattern $\dfrac{(x_1)}{(x_1)}$.
4. $(y_{x_1})(x_1)Rx_1y_{x_1}$ — 3, reiteration, qr. – WRONG
5. $(\exists y)(x_1)Rx_1y$ — 4, EGp
6. $(\exists y)(x)Rxy$ — 5, $X_1 = X$, $\dfrac{\exists y}{\exists y}$ is qr.

In this example, our new system has an advantage and a disadvantage. The advantage is that instead of hiding the switching of quantifiers behind a wrong ordering of valid-looking steps, as the standard method does, the fact of switching is made naked in the move from 3 to 4.

But the disadvantage is that switching *universal* quantifiers looks awfully good. Isn't it true that $(y)(x) = (x)(y)$? Isn't $\frac{(x)(y)}{(y)(x)}$ the very paradigm of a responsive pattern?

The solution to our puzzlement is not far to seek. The reason we standardly cannot go from $(x)\exists y$ to $\exists y(x)$ is that when we say $(x)\exists y$, there may for each entity x be a *different* entity y. Note that this is a different *entity* y. The situation with $(x_1)(y_{x_1})$ is similar but not exactly. For each x_1 in X_1, we define a different range $Y_{x_1} = Y \cap R(x_1 \ldots)$. Thus for each entity x_1, there is a different *variable* "y_{x_1}". In the expression "y_{x_1}", the "x_1" occurs free and must be bound. In the faulty statement $(y_{x_1})(x_1)Rx_1y_{x_1}$, the subscript "$x_1$" of the first "$y_{x_1}$" is unbound, and so the statement is ill-formed, since we do not allow free variables in our system. Whenever "(y_{x_1})" occurs, it must be *preceded* by either "(x_1)" or "$(\exists x_1)$".

To conclude our exploration of the possibility of the sort of system I am envisaging, let us do a pretend derivation for the Leblanc and Wisdom example *s*. I do the old and the new in parallel.

Since the problem results in switching quantifiers and we cannot do this directly, the approach is by indirect proof and instantiation, actual or pseudo.

1. $(x)(\exists y)(Px \supset Qy)$ $/\therefore (\exists y)(x)(Px \supset Qy)$

OLD WAY	NEW WAY
2. $\sim(\exists y)(x)(Px \supset Qy)$ assumed for reductio	2. $\sim(\exists y)(x)(Px \supset Qy)$ assumed for reductio
3. $(y)(\exists x)(Px \not\supset Qy)$ 2, negation in	3. $(y)(\exists x)(Px \not\supset Qy)$ 2, negation in
	Let $Z = Y$
4. $(\exists x)(Px \not\supset Qz)$ 3, UI	4. $(z)(\exists x)(Px \not\supset Qz)$ 3, UI*p* For each $z \in Z$, let $X_z = X \cap (P\_ \not\supset Qz)$
5. $Px_1 \not\supset Qz$ 4, EI	5. $(z)(x_z)(Px_z \not\supset Qz)$ 4, EI*p*, qr (of (z)).
6. $\sim Qz$ 5, PC (propositional calculus)	6. $(z)(\sim Qz)$ 5, PC, qr
7. Px_1 5, PC	7. $(z)(x_z)(Px_z)$ 5, PC, qr Here we must carry (z) along.
8. $(\exists y)(Px_1 \supset Qy)$ 1, UI The premise is now brought in.	8. $(z)(x_z)(\exists y)(Px_z \supset Qy)$ 1, UI*p* Instantiation is to x_z, but (z) is needed to cover. The justification is that $(z)(X_z \subset X)$. Now for each z, x_z, let $Y_{x_z} = Y \cap (Px_z \supset Q\_)$.

9. $Px_1 \supset Qy$	8, EI	9. $(z)(x_z)(y_{x_z})(Px_z \supset Qy_{x_z})$	8, EI*p*, qr
		Instantiation is to y_{x_z}, with $(z)(x_z)$ covering. This is done under qr of $(z)(x_z)$. (Or should there really be "qr" here??)	
10. Qy	7, 9 MP	10. $(z)(x_z)(y_{x_z})Qy_{x_z})$	7, 9 MP, qr
		But $Y = Z$ (see after 3) so, by 6.	
11. $(y)\sim Qy$	6, UG	11. $(y)(\sim Qy)$	6, UG*p* (if 4 was UI*p*)
12. $\sim Qy$	11, UI	12. $(z)(x_z)(y_{x_z})\sim Qy_{x_z}$	11, UI*p*
		Instantiation is to y_{x_z}, with $(z)(x_z)$ covering.	
13. Qy & $\sim Qy$	10, 12 conj.	13. $(z)(x_z)(y_{x_z})(Qy_{x_z} \& \sim Qy_{x_z})$	10, 12 conj. qr.
And this is inconsistent.		And this is inconsistent by matrix and qr.	

I began my exploration of this new kind of natural deduction system because of the idea that eliminating instantiation and generalization, and thus free variables, might give us a conceptually clearer and less complicated approach to natural deduction. Unfortunately, though the kind of system envisaged does look intriguing, it does not seem to free us notably from the usual bugs and complications.

Let me close by returning to the main point of my paper. I have argued that the Medieval method, that of the distribution theorists, can be extended to all of quantificational logic. I have thus attempted to refute Geach's claim that distribution theory was a blot on the history of logic.

However, to speak of "refutation" here is a bit misleading. In my work in this paper I am conscious throughout of following the trail laid out by Geach's criticisms, and of continuing a work of understanding that Geach began. If this be refutation, I am sure it is a kind of refutation that Geach will be happy to receive.

NOTES

1. In [4], see especially chapter 1, and further in chapter 2 of [5]. He refers to the "fools" who invented it in [5], p. 8, and in [4], p. 95. And he refers to the doctrine as one of a series of "corruptions of logic" (see [5], pp. 55, 53).

2. For supposition theory, see [14]; [16]; [10], ch. 5; [13]; and [18]. As these sources indicate, Sherwood, Ockham, and Buridan were among the important supposition theorists. Unfortunately, we do not seem to know who the *distribution* theorists were. Presumably they were very late Medieval or early Modern scholastics.

3. That the doctrine of distribution fails to distinguish $(x)(\exists y)$ from $(\exists y)(x)$ is Geach's most important criticism of that doctrine. Of course, it is a correct criticism (it is elaborated especially in [4], chapter 1). However, we should note that the *supposition* theorists distinguished the $\exists y$ after (x) from the $\exists y$ before (x) by having two different terms for the $\exists y$, depending on its location. (Of course, I am describ-

ing the situation very roughly.) Thus when the distribution theorists went to the single term "undistributed", they came closer to having a concept of existential quantification as such.

4. Geach argues, although in a different way, that distribution theory cannot work in syllogistic-with-negative-terms. See [5], pp. 62–63.

5. These premises make $\bar{B}$ the complement of B with respect to $\bar{B} \cup A \cup B$, which is the sum of all ranges involved in this argument. *Absolute* complementation is, in modern set theory, impossible.

6. Modern scholars have recently worked out what Aristotle meant by "syllogism". When I say that "Aristotle said" a syllogism must have a certain property, I really mean that that property is included in the modern reconstruction of Aristotle's notion. See [17], [3], and also [11].

7. A definition due to Thom, generalizing a usage of Ladd Franklin (see [19], p. 181).

8. I call it "Medieval" although I don't really know whether the distribution theorists were still Medieval. At any rate, they based their work on that of the supposition theorists, who clearly were Medieval.

9. This is similar to what happens in free logic. See, e.g., [6], pp. 29 ff.

10. I believe this construction is similar to one I have seen in a proof of the Löwenheim-Skolem theorem.

11. Note that use of "universal forward" means here that (c_1) is before (a_2) in the ultimate prenex, even though (c) was *after* (a) in the original, and even though (c_1) was in the scope of (a_2) in the relettered expansion. These observations relate to the unsnarling of the variables that "tended to get snarled up" on an earlier approach (see p. 15).

12. In the Fifth Edition, this set is on p. 132. At any event, see the section called "Arguments Involving Relations".

13. That logic ought to bring out the self-evident basis of reasoning was the main point which Descartes and Locke brought against the logic of their time, thus causing a decay in logical studies among philosophers. See Locke, *Essay Concerning Human Understanding*, bk. IV, sec. xvii. And Descartes, *Rules for the Direction of the Mind*, Rule X and beginning of Rule XIV.

    Bochenski notes, in [1], p. 12, that logic tends historically to rise quickly and then decay as "Former gains are forgotten, the problems are no longer found interesting". The instability of logical studies is due perhaps to the problem that if logic sticks close to ordinary reasoning processes, it seems boring and trivial, whereas if it develops esoteric and complicated rules, it seems irrelevant to ordinary reasoning. In either case the question arises: why should people who naturally know how to reason be pestered with the study of logic?

    The problem of the self-evidence of the basis of modern quantification theory seems significant for the future of logic.

14. My method here is somewhat similar to the method of Skolem functions.

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