

On the Equivalence of Proofs Involving Identity

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In the following, we will consider relations of equivalence defined on natural deduction proofs for first-order logic with identity. Such equivalence relations can be derived from theories of normalization, and they are imposed in applications of category-theory to proofs. Our main concern here will be with principles of proof-equivalence for various choices of identity rules. Independent treatments of identity are not common either in accounts of normalization or in the relevant work in category-theory, and we will approach proof-equivalence directly, with only passing comments on its connection with these two fields.¹

The discussion will center on two topics. First, we will develop a pair of moderately strong relations for each of the two usual sets of rules for identity, rules characterizing it as a congruence and as supporting replacement in all contexts. We will also consider two sets of rules for identity that are more analogous to the introduction and elimination rules employed for other constants. Each of these sets of rules suggests principles of proof-equivalence, but the resulting relations prove to be different from those developed for the congruence and replacement rules and are in some ways less satisfactory.

1 Derivations and proof equivalence This section is devoted to concepts and notation for derivations, and to background assumptions concerning proof-equivalence. We fix a first-order language whose nonlogical vocabulary may include sentential, individual, predicate, and function constants. The primitive logical constants are to be \perp , \supset , \wedge , \forall , and $=$, with $\neg\phi$ defined as $\phi \supset \perp$.² It is convenient to employ both predicate and function abstracts in our syntactic analysis; as notation, we use " $x.\phi$ ", " $x.t$ ", and the like. A universal formula $\forall x\phi$ is understood as the application of the constant \forall to an abstract $x.\phi$, and substitution is understood to be an operation which applies to an abstract $x.\phi$ (or $x.u$) and a term t to yield the substitution $\phi(t/x)$ (or $u(t/x)$). We will often use the abbreviated notation " $\phi(t)$ " and " $u(t)$ ", where " $\phi(-)$ " and " $u(-)$ " can be understood as notation for abstracts $x.\phi$ and $x.u$ with the abstracted variable

suppressed. We regard alphabetically variant (or congruent) abstracts as syntactically identical. This enables us to assume, without loss of generality, that bound variables are chosen so as to avoid clashes upon substitution; and such assumptions will not always be explicitly stated. It also enables us to present any pair of abstracts using the same bound variable.

We will use boldface to abbreviate notation for sequences of various sorts (e.g., “ \mathbf{t} ” for t_1, \dots, t_n). Certain related uses of boldface will be explained as they arise. We usually assume that sequences are nonempty, and the exceptions will be noted explicitly. When the empty sequence $()$ is included, the expanded notation “ t_1, \dots, t_n ” is understood to admit the case $n = 0$. The polyadic abstracts $\mathbf{x}.\phi$ and $\mathbf{x}.\mathbf{t}$ are to be the results of simultaneous abstraction, so the variables \mathbf{x} must all be distinct. Simultaneous substitution applies to an abstract $\mathbf{x}.\phi$ or $\mathbf{x}.\mathbf{t}$ and a sequence of terms \mathbf{t} of the right length.

The proofs we consider will be in tree-form, like Gentzen’s N -systems and those of Prawitz [6] and [7]. We use “C”, “D”, and “E” for proofs, writing

$$\begin{array}{c} \mathbf{D} \\ \phi \end{array} \quad \text{or} \quad \mathbf{D} : \phi$$

to display a conclusion. An *assumption* will be a pair consisting of a formula and a natural number index, and we adapt Prawitz’s notation to write “[ϕ]” and the like for assumptions, understanding the brackets (perhaps with primes or subscripts) to stand for the index. Assumptions can thus appear in more than one derivation, making them analogous to variables, and we will pursue this analogy in much of our notation and terminology.

We distinguish the discharge or binding of an assumption from the application of a rule like $\supset I$, writing

$$\begin{array}{c} [\phi] \\ \mathbf{D} \end{array} \quad \text{or} \quad [\phi].\mathbf{D}$$

for an assumption abstract. The latter notation will sometimes be used even for proofs in tree-form in order to indicate the abstract’s scope. On analogy with our conventions for bound variables, we identify abstracts that are alphabetic variants with respect to the indices of bound assumptions.³ The substitution of a proof $\mathbf{D}:\phi$ for an assumption [ϕ] in a derivation \mathbf{E} is written

$$\begin{array}{c} \mathbf{D} \\ [\phi] \quad \text{or} \quad \mathbf{E}(\mathbf{D}/[\phi]) \\ \mathbf{E} \end{array}$$

with indication of the assumption sometimes suppressed in the latter. These stand for the result of grafting \mathbf{D} to the tips of branches at which [ϕ] has free occurrences; the free assumptions of \mathbf{D} are to remain free. In a polyadic abstract $[\phi].\mathbf{D}$, the assumptions [ϕ] must all be distinct though the formulas ϕ that are assumed need not be.

We treat the proper parameters of rules like $\forall I$ in a similar way. A parameter is an ordinary individual variable, and generalization on it appears first in an abstract

$$\begin{array}{c} x \\ \mathbf{D} \end{array} \quad \text{or} \quad x.\mathbf{D}$$

which must satisfy the usual restriction on proper parameters: x may not have free occurrences in any free assumption of D . Like other abstracts, parameter abstracts are distinguished only up to alphabetic variance. Our notation for substitution is analogous to that used in other cases, but the restriction on abstraction here precludes a general analysis of substitution as an operation applying to an abstract and a term. Still, we may look at substitution in this way in cases where the abstract is defined. Substitution in proofs with discharged assumptions does not alter the locations at which top-formulas are discharged, though it may alter the formulas appearing at these locations. For example, the substitution of t for a parameter x in the proof on the left below gives the proof on the right

$$\begin{array}{ccc} [\phi] & & [\phi(t/x)] \\ D & & D(t/x) \\ \psi & & \psi(t/x) \\ \supset I \frac{}{\phi \supset \psi} & & \supset I \frac{}{\phi(t/x) \supset \psi(t/x)} \end{array}$$

where we assume that the index of $[\phi]$ is *distinctive* in the sense of differing from the indices of all assumptions free in $[\phi].D$ (so that any assumption $[\phi(t/x)]'$ already appearing free in $[\phi].D$ has an index different from that of $[\phi(t/x)]$).

Although successive abstraction of assumptions and parameters could come in any order, we will be mainly concerned with proofs and abstracts of the general form

$$\mathbf{x}. [\phi]. D = x_1 \dots x_m. [\phi_1]_1 \dots [\phi_n]_n. D,$$

where \mathbf{x} and $[\phi]$ may each be the empty sequence; we call these *derivations*. When the proof D has a conclusion ψ , the derivation $\mathbf{x}. [\phi]. D$ has the *type* $\mathbf{x}. \phi \rightarrow \mathbf{x}. \psi$; the sequence $\mathbf{x}. \phi$ is its *domain* and the length of \mathbf{x} is its *dimension*. The proof D itself is a derivation with type $\rightarrow \psi$, domain $()$, and dimension 0. A derivation of type $\mathbf{x}. \phi \rightarrow \mathbf{x}. \psi$ can be understood to show that the intersection of the relations expressed by the abstracts $\mathbf{x}. \phi_i$ is included in the relation expressed by $\mathbf{x}. \psi$.

The systems of proof we consider differ only in their rules for identity. All will include the usual introduction and elimination rules for \supset , \wedge , and \forall as well as classical indirect proof (IP)

$$\text{IP} \frac{\begin{array}{c} [\neg \phi] \\ D \\ \perp \end{array}}{\phi} .$$

The result of applying a rule will sometimes be written in linear form – “&I(D , E)” for example. Except for one of the rules of Section 3, the application of any of those we consider will fit the pattern

$$\text{Rule}(\mathbf{x}_1. [\phi]. D_1, \dots, \mathbf{x}_n. [\phi]_n. D_n),$$

where n may be 0.

We take an instance of a rule to be identified not by its figure but rather by the rule (or a label for it) together with the various parameters (such as component expressions) which combine to determine the instance. Thus we distinguish the various instances of $\forall E$ by the abstract and term which combine to determine the conclusion, and this is true even when the abstract is vacuous so the term does not appear in the figure. Similarly, we distinguish the various instances of $\&E$ by the components of the conjunction and whether the first or second is concluded. In linear notation, such determinants of the instance of a rule will sometimes be displayed as subscripts (e.g., “ $\forall E_{x,\phi,t}$ ” or “ $\&E_{1,\phi,\psi}$ ”).

1.1 Definition Given a system of proof all of whose rules take the form indicated above and which includes the usual rules for \supset , \wedge , and \forall , as well as IP, a *proof-equivalence* for the system is a dyadic relation \sim holding between proofs of the same conclusion which satisfies the following:

- (1) \sim is an equivalence relation
- (2) for each rule of the system, we have

Rule $(x_1.[\phi]_1.D_1, \dots, x_n.[\phi]_n.D_n) \sim$ Rule $(x_1.[\phi]_1.D'_1, \dots, x_n.[\phi]_n.D'_n)$

if $D_i \sim D'_i$ for all $1 \leq i \leq n$

- (3) if $D \sim D'$, then $D(E/[\phi]) \sim D'(E/[\phi])$ for any $[\phi]$ and $E:\phi$ and $D(t/x) \sim D'(t/x)$ for any t and x
- (4) (a) $\supset E(\supset I[\phi].D, E) \sim D(E/[\phi])$
 (b) $\supset I[\phi].\supset E(D, [\phi]) \sim D$ provided that $[\phi]$ is not free in D
 (c) $\wedge E_i(\wedge I(D_1, D_2)) \sim D_i$ for $i = 1, 2$
 (d) $\wedge I(\wedge E_1(D), \wedge E_2(D)) \sim D$
 (e) $\forall E_t(\forall Ix.D) \sim D(t/x)$
 (f) $\forall Ix.\forall E_x D \sim D$ provided that x is not free in D
- (5)
$$\frac{\begin{array}{c} D \\ [\neg\phi] \\ \phi \end{array}}{\supset E} \quad \frac{\perp}{\phi} \quad \sim \quad \frac{D}{\phi}$$
 provided that $[\neg\phi]$ is not free in D .

A proof-equivalence extends to all derivations by the stipulation that $x.[\phi].D \sim x.[\phi].D'$ if and only if $D \sim D'$.

By an obvious induction on the structure of derivations, clause (2) implies a strong replacement principle for \sim . If $D \sim D'$, then $(\dots D \dots) \sim (\dots D' \dots)$ when both are defined, where $(\dots - \dots)$ may be a context which binds free assumptions or parameters of D or D' . Clause (3) is included primarily for ease of reference. It holds without special assumption for the specific proof-equivalences we will define later, and its first part follows in any case from clauses (2) and (4)(a).

In (4), $\wedge E_i$ is an instance of \wedge -elimination which derives ϕ_i from $\phi_1 \& \phi_2$, and $\forall E_t$ is an instance of \forall -elimination which derives $\phi(t/x)$ from $\forall x\phi$. Parts (a), (c), and (e) correspond to the “immediate reductions” of Prawitz [7] while (b), (d), and (f) (read right to left) correspond to his “immediate expansions”. There is also an analogy between the first group and the principle of β -

conversion in the λ -calculus and between the second group and the principle of η -conversion. Because of this, we will refer to the two groups as β -principles and η -principles, respectively. The analogy with the λ -calculus carries over to the possibility of replacing any η -principle in this context by a kind of extensionality principle. For example, the alternative ζ -principle for \supset (again adapting Curry's terminology) says that proofs $D, D': \phi \supset \psi$ are equivalent if $\supset E(D, [\phi]) \sim \supset E(D', [\phi])$ for $[\phi]$ not free in either D or D' .

Clause (5) should be compared to (4)(b). It is a plausible and consistent principle specific to negation. One consequence of it is the ζ -principle that derivations $D, D': \phi$ are equivalent if $\supset E([\neg\phi], D) \sim \supset E([\neg\phi], D')$ for $[\neg\phi]$ not free in either D or D' . This is equivalent (in this context) to saying that $\neg\neg I(D) \sim \neg\neg I(D')$ only if $D \sim D'$ (where $\neg\neg I$ is a derived rule defined by $\neg\neg I(D) = \supset I[\neg\phi]. \supset E([\neg\phi], D)$ for any $[\neg\phi]$ not free in D). These consequences of (5) do not mention IP and imply (5) only in the presence of an appropriate β -principle; however, such a principle must be foresworn on pain of rendering equivalent any two derivations of the same conclusion.⁵

The significance of principles of proof-equivalence is often clearest when they are stated as properties of the structure of derivations modulo \sim . It follows from Definition 1.1(2) that each rule (instance) induces an operation Rule/\sim defined on equivalence classes of derivations of certain types. The principles 1.1(4) and 1.1(5) state algebraic properties of these operations. The β -principles of Definition 1.1(4) characterize the operation $\supset I/\sim$, $\wedge I/\sim$, and $\forall I/\sim$ as injections with left inverses given by

$$\begin{aligned} D/\sim &\mapsto [\phi]. \supset E(D, [\phi])/\sim \quad \text{for } [\phi] \text{ not free in } D: \phi \supset \psi, \\ D/\sim &\mapsto (\wedge E_1(D)/\sim, \wedge E_2(D)/\sim), \text{ and} \\ D/\sim &\mapsto x.\forall E_x(D)/\sim \quad \text{for } x \text{ not free in } D. \end{aligned}$$

The η -principles say that the latter operations are also right inverses for the former, implying that both are bijections. The alternate ζ -principles say instead that the operations of the second group are injections, which has the same implication. The principle (5) says that an operation IP/\sim has the right inverse

$$D/\sim \mapsto [\neg\phi]. \supset E([\neg\phi], D)/\sim \quad \text{for } [\neg\phi] \text{ not free in } D:\phi,$$

so the latter is an injection (as is the operation $\neg\neg I/\sim$ which results from composing it with $\supset I/\sim$).

There is a still richer structure that is most economically described using the concepts of category theory. Of the possible approaches, we will consider one that is close to the constructions of [3], [4], and [8]. Beginning with sequences of equivalence classes of derivations, it is possible to construct for each $n \geq 0$ a category $\mathbf{DER}n$ (with finite products) whose objects are the sequences $\mathbf{x}.\phi$ where \mathbf{x} has length n and whose morphisms $f: \mathbf{x}.\phi \rightarrow \mathbf{x}.\psi$ are the sequences $(\mathbf{x}.[\phi].D_i/\sim)_{1 \leq i \leq k}$, where $D_i:\psi_i$ for each $1 \leq i \leq k$ (and k may be 0). These morphisms represent abstract demonstrations of inclusion between the intersections of the n -adic relations expressed by the sequences of abstracts $\mathbf{x}.\phi$ and $\mathbf{x}.\psi$, respectively. The rule $\forall I$ can be used to define a bijection from each set $\text{hom}(\mathbf{x}y.\phi, \mathbf{x}y.\psi)$ ($=\{f \mid f: \mathbf{x}y.\phi \rightarrow \mathbf{x}y.\psi\}$), where y is not free in ϕ , to the set $\text{hom}(\mathbf{x}.\phi, \mathbf{x}.\forall y\psi)$ (where $\forall y\psi$ is the sequence of formulas $\forall y\psi_i$). This bijection

constitutes an adjunction of functors, one of which (from $\mathbf{DER}n$ to $\mathbf{DER}n + 1$) corresponds to an increase of dimension by vacuous abstraction and the other of which (from $\mathbf{DER}n + 1$ to $\mathbf{DER}n$) corresponds to a decrease by universal quantification. The rules for \supset and \wedge also determine adjunctions which give each category $\mathbf{DER}n$ exponentials and a second sort of product. We might generalize the situation here to say that, from the point of view of proof-equivalence, the defining characteristic of true introduction and elimination rules is the existence of an appropriate adjunction. The rules for identity considered in the next section do not take this form, but the rules to be considered in Section 3 do.

2 Congruence and replacement In this section, we will consider proof-equivalence relations for two common choices of identity rules. Two relations will be defined for each of the systems, a “basic” and a “groupoid” equivalence. Our main result will be that natural interpretations of each system within the other by way of derived rules determines an isomorphism between the equivalence classes modulo each pair of corresponding relations. That is, we will see how to regard the congruence and replacement rules as alternatives not only with respect to provability but also with respect to the equivalence of proofs.

2.1 Definition The *congruence* system has (in addition to the basic rules required by Definition 1.1) identity rules of transitivity, reflexivity, symmetry, congruence for function symbols, and congruence for (nonlogical) predicates. Their figures are as follows:

$$\begin{array}{ccc} \text{Trn} \frac{u = v \quad t = u}{t = v} & \text{Rfl} \frac{}{t = t} & \text{Sym} \frac{t = u}{u = t} \\ \\ \text{FC} \frac{t_1 = u_1 \dots t_n = u_n}{ft = fu} & & \text{PC} \frac{t_1 = u_1 \dots t_n = u_n \quad Pt}{Pu}, \end{array}$$

where f is an n -place function symbol and P is an n -place predicate.

2.2 Definition The *replacement* system has, as identity rules, Rfl and a rule which licenses both direct and inverse replacement

$$\text{Rpl} \frac{t = u \quad \phi(t)}{\phi(u)} \quad \frac{u = t \quad \phi(t)}{\phi(u)}$$

in which the abstract $\phi(-)$ may be vacuous (i.e., of the form $x.\phi$ for x not free in ϕ).

We will define proof equivalences for each of these systems by adding to the principles of Definition 1.1 a group of basic principles for the identity rules and a further “groupoid assumption”.

2.3 Definition The *basic* equivalence for the congruence system is the least proof-equivalence satisfying (1)–(5) below; the *groupoid* equivalence is the least proof-equivalence satisfying (1)–(6).⁶

$$(1) \quad \text{Trn} \frac{\text{C} \quad \text{Trn} \frac{\text{D} \quad \text{E}}{t = u \quad s = t}}{u = v \quad s = u} \sim \text{Trn} \frac{\text{C} \quad \text{D}}{u = v \quad t = u} \frac{\text{E}}{s = t} \text{Trn} \frac{}{s = v}$$

$$(2) \quad \text{Trn} \frac{\text{Rfl} \frac{}{u = u} \quad \text{D}}{t = u} \sim \text{D} \quad \text{Trn} \frac{\text{D} \quad \text{Rfl} \frac{}{t = t}}{t = u}$$

$$(3) \quad (a) \quad \text{Trn} \frac{\text{D} \quad \text{E}}{u = v \quad t = u} \quad \text{Sym} \frac{\text{E}}{t = u} \quad \text{Sym} \frac{\text{D}}{u = v} \quad \text{Trn} \frac{}{v = t} \sim \text{Trn} \frac{\text{E} \quad \text{D}}{u = t \quad v = u} \text{Trn} \frac{}{v = t}$$

$$(b) \quad \text{Sym} \frac{\text{Rfl} \frac{}{t = t}}{t = t} \sim \text{Rfl} \frac{}{t = t}$$

$$(c) \quad \text{Sym} \frac{\text{D}}{t = u} \quad \text{Sym} \frac{u = t}{t = u} \sim \text{D} \quad t = u$$

$$(4) \quad (a) \quad \text{Trn} \frac{\text{D} \quad \text{E}}{u = v \quad t = u} \quad \text{FC} \frac{\text{D}}{u = v} \quad \text{FC} \frac{\text{E}}{t = u} \quad \text{FC} \frac{}{ft = fv} \sim \text{Trn} \frac{\text{D} \quad \text{E}}{fu = fv \quad ft = fu} \text{FC} \frac{}{ft = fv}$$

$$(b) \quad \text{FC} \frac{\text{Rfl} \frac{}{t = t}}{ft = ft} \sim \text{Rfl} \frac{}{ft = ft}$$

$$(c) \quad \text{FC} \frac{\text{D}}{t = u} \quad \text{FC} \frac{\text{D}}{t = u} \quad \text{FC} \frac{u = t}{fu = ft} \sim \text{Sym} \frac{ft = fu}{fu = ft}$$

$$(5) \quad (a) \quad \text{PC} \frac{\text{C} \quad \text{D}}{u = v \quad t = u} \quad \text{E} \quad \text{Pt} \quad \text{PC} \frac{}{Pv} \sim \text{PC} \frac{\text{C} \quad \text{PC} \frac{\text{D} \quad \text{E}}{t = u \quad Pt}}{u = v \quad Pv} \text{PC} \frac{}{Pv}$$

$$(b) \quad \text{PC} \frac{\text{Rfl} \frac{}{t = t} \quad \text{D}}{Pt} \sim \text{D} \quad \text{Pt}$$

(6) (Groupoid assumption)

$$\text{Sym} \frac{\text{D} \quad t = u}{u = t} \quad \text{D} \quad \text{Trn} \frac{u = t \quad t = u}{t = t} \sim \text{Rfl} \frac{}{t = t} .$$

(Here and in the following we extend the boldface notation by use of $t = u$ for a sequence of equations $t_i = u_i$, use of $\text{Trn}(\mathbf{D}, \mathbf{E})$ for a sequence of derivations $\text{Trn}(\mathbf{D}_i, \mathbf{E}_i)$, and other similar abbreviations.)

As with the principles of Definition 1.1, these conditions can be understood to state properties of the operations Rule/\sim . Thus (1) describes Trn/\sim as associative and (2) says that the various Rfl/\sim are identities. Together they tell us that these rules provide the composition operation and identities for a category \mathbf{EQ} whose objects are terms and whose morphisms are equivalence classes \mathbf{D}/\sim of proofs $\mathbf{D}: t = u$. In this category-theoretic vocabulary, (3) describes Sym/\sim as a contravariant function from \mathbf{EQ} to \mathbf{EQ} which is its own inverse. And (4) describes FC/\sim for each n -place f as a covariant functor from the product category \mathbf{EQ}^n to \mathbf{EQ} which commutes with Sym/\sim . Similarly, (5) enables us to define, for each n -place P , a functor

$$\mathbf{D}_1/\sim, \dots, \mathbf{D}_n/\sim \mapsto [Pt].\text{PC}(\mathbf{D}_1, \dots, \mathbf{D}_n, [Pt])/\sim$$

(where $[Pt]$ is not free in \mathbf{D}) from \mathbf{EQ}^n to the category $\mathbf{DER0}$ mentioned in the first section. Alternatively, the structure \mathbf{EQ} can be thought of as a many-sorted monoid with an added operation Sym/\sim . For n -place f and P , FC/\sim and PC/\sim are then an n -place homomorphism and n -place action, respectively, in the senses appropriate for such a structure. Finally, the groupoid assumption tells us that the operations Sym/\sim are inverses with respect to Trn/\sim and Rfl/\sim , making \mathbf{EQ} into a many-sorted group, or a category whose morphisms are all invertible – in short, a “groupoid” (on one usage of that term). Clauses (3)(a)–(c) and (4)(b) and (c) then become redundant. Although the groupoid assumption simplifies the structure of proofs under \sim without trivialization, it has been kept separate because it rules out some natural representations of the structure of \mathbf{EQ} .⁷

The principles for the replacement system can be stated most compactly by introducing some new notation. Let $=_1$ be $=$ and let $=_2$ be its converse (so that $t =_2 u$ is $u = t$). We use “ $=_i$ ”, etc., as variables ranging over these two, enabling us to capture both direct and inverse replacement with a single figure, e.g.,

$$\text{Rpl} \frac{t =_i u \quad \phi(t)}{\phi(u)} .$$

2.4 Definition The *basic* equivalence for the replacement system is the least proof-equivalence satisfying (1)–(4) below; and the *groupoid* equivalence is the least proof-equivalence satisfying (1)–(5).

$$(1) \quad (a) \quad \text{Rfl} \frac{\text{---}}{t=t} \quad \text{D} \quad \text{Rpl} \frac{\phi(t)}{\phi(t)} \sim \text{D} \quad \phi(t)$$

$$(b) \quad \text{D} \quad \text{Rfl} \frac{\text{---}}{t=t} \quad \text{Rpl} \frac{t=_i u}{t=_i u} \sim \text{D} \quad t=_i u$$

$$(2) \quad (a) \quad \text{Rpl} \frac{\text{D} \quad \text{E} \quad t=_i u \quad \phi(t/x)}{\phi(u/x)} \sim \text{E} \quad \phi \quad \text{if } x \text{ is not free in } \phi$$

$$(b) \quad \text{Rpl} \frac{\text{D} \quad \text{E} \quad t=_i u \quad \phi(t, t)}{\phi(u, u)} \sim \text{Rpl} \frac{\text{D} \quad \text{Rpl} \frac{\text{D} \quad \text{E} \quad t=_i u \quad \phi(t, t)}{\phi(u, t)}}{t=_i u \quad \phi(u, u)}$$

$$(3) \quad (a) \quad \text{Rpl} \frac{\text{C} \quad \text{Rpl} \frac{\text{D} \quad \text{E} \quad s=_j t \quad \phi(s, u)}{\phi(t, u)}}{u=_i v \quad \phi(t, v)} \sim \text{Rpl} \frac{\text{D} \quad \text{Rpl} \frac{\text{C} \quad \text{E} \quad u=_i v \quad \phi(s, u)}{\phi(s, v)}}{s=_j t \quad \phi(t, v)}$$

$$(b) \quad \text{Rpl} \frac{\text{C} \quad \text{Rpl} \frac{\text{D} \quad \text{E} \quad s=_j t(u) \quad \phi(s)}{\phi(t(u))}}{u=_i v \quad \phi(t(v))} \sim \text{Rpl} \frac{\text{C} \quad \text{D} \quad u=_i v \quad s=_j t(u)}{s=_j t(v) \quad \phi(s)} \quad \text{E} \quad \phi(t(v))$$

$$(c) \quad \text{Rpl} \frac{\text{C} \quad \text{Rpl} \frac{\text{D} \quad \text{E} \quad u=_j v \quad \phi(s(u))}{\phi(s(v))}}{s(v)=_i t \quad \phi(t)} \sim \text{Rpl} \frac{\text{D} \quad \text{C} \quad u=_j v \quad s(v)=_i t}{s(u)=_i t \quad \phi(s(u))} \quad \text{E} \quad \phi(t)$$

$$(4) \quad (a) \quad \text{Rpl} \frac{\text{D} \quad t=_i u \quad [\phi(u)]}{[\phi(t)]} \quad \text{E} \quad \psi(t) \quad \text{D} \quad \text{E} \quad t=_i u \quad \psi(t) \quad \text{Rpl} \frac{\psi(u)}{\phi(u) \supset \psi(u)}$$

$$\text{Rpl} \frac{\text{D} \quad \supset \text{I}[\phi(t)]. \quad \frac{\psi(t)}{\phi(t) \supset \psi(t)}}{t=_i u \quad \phi(u) \supset \psi(u)} \sim \supset \text{I}[\phi(u)]. \quad \frac{\psi(u)}{\phi(u) \supset \psi(u)}$$

$$(b) \quad \text{Rpl} \frac{\text{C} \quad t =_i u \quad \text{D} \quad \text{E} \quad \frac{\phi(t) \quad \psi(t)}{\phi(t) \wedge \psi(t)}}{\phi(u) \wedge \psi(u)} \sim \frac{\text{Rpl} \frac{\text{C} \quad \text{D} \quad t =_i u \quad \phi(t)}{\phi(u)} \quad \text{Rpl} \frac{\text{C} \quad \text{E} \quad t =_i u \quad \psi(t)}{\psi(u)}}{\wedge \text{I} \frac{\phi(u) \quad \psi(u)}{\phi(u) \wedge \psi(u)}}$$

$$(c) \quad \text{Rpl} \frac{\text{D} \quad \forall \text{I}x. \frac{\text{E} \quad \phi(t, x)}{\forall x \phi(t, x)}}{\forall x \phi(u, x)} \sim \frac{\text{Rpl} \frac{\text{D} \quad \text{E} \quad t =_i u \quad \phi(t, x)}{\phi(u, x)}}{\forall \text{I}x. \frac{\phi(u, x)}{\forall x \phi(u, x)}}$$

(5) (Groupoid assumption)

$$\text{Rpl} \frac{\text{D} \quad \text{Rpl} \frac{\text{D} \quad \text{E} \quad t =_i u \quad \phi(t)}{\phi(u)}}{t =_i u \quad \phi(t)} \sim \text{E} \frac{\phi(t)}{\phi(t)}$$

where, in (4)(a), the index of $[\phi(t)]$ is chosen distinct from those free in D or $[\phi(t)].E$ and, in (4)(c), the parameter x is chosen not to be free in D . It should also be noted that these principles are stated only for instances of the rules which are associated with the displayed figures in the natural way (which should be clear in each case).⁸

The basic and groupoid equivalences defined here will shortly prove to induce a structure on derivations comparable to that described above in motivating the definitions of these relations for the congruence system. However, the most direct motivation of Definition 2.4 follows other lines. First, we can compare (1)(a) and (b) to the principles of Definition 1.1(4). Lacking a true introduction rule for $=$, the analogy cannot be exact; but (1)(a) can be thought of as a β -principle for introductions by Rfl (with the natural attempt to generalize it leading us to the indiscernibility system of Section 3). And (1)(b) bears some resemblance to η -principles. It licenses a way of expanding any proof of an equation (albeit not to the result of applying an introduction rule), and it is equivalent to a kind of extensionality principle. Using (3)(b) or (c) and (1)(a), any replacement by the left side of (1)(b) can be restated as a replacement by the right side. So (1)(b) is equivalent in this context to the principle that D, D' : $t_1 = t_2$ are equivalent whenever $\text{Rpl}_{x,\phi}(\text{D}, [\phi(t_i)]) \sim \text{Rpl}_{x,\phi}(\text{D}', [\phi(t_i)])$ for $i = 1, 2$, each abstract x,ϕ , and $[\phi(t_i)]$ not free in D or D' .

However, the principles of (1) are too weak in isolation to provide an adequate theory of proof equivalence. The further basic principles adopted derive from consideration of normalization for proofs with identity. Those of (2) enable us to restrict all replacements to a single occurrence of a term, eliminating vacuous replacements by (2)(a) and separating multiple replacements by (2)(b). The principles of (3) license the transformations “switch” and “shift” of Statman [9] (comparable transformations are applied to sequent proofs in Lifšic [5]). Such

transformations may be used to minimize the complexity of terms appearing in a proof. To see how, first note the following principle, which can be used to reduce the complexity of certain instances of Rfl:

$$\text{Rpl} \frac{\text{D} \quad \text{Rfl} \frac{u =_i v}{t(u) = t(u)}}{t(u) = t(v)} \sim \text{Rpl} \frac{\text{D} \quad \text{Rfl} \frac{u =_i v}{t(v) = t(v)}}{t(u) = t(v)}$$

(This can be shown by using (1)(a) and (b), respectively, to reduce the left and right sides of an instance of (3)(c).) Now consider the following adaptation of one of Lifšic's examples ([5], p. 17):

$$\text{Rpl} \frac{\text{D} \quad \text{Rpl} \frac{\text{Rpl} \frac{\text{E} \quad \text{Rpl} \frac{\text{F} \quad \text{G}}{a = fa \quad a = fa}}{a = f^2 a}}{a = f^3 a}}{a = fa \quad f^3 a = a}}{f^2 a = a}$$

(where powers are used to abbreviate iterated applications of f). This proof can be reduced to

$$\text{Rpl} \frac{\text{D} \quad \text{Rpl} \frac{\text{Rpl} \frac{\text{E} \quad \text{Rpl} \frac{\text{F} \quad \text{Rpl} \frac{\text{G} \quad \text{Rfl} \frac{a = fa}{a = a}}{fa = a}}{f^2 a = a}}{a = fa \quad f^3 a = a}}{a = fa \quad f^3 a = a}}{f^2 a = a}$$

by the principle above and two uses of (3)(b) (run right to left), and thence reduced to

$$\text{Rpl} \frac{\text{D} \quad \text{E} \quad \text{Rpl} \frac{\text{F} \quad \text{Rpl} \frac{\text{G} \quad \text{Rfl} \frac{a = fa}{a = a}}{fa = a}}{a = fa \quad a = fa \quad f^2 a = a}}{a = a \quad f^2 a = a}}{f^2 a = a}$$

by (3)(b) (run left to right). The result contains no term more complex than those appearing in the assumptions and the final conclusion.

The principles of (4) license permutation of Rpl with introduction rules for the constants \supset , \wedge , and \forall ; and appropriate permutation principles for their elimination rules follow by Definition 1.1(4). In the congruence system, principles like these will be consequences of the definition of replacement for compound formulas. In the present case too, they can be thought of as imposing a relation between replacement in compounds and replacement in their components (recalling that by the η -principles of Definition 1.1(4) any proof of a compound can be restated as a proof by its introduction rule). A principle licensing permutation with IP is of some interest for normalization, but it would not follow from the definition of replacement in the congruence system and (as will be noted in Section 3), it has trivializing consequences in the presence of the groupoid assumption.

The principle (5) expresses the groupoid assumption for the replacement system. It renders the operations of direct and inverse replacement by a given proof $D: t = u$ genuine inverses modulo \sim . As in the congruence system, this assumption permits a simpler statement of the principles of proof equivalence. The groupoid equivalence for the replacement system can be characterized as the least proof-equivalence which satisfies the cases of (1)(a) and (b), (2)(a), and the permutation principles (4) for direct replacement alone (eliminating inverse replacement from the right of (4)(a) along the lines of (*) below) as well as satisfying two further principles, a characterization of inverse replacement in terms of direct replacement

$$(*) \text{ Rpl } \frac{\text{D} \quad \text{E} \quad u = t \quad \phi(t)}{\phi(u)} \sim \text{Rpl } \frac{\text{D} \quad \text{Rfl} \frac{u = t \quad u = u}{t = u} \quad \text{E} \quad \phi(t)}{\phi(u)}$$

and the following principle for permutation of direct replacement with itself

$$\text{Rpl } \frac{\text{C} \quad \text{Rpl } \frac{\text{D} \quad \text{E} \quad s(u) = t(u) \quad \phi(s(u), u)}{\phi(t(u), u)} \quad u = v}{\phi(t(v), v)} \sim \text{Rpl } \frac{\text{C} \quad \text{D} \quad \text{C} \quad \text{E} \quad u = v \quad s(u) = t(u) \quad \text{Rpl } \frac{u = v \quad \phi(s(u), u)}{\phi(s(v), v)}}{\phi(t(v), v)}$$

The use of (*) alone would enable us to simplify the definition of the basic equivalence by dropping (3)(c) and eliminating most cases for inverse replacement (apparent exceptions being (1)(b), and (3)(b) for $j = 2$).

We wish to show that the basic and groupoid equivalences defined in Definitions 2.3 and 2.4 yield the same abstract structure for equivalence classes of

derivations. The first step is to show that the special identity rules at each system are derived rules of the other. There is nothing new here, but particular definitions will be noted for reference in later arguments. Both systems may be extended so that they have a common set of identity rules—Trn, Rfl, Sym, a general congruence rule for function abstracts, and a polyadic replacement rule

$$\text{TC} \frac{u_1 = v_1 \dots u_n = v_n}{t(\mathbf{u}) = t(\mathbf{v})} \quad \text{Rpl} \frac{u_1 =_i v_1 \dots u_n =_i v_n \quad \phi(\mathbf{u})}{\phi(\mathbf{v})} .$$

These will be defined so that TC generalizes FC and Rpl generalizes both PC and monadic replacement.

2.5 Definition We define derived rules TC and Rpl in the congruence system, doing this in the case of TC and direct replacement by recursion on the structure of the abstract in question. (Subscripts are used to distinguish direct and inverse replacement, so that Rpl_i has the figure exhibited above, and they are also used to indicate certain other determinants of the instances of rules.)

- (1) (a) $\text{TC}_{x,t}(\mathbf{D}) = \text{Rfl}_t$ if t is a constant or variable not among x
 (b) $\text{TC}_{x,x_i}(\mathbf{D}) = D_i$
 (c) $\text{TC}_{x,f_t}(\mathbf{D}) = \text{FC}_f(\text{TC}_{x,t_1}(\mathbf{D}), \dots, \text{TC}_{x,t_n}(\mathbf{D}))$
- (2) (a) $\text{Rpl}_{1,x,\phi}(\mathbf{D}, E) = E$ if ϕ is a sentential constant
 (b) $\text{Rpl}_{1,x,P_t}(\mathbf{D}, E) = \text{PC}_P(\text{TC}_{x,t_1}(\mathbf{D}), \dots, \text{TC}_{x,t_n}(\mathbf{D}), E)$
 (c) $\text{Rpl}_{1,x,s=t}(\mathbf{D}, E) = \text{Trn}(\text{TC}_{x,t}(\mathbf{D}), \text{Trn}(E, \text{Sym}(\text{TC}_{x,s}(\mathbf{D}))))$
 (d) $\text{Rpl}_{1,x,\phi \supset \psi}(\mathbf{D}, E) = \supset I[\phi(\mathbf{v})].\text{Rpl}_{x,\psi}(\mathbf{D}, \supset E(E, \text{Rpl}_{x,\phi}(\text{Sym}(D_1), \dots, \text{Sym}(D_n), [\phi(\mathbf{v})])))$ where $\mathbf{D}: \mathbf{u} = \mathbf{v}$ for some \mathbf{u} and $[\phi(\mathbf{v})]$ is chosen not to be free in \mathbf{D} or E
 (e) $\text{Rpl}_{1,x,\phi \wedge \psi}(\mathbf{D}, E) = \wedge I(\text{Rpl}_{x,\phi}(\mathbf{D}, \wedge E_1(E)), \text{Rpl}_{x,\psi}(\mathbf{D}, \wedge E_2(E)))$
 (f) $\text{Rpl}_{1,x,\forall y\phi}(\mathbf{D}, E) = \forall I y.\text{Rpl}_{x,\phi}(\mathbf{D}, \forall E_y(E))$ where y is chosen distinct from x and the free parameters of \mathbf{D} and E
- (3) $\text{Rpl}_{2,x,\phi}(\mathbf{D}, E) = \text{Rpl}_{1,x,\phi}(\text{Sym}(D_1), \dots, \text{Sym}(D_n), E)$.

2.6 Definition We define derived rules Trn, Sym, polyadic Rpl, and TC in the replacement system as follows:

- (1) $\text{Trn}(\mathbf{D}, E) = \text{Rpl}_{1,x,t=u}(\mathbf{D}, E)$ where E has the conclusion $t = u$ for some u and x is not free in t
- (2) $\text{Sym}(\mathbf{D}) = \text{Rpl}_{1,x,x=u}(\mathbf{D}, \text{Rfl}_u)$ where \mathbf{D} has the conclusion $u = v$ for some v and x is not free in u
- (3) $\text{Rpl}_{i,xy,\phi}(\mathbf{C}, \mathbf{D}, E) = \text{Rpl}_{i,x,\phi(v/y)}(\mathbf{C}, \text{Rpl}_{i,y,\phi(s/x)}(\mathbf{D}, E))$ where \mathbf{C} and \mathbf{D} have the conclusions $s = t$ and $\mathbf{u} = \mathbf{v}$ for some t and \mathbf{u} , x is not free in \mathbf{v} , and \mathbf{y} are not free in s
- (4) $\text{TC}_{x,t}(\mathbf{D}) = \text{Rpl}_{1,x,t(u/x)=t}(\mathbf{D}, \text{Rfl}_{t(u/x)})$ where \mathbf{D} have conclusions $\mathbf{u} = \mathbf{v}$ for some \mathbf{v} , and x are not free in \mathbf{u} .

(Here (3) is the inductive clause of a definition by recursion on the length of the replaced sequence.)

Since we can regard FC and PC as special cases of TC and Rpl for the abstracts $\mathbf{x}.fx$ and $\mathbf{x}.Px$, the two systems thus extended have a common set of rules. Next we show that they are identical also in their structure modulo proof-equivalence (both basic and groupoid). Specifically, we show, for each system and each relation, both the principles defining the corresponding relation of the other system and the definitions of derived rules in the latter (stated as equivalences). We show this for the two systems in turn, in each case following a preliminary lemma.

2.7 Lemma *The following hold for both proof-equivalences of the congruence system:*

- (1) (a) *a generalization of Definition 2.3(4) to TC*
 (b) *generalizations of Definition 2.3(5)(b) to Rpl_i and of Definition 2.3(5)(a) to Rpl_i as well as the following analogue for Rpl_2 :*

$$\text{Trn} \frac{\text{D} \quad \text{C}}{u = t \quad v = u} \quad \text{E} \quad \text{Rpl} \frac{v = t \quad \phi(t)}{\phi(v)} \sim \text{Rpl} \frac{\text{C} \quad \text{Rpl} \frac{\text{D} \quad \text{E}}{u = t \quad \phi(t)}}{v = u \quad \phi(u)}{\phi(v)}$$

- (2) (a) *an equivalence generalizing Definition 2.5(1)(a) to each vacuous abstract $\mathbf{x}.t$*
 (b) *an equivalence generalizing Definition 2.5(2)(a) to each vacuous abstract $\mathbf{x}.\phi$*
 (3) (a) *an equivalence generalizing Definition 2.5(1)(c) from FC to TC for each function abstract*
 (b) *an equivalence generalizing Definition 2.5(2)(b) from PC to Rpl_i for each predicate abstract*
 (4) (a) $\text{TC}_{xy.t}(\mathbf{C}, \mathbf{D}) \sim \text{Trn}(\text{TC}_{x.t(v/y)}(\mathbf{C}), \text{TC}_{y.t(s/x)}(\mathbf{D}))$ *where \mathbf{C} and \mathbf{D} have conclusions $s = s'$ and $u = v$ for some s' and u , x is not free in v , and y are not free in s*
 (b) *an equivalence for Rpl_i analogous to Definition 2.6(3).*

Proof: The arguments for parts (a) and the cases of parts (b) for Rpl_1 are straightforward inductions on the structure of the abstracts in which replacement is made. The cases for Rpl_2 are immediate consequences or follow by simple calculation.

2.8 Proposition *Clauses (1)–(4) of Definition 2.4 and all clauses of Definition 2.6 (written as equivalences) hold for both equivalence relations of the congruence system. Clause (5) of Definition 2.4 holds for the groupoid equivalence.*

Proof: All arguments are by calculation, and we will only look at examples. Using Definitions 2.5(3) and 2.7(4)(b), 2.4(2)(b) and (3)(a) can be reduced to principles in which each side consists of a single direct replacement; the latter follow from Definition 2.7(3)(b) for appropriate choices of the abstracts $\mathbf{x}.t_i$. As an example of the arguments used for Definition 2.4(3)(b) and (c), we consider the case of (c) where $i = j = 1$. By Definition 2.7(3)(b), this follows from

$$\text{Rpl} \frac{\text{C} \quad \text{Rpl} \frac{\text{D} \quad \text{TC} \frac{u=v}{s(u)=s(v)} \quad \text{E} \quad \phi(s(u))}{\phi(s(v))}}{\phi(t)}$$

$$\text{Rpl} \frac{\text{D} \quad \text{TC} \frac{u=v}{s(u)=s(v)} \quad \text{C} \quad s(v)=t \quad \text{E} \quad \phi(s(u))}{\text{Rpl} \frac{s(u)=t \quad \phi(s(u))}{\phi(t)}}$$

And this follows in full generality from the special case where the abstract $s(-)$ is the identity. For that case, expansion of the right by Definitions 2.5(2)(c) and (3) and reduction by Definitions 2.3(2) and (3)(c) gives

$$\text{Rpl} \frac{\text{C} \quad \text{Rpl} \frac{\text{D} \quad \text{E} \quad u=v \quad \phi(u)}{\phi(v)}}{\phi(t)} \quad \sim \quad \text{Trn} \frac{\text{C} \quad \text{D} \quad v=t \quad u=v \quad \text{E} \quad \phi(u)}{\text{Rpl} \frac{u=t \quad \phi(u)}{\phi(t)}}$$

which is a consequence of Definition 2.7(1)(b).

2.9 Lemma *Clauses (1)(a) and (2)–(4) of Definition 2.4 generalize to polyadic replacement.*

Proof: The arguments are all straightforward inductions on the length of the sequence of replacements (using the generalization of (3)(a) for a number of the others).

2.10 Proposition *Clauses (1)–(5) of Definition 2.3 and all clauses of Definition 2.5 (stated as equivalences) hold for both equivalence relations of the replacement system. Clause (6) of Definition 2.3 holds for the groupoid equivalence.*

Proof: The arguments are all calculations using Definitions 1.1, 2.4 as generalized in 2.9, and the definitions of 2.6. It is also convenient to use a generalized form of the principle (*) noted in the discussion of Definition 2.4. We consider only the case of Definition 2.3(4)(c) for a monadic function (the general case being only notationally different). Using 2.4(1)(a) to expand the left of this principle for an inverse replacement by $ft = ft$ and 2.4(1)(b) to expand its right to a direct replacement in $fu = fu$ and then eliminating TC and Sym by definition, we get

$$\begin{array}{c}
 \begin{array}{c}
 \text{D} \quad \text{Rfl} \text{-----} \\
 t = u \quad t = t \\
 \text{Rpl} \frac{\quad}{u = t} \quad \text{Rfl} \text{-----} \\
 \text{Rfl} \text{-----} \quad \text{Rpl} \frac{\quad}{fu = fu} \\
 ft = ft \quad fu = ft \\
 \text{Rpl}_2 \frac{\quad}{fu = ft}
 \end{array} \\
 \\
 \begin{array}{c}
 \text{D} \quad \text{Rfl} \text{-----} \\
 t = u \quad ft = ft \\
 \text{Rpl} \frac{\quad}{ft = fu} \quad \text{Rfl} \text{-----} \\
 \text{Rpl} \frac{\quad}{fu = ft} \quad \text{Rfl} \text{-----} \\
 \quad \quad \quad fu = fu \\
 \sim \quad \text{Rpl} \frac{\quad}{fu = ft}
 \end{array}
 \end{array}$$

Applying (*), run right to left, to each side gives

$$\begin{array}{c}
 \begin{array}{c}
 \text{D} \quad \text{Rfl} \text{-----} \\
 t = u \quad fu = fu \\
 \text{Rpl} \frac{\quad}{fu = ft} \\
 \text{Rfl} \text{-----} \\
 ft = ft \\
 \text{Rpl}_2 \frac{\quad}{fu = ft}
 \end{array} \\
 \\
 \begin{array}{c}
 \text{D} \quad \text{Rfl} \text{-----} \\
 t = u \quad ft = ft \\
 \text{Rpl} \frac{\quad}{ft = fu} \quad \text{Rfl} \text{-----} \\
 \quad \quad \quad fu = fu \\
 \sim \quad \text{Rpl} \frac{\quad}{fu = ft}
 \end{array}
 \end{array}$$

and this is a case of 2.4(3)(c) (where $i = j = 2$ and $s(-)$, t , u , and v are $x.fx$, ft , u , and t , respectively).

Propositions 2.8 and 2.10 can be understood to show that, as theories of either the basic or groupoid equivalence, the congruence and replacement systems have a common definitional extension, the *extended system*, whose identity rules are Trn, Rfl, Sym, TC, and polyadic Rpl.⁹ We can regard Definitions 2.5 and 2.6 as defining translations $D \mapsto D^0$ from the extended system to the congruence system and $D \mapsto D^*$ from the extended system to the replacement system. And a straightforward argument from 2.8 and 2.10 shows that $D^0 \sim D \sim D^*$ in the extended system. This enables us to complete the task of this section.

2.11 Theorem *There is a translation $D \mapsto D^\dagger$ from the replacement system to the congruence system which induces a bijection $D/\sim \mapsto D^\dagger/\sim$ on abstract proofs.*

Proofs: Let $(\)^\dagger$ be the restriction of $(\)^0$ to derivations of the replacement system. If $D \sim E$ in the replacement system, we have $D^\dagger = D^0 \sim D \sim E \sim E^\dagger$ in the extended system. So $D^\dagger \sim E^\dagger$ in the congruence system since the extended

system is a definitional and hence conservative extension of it. On the other hand, if $D^\dagger \sim E^\dagger$ in the congruence system, then $D \sim D^\dagger \sim E^\dagger \sim E$ in the extended system, so $D \sim E$ in the replacement system. Further, given D in the congruence system, we have $D \sim D^* \sim D^{*0} = D^{*\dagger}$ in the extended system, so $D \sim D^{*\dagger}$ in the congruence system.

Although Theorem 2.11 claims the existence of a bijection only in the case of 0-dimensional abstract proofs with domain $()$, bijections are easily defined on this basis for abstract deviations of all types.

3 Indiscernibility and reflexivity In this section, we will consider two ways of looking at identity which, unlike the concepts of congruence and replacement, motivate introduction and elimination rules whose natural principles of equivalence are analogous to the β - and η -principles of Definition 1.1(4). However, neither approach gives abstract proofs the structure provided by the congruence and replacement systems (for either the basic or groupoid equivalence), and the structures these new systems do provide are in some ways unsatisfactory. The first approach is an adaptation of the usual second-order definition of identity as a relation that holds between objects which have all their properties in common. This suggests an introduction rule to pair with replacement, but the resulting structure of abstract proofs has features that make it seem impossible to capture by congruence rules. The second approach derives from a category-theoretic representation of identity suggested by Lawvere. It provides an elimination rule to pair with Rfl by presenting identity as the narrowest reflexive relation. However, the natural principles of equivalence for these rules reduce the variety of nonequivalent derivations in ways that seem unacceptable.¹⁰

To accommodate the first approach, we enlarge the first-order language to include monadic predicate variables (our notation is “ X ”), but we do not introduce second-order quantifiers. The rules of proof of the replacement system have natural extensions to this language and remain complete (assuming predicate variables are treated in the same way as constants when defining semantic consequence). We recognize abstracts $X.\phi$ and $X.D$ of formulas and derivations by predicate variables (with restrictions and conventions entirely analogous to those for individual variables) as well as substitution of monadic predicate abstracts for them (defining $Xt(x.\phi/X)$ as $\phi(t/x)$ and extending this in the usual way). Clauses (2) and (3) of Definition 1.1 are easily modified to take account of abstraction by and substitution for predicate variables, as is the discussion of the structure of abstract derivations which 1.1 imposes. Within this framework, we consider the following system of proof:

3.1 Definition The *indiscernibility system* has, in addition to basic rules, identity rules of direct replacement (only) and *identity of indiscernibles* (II)

$$\text{II} \frac{\begin{array}{c} X \\ [Xt] \\ D \\ Xu \end{array}}{t = u} .$$

The *indiscernibility equivalence* is the least proof equivalence for the derivations of this system which satisfies the following:

$$(1) \quad \text{Rpl} \frac{\text{II} \frac{X}{[Xt]} \quad \text{E} \frac{Xu}{t = u} \quad \phi(t/x)}{\phi(u/x)} \sim \text{E} \frac{D \frac{[\phi(t/x)]}{D(x, \phi/X)}}{\phi(u/x)}$$

where the index of $[Xt]$ is chosen distinct from those of any other free assumptions of D ;

$$(2) \quad \text{Rpl} \frac{D \frac{t = u \quad [Xt]}{Xu}}{t = u} \sim D \frac{t = u}{t = u}$$

provided that X (and thus $[Xt]$) is not free in D .

With respect to provability, the indiscernibility system is equivalent to the replacement system. We can define a derived rule of reflexivity in the former by

$$\text{Rfl}_t = \text{II } X.[Xt].[Xt],$$

defining inverse replacement from this and direct replacement along the lines of the equivalence (*) of Section 2. And we can introduce II in the replacement system by

$$\text{II}' \frac{X}{[Xt]} \frac{D \frac{Xu}{t = u}}{t = u} = \text{Rfl} \frac{[t = t]}{D(x, t = x/X)} \frac{t = u}{t = u}$$

where the index of $[Xt]$ is chosen distinct from those of all other free assumptions of D . With II defined in this way, the η -principle 3.1(2) holds in the replacement system by Definition 2.4(1)(b). However, the β -principle (1) does not hold, and we will see shortly that adding it would trivialize the account of proof equivalence.

As a first indication of the character of the indiscernibility system, note that by choosing E in 3.1(1) identical to the assumption $[Xt]$, we can see that any derivation $D: Xu$ which has no occurrences of X in assumptions other than $[Xt]$ is equivalent to a replacement of t by u in Xt . Consequently, principles of equivalence that relate replacement by the same equation in varying contexts (like those of Definition 2.4(4)) imply constraints relating the various instances

$D(x.\phi/X)$ of the derivation D . Such constraints impose a degree of uniformity on the operation

$$x.\phi \mapsto [\phi(t)].D(x.\phi/X)/\sim,$$

which assigns a derivation of type $\phi(t) \rightarrow \phi(u)$ to each abstract $x.\phi$. For example, letting f be such an operation, 2.4(4)(b) implies that

$$f(x.\phi \wedge \psi)(C/\sim) = (\wedge I/\sim)(f(x.\phi)(\wedge E_1 C/\sim), f(x.\psi)(\wedge E_2 C/\sim))$$

for each $C:\phi(t) \wedge \psi(t)$ (defining application for abstract derivations by way of substitution of representatives).

The full range of constraints on replacement present in the equivalences of the last section can appear along with 3.1(1) only on pain of trivializing proof-equivalence by rendering all proofs of the same conclusion equivalent. The problem arises for derivations D which can be abbreviated using a rule of *ex falso quodlibet* (EFQ) defined as follows:

$$\text{EFQ} \frac{C}{\perp} = \text{IP} \frac{[\neg\phi]}{C} \frac{\perp}{\phi}$$

for $[\neg\phi]$ not free in C .

3.2 Proposition *Let \sim be a proof equivalence for the rules of the indiscernibility systems which obeys 3.1(1) and 2.4(2)(a) (for direct replacement.) Then $E \sim E'$ for any E and E' with the same conclusion.*

Proof: In 3.1(1) choose D as $\text{EFQ}_{Xu}([\perp]')$ and let the abstract $x.\phi$ be vacuous. Then, applying 2.4(2)(a) to the left side, we get

$$E \sim \text{EFQ} \frac{[\perp]'}{\phi}$$

for any $E:\phi$. So any $E, E':\phi$ are equivalent.

Now 2.4(2)(a) is a rather degenerate analogue to the principles of 2.4(4), and vacuous replacement itself is dispensable. However, the same problem arises with other uniformity constraints, like the following:

$$\text{Rpl} \frac{\text{D} \quad \text{E}}{t = u \quad \phi \wedge \psi(t)} \sim \frac{\wedge E \frac{\text{E}}{\phi \wedge \psi(t)} \quad \text{Rpl} \frac{\text{D} \quad \wedge E \frac{\text{E}}{\phi \wedge \psi(t)}}{t = u \quad \psi(t)}}{\wedge I \frac{\phi \quad \psi(u)}{\phi \wedge \psi(u)}}$$

which we would expect to hold for any definition of Rpl on the basis of congruence rules. Consequently, it is hard to see how the principle 3.1(1) could be satisfied by a nontrivial proof-equivalence for the congruence system.

The second approach to introduction and elimination rules for identity derives from the representation of quantifiers in Lawvere [3]. Quantifiers appear as special cases of two general operators, with the one corresponding to \exists of interest here. In the present context, it can be thought of as a logical constant Σ which applies to a sequence $\mathbf{x}.t = \mathbf{x}.t_1, \dots, \mathbf{x}.t_m$ of k -adic function abstracts and a sequence $\mathbf{x}.\phi = \mathbf{x}.\phi_1, \dots, \mathbf{x}.\phi_n$ of k -adic predicate abstracts to yield an m -adic predicate whose application, $(\Sigma(\mathbf{x}.t)(\mathbf{x}.\phi))\mathbf{u}$, is intended to capture the content of $\exists x_1 \dots \exists x_k (\phi_1(\mathbf{x}) \wedge \dots \wedge \phi_n(\mathbf{x}) \wedge t_1(\mathbf{x}) = u_1 \wedge \dots \wedge t_m(\mathbf{x}) = u_m)$ (where we assume that \mathbf{x} are not free in \mathbf{u}); we allow any of k, m , and n to be 0. If $m = 0$, the operator Σ produces a formula $\Sigma(\mathbf{x}.\phi)$ and, when $k = n = 1$, this is the ordinary existential $\exists x\phi$. Indeed, the rules and equivalences for Σ that correspond to Lawvere's characterization of it as a certain sort of adjoint functor are natural generalizations of the usual rules for \exists together with equivalences analogous to those for \supset, \wedge , and \forall of Definition 1.1(4).

Lawvere [4] considers the treatment of $=$ as the special case $\Sigma(x.x, x.x)$ —that is, as the case for $k = 1, m = 2$, and $n = 0$ where xt_1 and xt_2 are both the identity $x.x$. When the rules and equivalences for Σ are specialized for this case we get the following system:

3.3 Definition The *reflexivity system* has as identity rules Rfl and *minimality*

$$M \frac{\begin{array}{c} x \\ D \\ \phi(x, x) \end{array} \quad \begin{array}{c} E \\ t = u \end{array}}{\phi(t, u)} \quad (\text{where } x \text{ is not free in } \phi(-, -)).$$

And the *reflexivity equivalence* is the least proof equivalence for the resulting derivations which obeys the following:

(1)

$$M \frac{\begin{array}{c} x \\ D \\ \phi(x, x) \end{array} \quad \begin{array}{c} \text{Rfl} \text{---} \\ t = t \end{array}}{\phi(t, t)} \sim \begin{array}{c} D(t/x) \\ \phi(t, t) \end{array}$$

(2)

$$\text{Rfl} \frac{x}{[x = x]} \quad \begin{array}{c} D(x, x) \\ \phi(x, x) \end{array} \quad \begin{array}{c} E \\ t = u \end{array} \quad \begin{array}{c} E \\ [t = u] \\ D(t, u) \end{array} \\ M \frac{\phi(x, x) \quad t = u}{\phi(t, u)} \sim \begin{array}{c} \phi(t, u) \end{array}$$

provided x is not free in the abstract $yz.[y = z].D$ of which $[x = x].D(x, x)$ and $[t = u].D(t, u)$ are instances (where we assume that the index of $[y = z]$ is chosen distinct from that of any other free assumption of D).

This system may be motivated by the concept of identity as the minimal reflexive relation. The introduction rule Rfl shows us that identity is reflexive and the elimination rule M implies that it is included in any reflexive relation. According to the $\beta\eta$ -principles (1) and (2), M induces a bijection from abstract derivations of type $\rightarrow x.\phi(x, x)$ to those of type $yz.y = z \rightarrow yz.\phi(y, z)$ (where $x, y,$ and z are not free in $\phi(-, -)$), with an inverse constructed by diagonalization and application to Rfl . Derivations of the first type can be thought of as demonstrations of reflexivity and those of the second as showing that identity is included in the relation expressed by $yz.\phi(y, z)$.

It was noted in Section 1 that the η -principles for $\supset, \wedge,$ and \forall could be replaced by ζ -principles. Here the analogous substitute is the following:

Let $yz.[y = z].D$ and $yz.[y = z].D'$ be abstracts with a distinctive index chosen for $[y = z]$ and without free occurrences of x . Then, if $D(x, x)(Rfl_x) \sim D'(x, x)(Rfl_x)$, we have $D(t, u)(E) \sim D'(t, u)(E)$ for any $t, u,$ and $E: t = u$.

Since there is no mention of M , this principle can also be considered for the systems of Section 2. In both contexts it tells us that the abstract $x.Rfl_x$ is initial in the category-theoretic sense in a certain category whose objects are abstracts with types of the form $\rightarrow x.\phi(x, x)$ (for x not free in $\phi(-, -)$) and whose morphisms are abstracts whose types have the form $yz.\phi \rightarrow yz.\psi$. Accordingly, we will refer to the principle above as the assumption of *initiality for Rfl* and refer to the least proof equivalence for the replacement system which satisfies it along with Definition 2.4(1)(a) as the *initiality equivalence*.

3.4 Proposition *The initiality equivalence for the replacement system extends the groupoid equivalence. Furthermore, the replacement system under the initiality equivalence and the reflexivity system under its natural equivalence have a common definitional extension.*

Proof: First, we remark that the assumption of initiality for Rfl can be strengthened to apply in cases where y and z have free occurrences in assumptions of D or D' other than $[y = z]$. For we can apply initiality as stated to the results of discharging such assumptions with $\supset I$, recovering $D(t, u)(E)$ and $D'(t, u)(E)$ from the result by way of the β -principle for the \supset -rules. So, given this assumption, we can show that two derivations are equivalent by regarding them as instances $D(t, u)(E)$ and $D'(t, u)(E)$ of a pair of derivations $D(y, z)([y = z])$ and $D'(y, z)([y = z])$ (where $[y = z]$ has a distinctive index) that have equivalent instances $D(x, x)(Rfl_x)$ and $D'(x, x)(Rfl_x)$ for a new variable x . All the principles of 2.4, including the groupoid assumption, follow easily from this extended principle and 2.4(1)(a).

To establish the second part of the proposition, we use the following definitions of M in terms of Rpl and vice versa:

$$\begin{array}{c}
 \begin{array}{c} x \\ D \quad E \\ \phi(x, x) \quad t = u \\ \hline M \quad \phi(t, u) \end{array} = \text{Rpl} \frac{\begin{array}{c} E \quad D(t/x) \\ t = u \quad \phi(t, t) \\ \hline \phi(t, u) \end{array}}{\phi(t, u)} \\
 \\
 \begin{array}{c} \text{Rpl} \frac{\begin{array}{c} D \quad E \\ t =_i u \quad \phi(t) \\ \hline \phi(u) \end{array}}{\phi(u)} = \begin{array}{c} x. \supset I \frac{[\phi(x)]}{[\phi(x)]} \quad D \\ M \frac{\phi(x) \supset \phi(x) \quad t =_i u}{\phi(t) \supset \phi(u)} \quad E \\ \hline \supset E \quad \phi(u) \end{array} \end{array}$$

When the replacement system under the initiality equivalence is extended by the first, we can derive the principles 3.3(1) and (2) as well as the second definition above as an equivalence. And, when the reflexivity system is extended by the second definition, we can derive the assumption of initiality for Rfl, 2.4(1)(a), and the second of the definitions as an equivalence.

The ease with which the initiality assumption enables us to prove a variety of principles of proof equivalence suggests that the reflexivity system is rather strong. This is confirmed by the following alternative characterization of the initiality equivalence for the replacement system.¹¹

- 3.5 Proposition** *The initiality equivalence is the least proof equivalence for the replacement system which satisfies Definition 2.4(1)(a) and the following:*
- (1) for any D, D': $t = u, D \sim D'$
 - (2) for any D, D': ϕ , if $D(t/x) \sim D'(t/x)$ for some x and t , then $D \sim D'$.

Proof: To see that the initiality assumption follows from (1) and (2), suppose that $D(x, x) (\text{Rfl}_x) \sim D'(x, x) (\text{Rfl}_x)$. By (1), $\text{Rfl}_x \sim [x = x]$ so $D(x, x) \sim D'(x, x)$. And thus $D \sim D'$ by (2), so $D(t, u) (E) \sim D'(t, u) (E)$ by Definition 1.1(3).

The argument in the other direction is somewhat longer. We first derive the principle (1) from the initiality assumption in two steps. Putting the strengthened form of the initiality assumption noted in the proof of Proposition 3.4 together with Definition 2.4(1)(a), we can establish the following permutation principle for Rpl and IP:

$$\begin{array}{c}
 \begin{array}{c} D \\ \text{Rpl} \frac{\begin{array}{c} [\neg\phi(t)] \\ E \\ \perp \\ D \quad \text{IP} \frac{\perp}{\phi(t)} \\ t =_i u \quad \phi(t) \\ \hline \phi(u) \end{array}}{\phi(u)} \end{array} \sim \begin{array}{c} D \\ \text{Rpl} \frac{\begin{array}{c} t =_i u \quad [\neg\phi(u)] \\ [\neg\phi(t)] \\ E \\ \perp \\ \text{IP} [\neg\phi(u)] \end{array}}{\phi(u)} \end{array}
 \end{array}$$

Now take a case of this where $i = 1$, $\phi(-)$ is $x.v = x$ (with x not free in v), and E is $[\perp]$. We get $\text{Trn}(D, \text{EFQ}_{v=t}([\perp])) \sim \text{EFQ}_{v=u}([\perp])$. Applying Trn to each side along with $\text{Sym}(\text{EFQ}_{v=t}([\perp]))$, the left side reduces to D (using a number

of principles including the groupoid assumption), rendering D equivalent to $\text{Trn}(\text{EFQ}_{v=u}([\perp]), \text{Sym}(\text{EFQ}_{v=t}([\perp])))$. Since D can be chosen arbitrarily, the trivialization principle (1) follows.¹²

In the case of the second principle, note first that

$$\frac{\frac{\frac{\text{Trn} \frac{y = z}{y = z}}{\forall E \frac{[\forall w y = w]}{y = y}}}{\text{Sym} \frac{y = y}{y = y}}}{\text{Rpl} \frac{y = z \quad \frac{D(y/x)}{\phi(y)}}{\phi(z)}} \sim \frac{D(z/x)}{\phi(z)}$$

by the initiality assumption since the left side reduces to a replacement by Rfl (using the groupoid assumption) when x is substituted for both y and z. The equivalence above is preserved if we substitute t for y and x for z, and thus (2) follows.

Given Proposition 3.4, the principles 3.5(1) and (2) hold for the reflexivity system and indicate the difficulties with that approach to identity. There are no nonequivalent proofs of the same equation, and we can have nonequivalent proofs of the same formula in other cases only when any substitution of terms leaves them nonequivalent.

The proof of 3.5, like that of 3.2, involves reference to derivations containing free assumptions of formulas like \perp or $\forall w y = w$ in the course of showing the equivalence of derivations in which such assumptions do not appear. It might be hoped that we could avoid the trivialization results of 3.2 and 3.5 by restricting the transitivity of \sim in some way to preclude such arguments. This may be true if the notion of a proof-equivalence is otherwise weakened, but a limitation on transitivity alone would not be enough. This can be shown without developing an alternative to transitivity in any detail by exhibiting ways of avoiding its more dubious uses.

Suppose that we have shown that $D \sim D'$ by way of a derivation E whose equivalence to each is shown in unproblematic ways but which contains a free assumption $[\perp]$ not appearing in D or D' (and clearly all free assumptions of E not in D or D' could be replaced by some one $[\perp]$). Now even with a restriction on transitivity we could expect to show the equivalence of $\supset I[\perp].D$ and $\supset I[\perp].D'$ by way of the equivalence of each to $\supset I[\perp].E$ since the assumption $[\perp]$ is no longer free. By the β -principle for the \supset -rules we have

$$\supset E \frac{\supset E \frac{[\neg\phi] \quad D}{\phi}}{\perp} \sim \supset E \frac{\supset E \frac{\supset E \frac{\supset I \frac{D}{\phi}}{\perp \supset \phi} \quad \supset E \frac{[\neg\phi] \quad C}{\phi}}{\perp}}{\perp}$$

and a similar equivalence for D' where in both cases $[\neg\phi]$ is not free in D or D' and C is any proof of ϕ . The right sides of the two cases are equivalent by assumption so we can abstract $[\neg\phi]$ and apply $\supset I$ to the left sides to get $\neg\neg I(D) \sim \neg\neg I(D')$. And thus $D \sim D'$ by the ζ -principle that follows from Definition 1.1(5). Without that principle, we might not be able to prove D and D' equivalent, but a trick like that used above would enable us to replace one by the other in any proof of a negation (and like replacement by \sim , this extends to cases where assumptions or parameters of D and D' are bound).

NOTES

1. For issues relevant to normalization, see [2], [5], and [9]. Lawvere [4] suggests a way of construing identity by way of an adjoint functor comparable to those employed in category-theoretic treatments of other logical constants.
2. The omission of \vee and \exists is not entirely trivial since the principles of proof equivalence that would be implied by their usual definitions are not dual to those for \wedge and \forall —and thus arguably not the right ones. However, it is not hard to add \vee and \exists as further primitives whose rules obey the dual principles.
3. As a result, our abuse of Prawitz's notation appears only with free assumptions; all we know of a bound assumption is the set of locations at which it appears.
4. Recall here that the conventions governing the identity of abstracts enable us to choose identical abstraction prefixes for any two derivations with the same domain and dimension.
5. See [1] for an argument that applies here and see [10], Chapter 10, for a similar argument concerning a different set of background assumptions. In both cases, the trivialization rests on features of classical logic that appear in the present context with the admission of vacuous abstracts $[\phi].D$. It is avoidable in comparable treatments of relevance logics (see [11]).
6. This is essentially the relation \sim_K of [1].
7. For example, as a product $\mathbf{C} \times \mathbf{C}^{\text{op}}$ of a category and its opposite, with $\text{Sym}/$ appearing as the functor which reverses pairs.
8. So (2)(a) applies only to instances of Rpl determined by a vacuous abstract and thus does not subsume all cases of (1)(a). There need be no equivalence of a nonvacuous replacement $\text{Rpl}(D, E)$ with E when $D: t = t$ is not an instance of Rfl .
9. There are a number of ways of presenting the accounts of proof-equivalence as first-order theories with \sim as a primitive predicate. What is important for the following is only the obvious point that the extended system constitutes a conservative extension in its account of \sim (beginning with either the basic or groupoid principles).
10. It follows from the models constructed in [1] that the systems of the preceding section avoid this problem.
11. The construction mentioned in note 10 can be modified to show that the trivialization is limited to obvious consequences of the principles (1) and (2) of Proposition 3.5.

12. Note that the same argument would apply if we were to add this permutation principle to those of Definition 1.1(4) in the presence of the groupoid assumption. (We leave open the question of its consequences without the groupoid assumption.)

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